Robust Adaptive Control for a Class of Nonlinear Systems with Uncertainties and Input Constraints

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Abstract. Robust adaptive controllers for multivariable nonlinear systems with fast convergence rate are designed, in spite of unwanted effects of external disturbances and input constraints. The proposed adaptive controller guarantees the local stability of the plant in the presence of input constraints, with required conditions on the upper bound for controller parameter matrices and initial conditions of the plant. These results are applied to a simulation study.

1. Introduction

In practical control problems, one of the major issues that arises naturally while controlling a dynamic system is the inevitable constraint on the magnitude of its control input. Hence, any input that is determined to process such a system should meet a set of desired control objectives while remaining within certain bounded limits. Ignoring these requirements when designing a controller may lead to a poor performance or even instability for the underlying system.

Samples of recent publications in direct adaptive control of linear continuous-time plants with input constraint are [4] and [10]. To the best of our knowledge for practical problems involving nonlinear systems with input constraints, there are only a few papers in the field of adaptive control of nonlinear systems such as [2], and [6]. In [6], an adaptive control scheme for nonlinear uncertain systems with actuator amplitude and the rate constraint based on Lyapunov direct method is developed. While in this paper, we design a robust direct adaptive control for multivariable nonlinear systems in the presence of uncertainties and disturbance with input saturation by extending the adaptive controller in [1] and [6], using the concept of auxiliary input in [7] and [8]. Here, we find conditions for the upper bounds of parameter matrices and initial conditions of the plant, in order to guarantee its local stability in the presence of input saturation.

2. Robust Adaptive Controller for Nonlinear Systems

A multivariable nonlinear uncertain plant P is described as follows.
$$\dot{x}(t) = [A(x(t)) + \Delta A_1(x(t)) + \Delta A_2(x(t))]x(t) + [B(x(t)) + \Delta B(x(t))]u(t) + d(t), \quad (1)$$

where $A(x) \in R^{n \times n}$, $B(x) \in R^{n \times m}$ are state-dependent, time-varying matrices, for $n \geq m$; $x \in R^n$, $x(0) = x_0$, $t \geq 0$, is directly the measurable state vector; in addition we assume that the output vector is the same as this state vector; $u(t) \in R^m$ is the adaptive control input vector; $d(t) \in R^n$

is a disturbance vector such that
$$\|d(t)\| \leq \|d(t)\|_{\max} \triangleq \xi_d. \quad (2)$$
In (1), $\Delta A_1(x) \in R^{n \times n}$ is an unstructured uncertainty; while $\Delta A_2(x) \in R^{n \times n}$ is a structured uncertainty which belongs to the set D as follows.
$$D = \{\Delta A_2 : R^n \to R^{n \times n} : \Delta A_2 = A_\delta(x)\delta(h_\delta(x)), \\ x \in R^n, \, \delta(\cdot) \in \Delta\}, \quad (3)$$
with $\delta(\cdot)$ belonging to the following set
$$\Delta = \{\delta : R^{l_\delta} \to R^{p_\delta \times n} : \|\delta(h_\delta(x))\| \leq \|w(h_\delta(x))\|\}. \quad (4)$$

In (3), $A_\delta(x) \in R^{n \times p_\delta}$, $h_\delta(x) : R^n \to R^{l_\delta}$ are known functions of the state of the plant; $\delta \in R^{p_\delta \times n}$ is an uncertain function which is bounded by (4); $w \in R^{p_\delta \times n}$ is a known function and $\Delta B(x) \in R^{n \times m}$ is an unstructured uncertainty.

The reference model for (1) is described by
$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t), \quad (5)$$

and the reference input $r(t) \in R^m$ is bounded as
$$\|r(t)\| \leq \|r(t)\|_{\max} \triangleq \xi_r. \quad (6)$$
In (5), $A_r \in R^{n \times n}$ and $B_r \in R^{n \times m}$ are such that the pair $\{A_r, B_r\}$ is completely controllable, and A_r is chosen as an Hurwitzian matrix. The reference input $r(t) \in R^m$ has been chosen such that $x_r(t)$ represents the desired trajectory for the corresponding plant at time t.

Our objective is to determine the adaptive input $u(t)$ so that for all admissible $x(t)$ and $u(t)$, and a given set of initial conditions the system remains bounded. Furthermore system state $x(t)$ tracks $x_r(t)$ as closely as possible. To that end, first, we design the controller structure by assuming that the plant matrices are known; second, we consider the closed-loop system with the above mentioned control law. Subsequently, by using Lyapunov direct method we find the adaptation laws.

Lemma 2.1. Consider the plant P in (1) without uncertainties, disturbance and any constraint on the input, but with known plant matrices $A(x_p)$ and $B(x_p)$ as follows.
$$\dot{x}_p(t) = A(x_p)x_p(t) + B(x_p)u(t). \quad (7)$$

Then the control law
$$u(t) = \hat{B}(x_p)\hat{K}_1[r + \hat{K}_2\hat{A}(x_p)x_p(t)], \quad (8)$$
guarantees the convergence of the error
$$e_p(t) \triangleq x_p(t) - x_r(t), \quad (9)$$
to zero, if there exist gain matrices and state-dependent known matrices such as $\hat{K}_1 \in R^{m \times m}$, $\hat{K}_2 \in R^{m \times s}$ $\hat{B}(x_p) \in R^{m \times m}$ and $\hat{A}(x_p) \in R^{s \times n}$ such that
$$B(x_p)\hat{B}(x_p)\hat{K}_1 = B_r, \quad (10\text{-}a)$$
$$A(x_p) - A_r + B_r\hat{K}_2\hat{A}(x_p) = 0. \quad (10\text{-}b)$$

Further, the solution is globally asymptotically stable.

Proof. Substituting control law (8) in (7) and using (10-a), we obtain the error equation as follows.

$$\dot{e}_p = A_r e_p(t) + [A(x_p) - A_r + B_r \hat{K}_2 \hat{A}(x_p)] \, x_p(t)$$
$$+ [B(x_p)\hat{B}(x_p)\hat{K}_1 - B_r] \, r(t). \quad (11)$$

From (10) we find the dynamic error (11) as follows.

$$\dot{e}_p(t) = A_r e_p(t), \quad (12)$$

where A_r is Hurwitzian, and subsequently it follows that the error which is defined in (9) approaches zero. Thus, the solution is globally asymptotically stable. Q.E.D.

In Lemma 2.1, it is shown that there exist gain matrices $\hat{K}_1$, $\hat{K}_2$ and state-dependent known matrices $\hat{A}(x)$ and $\hat{B}(x)$ such that they provide perfect tracking with known plant matrices $A(x)$ and $B(x)$. Now, we find time-varying adaptive gain matrices $K_1(t)$ and $K_2(t)$ such that the system state $x(t)$ tracks $x_r(t)$ as closely as possible in the presence of uncertainties and disturbance.

Using a control law

$$u(t) = \hat{B}(x)K_1(t) \, [\, r + K_2(t)\hat{A}(x)x(t) + u_{aux}(t)], \quad (13)$$

in plant (1) yields

$$\dot{x}(t) = A_r x(t) + B_r(K_2 - \hat{K}_2)\hat{A}(x)x(t) + B_r u_{aux}(t)$$
$$+ B_r(\hat{K}_1^{-1} - K_1^{-1})\hat{B}^{-1}u(t) + B_r r(t)$$
$$+ [\Delta A_1(x) + \Delta A_2(x)]x(t) + \Delta B(x)u(t) + d(t), \quad (14)$$

where, $K_1(t) \in R^{m \times m}$ and $K_2(t) \in R^{m \times s}$ are time-varying gain matrices and $u_{aux}(t)$ is an auxiliary control that is designed subsequently [7] and [8]. We define the parameter and state errors $\Phi_1(t)$, $\Phi_2(t)$ and e(t) as follows.

$$\Phi_1(t) \triangleq \hat{K}_1^{-1} - K_1^{-1}(t), \quad \Phi_2(t) \triangleq K_2(t) - \hat{K}_2, \quad (15\text{-}a)$$

$$e(t) \triangleq x(t) - x_r(t). \quad (15\text{-}b)$$

The differential equation that represents state error defined in (15-b) with uncertainties and disturbance is as follows.

$$\dot{e} = A_r e + B_r \Phi_1(t)\hat{B}^{-1}u(t) + B_r \Phi_2(t)\hat{A}x(t) + B_r u_{aux}(t)$$
$$+ [\Delta A_1(x) + \Delta A_2(x)]x(t) + \Delta B(x)u(t) + d(t). \quad (16)$$

In order to proceed further with our constructional method, we use direct Lyapunov approach to find adaptation control policies such that the state of the system (1) tracks that of the reference model in (5) as closely as possible in the presence of uncertainties and disturbance.

Theorem 2.1. Consider the plant P in (1) with unstructured uncertainty $\Delta B(x) = 0$. Suppose that there exist the gain matrices $\hat{K}_1 \in R^{m \times m}$, $\hat{K}_2 \in R^{m \times s}$ with det $\hat{K}_1 \neq 0$, and state-dependent known matrices $\hat{B}(x) \in R^{m \times m}$, $\hat{A}(x) \in R^{s \times n}$ with det $\hat{B}(x) \neq 0$ such that they satisfy conditions in (10). Then the control input (13), for $u_{aux} = u_d + u_l + u_s + u_p$ such that

$$u_d = - \{B_r^T Pe \, \|B_r^\dagger\| \, \xi_d\} \, / \, \|B_r^T Pe\|, \quad (17\text{-}a)$$

$$u_l = - SB_r^T Pe, \quad (17\text{-}b)$$

$$u_s = -\{B_r^T Pe\|Pe\| \, \|w(h_\delta(x))\| \, \|A_\delta\| \|x\|\}/\|B_r^T Pe\|^2, \quad (17\text{-}c)$$

$$u_p = - \{B_r^T Pe \, \|Pe\| \, \|\Delta A_1\|_{\max} \, \|x\|\} \, / \, \|B_r^T Pe\|^2, \quad (17\text{-}d)$$

and adaptation laws

$$\dot{\Phi}_1(t) = - Q_1 B_r^T Pe \, u^T \hat{B}^{-T}, \quad (18\text{-}a)$$

$$\dot{\Phi}_2(t) = - Q_2 B_r^T Pe \, x^T \hat{A}^T, \quad (18\text{-}b)$$

guarantees the stability of (16). Here $0 < P = P^T \in R^{n \times n}$ is the solution of the following Riccati matrix equation.

$$- \dot{P} = A_r^T P + PA_r - 2PB_r SB_r^T P + Q, \quad (19)$$

where $0 < Q = Q^T \in R^{n \times n}$, $0 < S = S^T \in R^{m \times m}$, $0 < Q_1 = Q_1^T \in R^{m \times m}$ and $0 < Q_2 = Q_2^T \in R^{m \times m}$. The superscript † represents Penrose pesudo-inverse of B_r which exists if $B_r^T B_r$ is a nonsingular matrix [5].

Proof. Define a positive-definite function V_1 as the Lyapunov function

$$V_1 = e^T(t)Pe(t) + Tr\{\Phi_1^T(t)Q_1^{-1}\Phi_1(t)\}$$
$$+ Tr\{\Phi_2^T(t)Q_2^{-1}\Phi_2(t)\}. \quad (20)$$

Taking derivative of (20) along (16) when $\Delta B(x) = 0$, and for $u_{aux} = u_d + u_l + u_s + u_p$ with (17) to (19) yields

$$\dot{V}_1 \leq -e^T Qe. \quad (21)$$

Thus, $\dot{V}_1 \leq 0$, which shows (16) is stable. Q.E.D.

Remark 2.1. With respect to $K_1(t)$, the Lyapunov function defined in (20) is not radially unbounded. Therefore Theorem 2.1 yields local stability. In addition, it can be shown that $\dot{K}_1(t) = K_1(t)\dot{\Phi}_1(t)K_1(t)$.

In Theorem 2.1, we have assumed that the uncertainty $\Delta B(x) = 0$. In the next step, we remove this condition and consider the effect of uncertainty $\Delta B(x)$ on (1).

Theorem 2.2. Consider the plant P in (1) with $\Delta B(x) \neq 0$. Suppose that there exist the gain matrices $\hat{K}_1 \in R^{m \times m}$, with det $\hat{K}_1 \neq 0$, $\hat{K}_2 \in R^{m \times s}$ and state-dependent known matrices $\hat{B}(x) \in R^{m \times m}$, $\hat{A} \in R^{s \times n}$ with det $\hat{B}(x) \neq 0$, such that they satisfy the conditions in (10). If

$$\sqrt{(V_0)} < \sqrt{(\lambda_{\min}(Q_1^{-1})} \{ \frac{1}{\|\hat{K}_1\|} - \|B_r^\dagger\|\|\Delta B\|_{\max}\|\hat{B}\|_{\max}\}, (22\text{-}a)$$

$$\|\Delta B\|_{\max} < 1/\{\|B_r^\dagger\| \, \|\hat{B}\|_{\max} \, \|\hat{K}_1\|\}, \quad (22\text{-}b)$$

then the control input (13), for $u_{aux} = u_l + u_p + u_a$ such that

$$u_l = - SB_r^T Pe, \quad (23\text{-}a)$$

$$u_p = - [\gamma_1 + \frac{\gamma_2}{\|B_r^T Pe\|} + \frac{\gamma_3}{\|B_r^T Pe\|^2}] B_r^T Pe, \quad (23\text{-}b)$$

$$u_a = - \frac{B_r^T Pe\|Pe\|(\|w(h_\delta(x))\| \, \|A_\delta\| + \|\Delta A_2\|_{\max})\|x\|}{\|B_r^T Pe\|^2}, (23\text{-}c)$$

with

$$\beta^{-1} \triangleq 1 - \|B_r^\dagger\| \, \|\Delta B(x)\|_{\max} \, \|\hat{B}(x)\| \, \|K_1(t)\|, \quad (24)$$

$$\gamma_1 = \beta \, \|B_r^\dagger\| \, \|\Delta B(x)\|_{\max} \, \|\hat{B}(x)K_1(t)\| \, \|S\|, \quad (25\text{-}a)$$

$$\gamma_2 = \beta \, \|B_r^\dagger\| \times \{ \, \|\Delta B\|_{\max} \, \|\hat{B}K_1 K_2 \hat{A}\| \, \|x\|$$
$$+ \|\Delta B\|_{\max} \, \|\hat{B}K_1\| \, \xi_r + \xi_d \, \}, \quad (25\text{-}b)$$

$$\gamma_3 = \beta\|B_r^\dagger\|\|\Delta B\|_{\max}\|\hat{B}K_1\|\|Pe\| \times$$
$$(\, \|w(h_\delta(x))\| \, \|A_\delta\| + \|\Delta A_2\|_{\max})\|x\| \quad (25\text{-}c)$$

and adaptation laws in (18), guarantees the stability of (16). Here P, Q, S and $Q_{1\&2}$ are defined in Theorem 2.1. and V_0 is the initial condition of (20).

Proof. Using the derivative of (20) along (16) when $\Delta B(x) \neq 0$ and for $u_{aux} = u_l + u_p + u_a$ becomes

$$
\begin{aligned}
\dot{V}_1 &= e^T(\dot{P} + A_r^T P + P A_r)e \\
&\quad + 2Tr\{\Phi_1^T(Q_1^{-1}\dot{\Phi}_1 + B_r^T P e u(t)^T \hat{B}^{-T}(x))\} \\
&\quad + 2Tr\{\Phi_2^T(Q_2^{-1}\dot{\Phi}_2 + B_r^T P e x^T \hat{A}^T(x))\} \\
&\quad + (2PB_r S B_r^T P + 2e^T P B_r u_l) \\
&\quad + 2e^T P\{(\Delta A_1(x) + \Delta A_2(x)) x(t) \\
&\quad + \Delta B(x)\hat{B}(x)K_1(r + K_2\hat{A}(x)x + u_l + u_p + u_a) \\
&\quad + d(t)\} + 2e^T P B_r (u_p + u_a).
\end{aligned}
\tag{26}
$$

Substituting (18) and (23-a) in (26) yields.

$$
\begin{aligned}
&\dot{V}_1[e(t), K_1(t), K_2(t)] \\
&= 2e^T P\{(\Delta A_1(x) + \Delta A_2(x)) x(t) \\
&\quad + \Delta B(x)\hat{B}(x)K_1(r + K_2\hat{A}(x)x + u_l + u_p + u_a) \\
&\quad + d(t)\} + 2e^T P B_r (u_p + u_a) \\
&\leq -e^T Q e + 2\|e^T P\| (\|w(h_\delta(x))\| \|A_\delta\| + \|\Delta A_2\|_{max})\|x\| \\
&\quad + 2\|e^T P B_r\| \|B_r^\dagger\| \times \Big\{ \|\Delta B(x)\|_{max} \|\hat{B}(x)K_1\| \xi_r \\
&\quad + \|\Delta B(x)\|_{max} \|\hat{B}(x)K_1\| \|K_2\hat{A}(x)\| \|x\| \\
&\quad + \|\Delta B(x)\|_{max} \|\hat{B}(x)K_1\| \|S\| \|B_r^T P e\| + \xi_d \\
&\quad + \|\Delta B(x)\|_{max}\|\hat{B}(x)K_1\| \|u_{aux}\| \Big\} + 2e^T P B_r (u_p + u_a).
\end{aligned}
\tag{27}
$$

Substituting (23) to (25) in (27) yields $\dot{V}_1 \leq -e^T Q_1 e$, if $\beta^{-1} > 0$. To meet this requirement, using (20) we obtain.

$$
\|\Phi_1(t)\|_{max} < \sqrt{V_0 \lambda_{max}(Q_1)}.
\tag{28}
$$

From (22-a) and (28) we find that
$$
\|\Phi_1\|_{max} < [1/\|\hat{K}_1\|] - \|B_r^\dagger\| \|\Delta B\|_{max} \|\hat{B}\|_{max}.
\tag{29}
$$
From (15-a) we obtain
$$
K_1(t) - \hat{K}_1 = K_1(t)\Phi_1(t)\hat{K}_1,
\tag{30}
$$

and subsequently from norms properties and (29) we conclude that
$$
\|K_1(t)\| - \|\hat{K}_1\| \leq \|K_1(t)\| \|\Phi_1\|_{max} \|\hat{K}_1\|,
\tag{31}
$$

which yields
$$
\|K_1(t)\|_{max} = \|\hat{K}_1\|/\{1 - \|\hat{K}_1\| \|\Phi_1\|_{max}\}.
\tag{32}
$$

Clearly, by substituting $\|K_1(t)\|_{max}$ in (32) as $\|K_1(t)\|$ in (24), we obtain $\beta^{-1} > 0$. Inequality (22-a) yields
$$
\|\Phi_1\|_{max} < 1/\|\hat{K}_1\| - \|B_r^\dagger\| \|\Delta B\|_{max} \|\hat{B}\|_{max}.
\tag{33}
$$

Because $\|\Phi\|_{max} > 0$ and from (33) we have

$$
[1/\|\hat{K}_1\|] - \|B_r^\dagger\| \|\Delta B\|_{max} \|\hat{B}\|_{max} > 0,
\tag{34}
$$

that yields the condition in (22-b). Q.E.D.

3. Adaptive Control with Input Constraints

In the preceding section, we have proposed a robust adaptive controller for a class of nonlinear systems without any constraint on the functional input which is applied to the system. Here our objective is to develop an adaptive scheme for nonlinear systems with input constraints and uncertainties.

Consider the plant P in (1), with uncertainties ΔA_1, ΔA_2, ΔB and disturbance $d(t)$, which are defined in Section 2; in addition we assume that the functional input $u(t) \in R^m$ is constrained by the following requirement, where C_i is a known positive quantity.

$$
|u_i(t)| \leq C_i, \text{ with } i = 1, \ldots, m.
\tag{35}
$$

We define an adaptive controller input as

$$
v(t) = \hat{B}(x)K_1(t) [r + K_2(t)\hat{A}(x)x(t) + u_{aux}(t)],
\tag{36}
$$

where, $K_1(t) \in R^{m \times m}$, $K_2(t) \in R^{m \times s}$ are time-varying matrices and $u_{aux}(t)$ is an auxiliary input. Because the system functional input $u(t)$ in (1) is bounded, we define such an input in terms of our input $v(t)$ in (36) and its corresponding error as follows.

$$
u_i(t) = \begin{cases} v_i(t), & \text{if } |v_i(t)| \leq C_i, \\ C_i sgn(v_i(t)), & \text{if } |v_i(t)| > C_i, \end{cases}
\tag{37}
$$

where, $i = 1, \ldots, m$ and
$$
\Delta u(t) \triangleq u(t) - v(t).
\tag{38}
$$

Suppose, there exists the parameter matrix gain $\hat{K}_3 \in R^{n \times q}$ and $\hat{B}_\Delta(x) \in R^{q \times m}$ such that
$$
B(x) = \hat{K}_3 \hat{B}_\Delta(x),
\tag{39}
$$
for the purpose of removing the effect of $\Delta u(t)$, as in [4] we generate $e_\Delta(t)$ from the following differential equation.

$$
\dot{e}_\Delta(t) = A_r e_\Delta(t) + K_3(t)\hat{B}_\Delta(x)\Delta u(t), \text{ with } e_\Delta(0) = 0.
\tag{40}
$$

By defining
$$
\Phi_3(t) \triangleq \hat{K}_3 - K_3(t),
\tag{41}
$$
and an augmented error e_1 as
$$
e_1(t) \triangleq e(t) - e_\Delta(t),
\tag{42}
$$
and using (1), (15) and (41), the differential equation for the dynamical error $e_1(t)$ becomes

$$
\begin{aligned}
\dot{e}_1(t) &= A_r e_1(t) + B_r \Phi_1(t)\hat{B}^{-1}v(t) + B_r \Phi_2(t)\hat{A}(x)x(t) \\
&\quad + \Phi_3(t)\hat{B}_\Delta(x)\Delta u(t) + [\Delta A_1(x) + \Delta A_2(x)]x(t) \\
&\quad + B_r u_{aux}(t) + \Delta B(x)v(t) + \Delta B(x)\Delta u(t) + d(t).
\end{aligned}
\tag{43}
$$

In the next step, using Lyapunov direct method we construct the adaptation laws for parameter matrices Φ_i's, which guarantee the stability of the $e_1(t)$ with $\Delta B = 0$.

Lemma 3.1. Consider the plant P in (1) with unstructured uncertainty $\Delta B(x) = 0$ and bounded input $u(t)$. Suppose that there exist the gain matrices $\hat{K}_1 \in R^{m \times m}$, $\hat{K}_2 \in R^{m \times s}$, $\hat{K}_3 \in R^{n \times q}$ with $\det \hat{K}_1 \neq 0$ and state-dependent known matrices $\hat{B}(x) \in R^{m \times m}$, $\hat{A}(x) \in R^{s \times n}$ with $\det \hat{B}(x) \neq 0$ and $\hat{B}_\Delta(x) \in R^{q \times m}$ such that they satisfy conditions in (10) and (39). Then the control input (36) with functional input (37), for $u_{aux} = u_d + u_l + u_s + u_p$ such that

$$
u_d = -\{B_r^T P e_1 \|Pe_1\| \xi_d\} / \|B_r^T P e_1\|^2,
\tag{44-a}
$$
$$
u_l = -SB_r^T P e_1,
\tag{44-b}
$$
$$
u_s = -\{B_r^T P e_1 \|Pe_1\| \|w(h_\delta(x))\|\|A_\delta\|\|x\|\}/\|B_r^T P e_1\|^2,
\tag{44-c}
$$
$$
u_p = -\{B_r^T P e_1 \|Pe_1\| \|\Delta A_1\|_{max} \|x\|\} / \|B_r^T P e_1\|^2,
\tag{44-d}
$$

with adaptation laws

$$\dot{\Phi}_1(t) = -Q_1 B_r^T P e_1(t)\, u^T \hat{B}^{-T}, \tag{45-a}$$

$$\dot{\Phi}_2(t) = -Q_2 B_r^T P e_1(t)\, x^T \hat{A}^T, \tag{45-b}$$

$$\dot{\Phi}_3(t) = -Q_3 P e_1(t)\, \Delta u^T(t) \hat{B}_\Delta^T, \tag{45-c}$$

guarantees the boundedness of $e_1(t)$ and Φ_i's. Here P is the solution of (19), where $0 < Q = Q^T \in R^{n \times n}$, $0 < S = S^T \in R^{m \times m}$, $0 < Q_1 = Q_1^T \in R^{m \times m}$, $0 < Q_2 = Q_2^T \in R^{m \times m}$ and $0 < Q_3 = Q_3^T \in R^{n \times n}$.

Proof. The proof is similar to Theorem 2.1 by considering the positive-definite Lyapunov function $V_2(t)$

$$V_2(t) = e_1^T(t) P e_1(t) + Tr\{\Phi_1^T(t) Q_1^{-1} \Phi_1(t)\} \\ + Tr\{\Phi_2^T(t) Q_2^{-1} \Phi_2(t)\} + Tr\{\Phi_3^T(t) Q_3^{-1} \Phi_3(t)\}, \tag{46}$$

and substituting (44) and (45) into its derivative. Q.E.D.

In the next step, we develop a control law such that it stabilizes the error defined in (42) with bounded inputs and $\Delta B \neq 0$. Also we assume that, $\Delta A_{1\&2} = 0$ and $d(t) = 0$. The results can be extended to the plant (1).

Theorem 3.1. Consider the plant P in (1) with unstructured uncertainty $\Delta B(x) \neq 0$, $\Delta A_{1\&2} = 0$, $d(t) = 0$ and bounded input $u(t)$. Suppose that there exist the gain matrices $\hat{K}_1 \in R^{m \times m}$, $\hat{K}_2 \in R^{m \times s}$, $\hat{K}_3 \in R^{n \times q}$ with $\det \hat{K}_1 \neq 0$, and state-dependent known matrices $\hat{B}(x) \in R^{m \times m}$, $\hat{A}(x) \in R^{s \times n}$ with $\det \hat{B}(x) \neq 0$ and $\hat{B}_\Delta(x) \in R^{q \times m}$ such that they satisfy conditions in (10) and (39). If the conditions (22) are held, where $V_0 = V_2(0)$ and $V_2(0)$ is the initial condition of Lyapunov function in (46), then the control input (36), with functional input (37) for $u_{aux} = u_l + u_{p1} + u_\Delta$ that u_l and u_{p1} satisfying

$$u_l = -S B_r^T P e_1(t), \tag{47-a}$$

$$u_{p1} = -[\|\alpha^*\| \gamma_1 + \frac{\gamma_2}{\|B_r^T P e_1(t)\|}] B_r^T P e_1(t), \tag{47-b}$$

where, $\beta^{-1} = 1 - \|\alpha^*\| \|B_r^\dagger\| \|\Delta B(x)\|_{\max} \|\hat{B}(x)\| \|K_1(t)\|$, γ_1 is defined in (25) and

$$\gamma_2 = \beta \|B_r^\dagger\| \|\Delta B\|_{\max} \|\hat{B} K_1\| \|\alpha^*\| \times \{\|K_2 \hat{A}\| \|x\| \\ + \xi_r + \|B_r^\dagger\| \|\Delta B\|_{\max} \|C\|\}, \tag{47-c}$$

$$\alpha_i^* = \begin{cases} 1, & \text{if } \Delta u_i(t) = 0, \\ 0, & \text{if } \Delta u_i(t) \neq 0, \end{cases} \tag{47-d}$$

where, $i = 1, \ldots, m$, and $\alpha^* = diag[\alpha_i^*]$,

$$u_\Delta = -\{B_r^T P e_1 \|B_r^\dagger\| \|\Delta B\|_{\max} \|C\|\} / \|B_r^T P e_1\|, \tag{47-e}$$

and $C = (I_m - \alpha^*)[C_1, \ldots, C_m]^T$ with adaptation laws in (45), guarantees the boundedness of e_1 and Φ_i's. Here, P, Q, S and $Q_{1\&2\&3}$ are defined in Lemma 3.1.

Proof. Using Lyapuov function in (46), the proof is similar to that of Theorem 2.2. Q.E.D.

3.1. Improving Transient Response

Obviously, the behavior of the system in its transient trajectory is important, because the signals may exceed some limits that cause the steady state errors or even instability of the system. We can see this effect in systems with input

constraint, while the input fluctuates very fast and will take large values in the transient time. One way to improve the performance of the system is to decrease the transient response time for the system by increasing its convergence speed. Convergence speed of two or more different Lyapunov designs can be compared by a positive value $\eta_i = -\dot{V}_i/V_i$, $i = 1, 2, \cdots$ [3]. Therefore design A has a faster convergence rate than design B if $\eta_A > \eta_B$.

In [7] and [8], using an auxiliary input the lower bound of η has been maximized. In this design we maximize the lower bound of η, using parameter matrix Φ, in order to analyze the robustness of the system with constrained input. To proceed further, consider plant P in (1) without uncertainties ΔA_1, ΔA_2, ΔB and disturbance $d(t)$. Using control law (36) and (37) with $u_{aux}(t) = 0$ we obtain the dynamic equation for $e_w \triangleq x - x_r$ as follows.

$$\dot{e}_w(t) = A_r e_w(t) + B_r \Phi_1(t) \hat{B}^{-1} u(t) + B_r \Phi_2(t) \hat{A}(x) x(t). \tag{48}$$

Here, we define $e_2(t) = e_w - e_\Delta$ that yields

$$\dot{e}_2(t) = A_r e_2(t) + B_r \Phi_1(t) \hat{B}^{-1} v(t) + B_r \Phi_2(t) \hat{A}(x) x(t) \\ + \Phi_3(t) \hat{B}_\Delta(x) \Delta u(t), \tag{49}$$

Defining

$$\Phi(t) = [\Phi_1(t), \Phi_2(t)]^T, \tag{50}$$

$$\omega(t) = [(\hat{B}^{-1} v(t))^T, (\hat{A}(x) x(t))^T]^T, \tag{51}$$

the dynamic error (49) becomes

$$\dot{e}_2(t) = A_r e_2(t) + B_r \Phi^T(t) \omega(t) + \Phi_3 \hat{B}_\Delta \Delta u(t). \tag{52}$$

Lemma 3.2. Consider the differential equation (52) and suppose that there exist the gain matrices $\hat{K}_1 \in R^{m \times m}$, $\hat{K}_2 \in R^{m \times s}$, $\hat{K}_3 \in R^{n \times q}$ with $\det \hat{K}_1 \neq 0$, and state-dependent known matrices $\hat{B}(x) \in R^{m \times m}$, $\hat{A}(x) \in R^{s \times n}$ with $\det \hat{B}(x) \neq 0$ and $\hat{B}_\Delta(x) \in R^{q \times m}$ such that they satisfy conditions in (10) and (39). Then the adaptation law,

$$\dot{\Phi}(t) = -Q_4 \omega(t) e_2^T(t) P B_r - m(t) Q_4 U \Phi(t), \tag{53}$$

and $\dot{\Phi}_3(t) = -Q_3 P e_2(t)\, \Delta u^T(t) \hat{B}_\Delta^T$ with

$$m(t) > -\{\lambda_{\min}(Q) \|e_2\|^2\} / Tr\{\Phi^T U \Phi\}, \tag{54}$$

guarantees the boundedness of e_2 and Φ. Here, $0 < P = P^T \in R^{n \times n}$ such that

$$A_r^T P + P A_r = -Q, \tag{55}$$

where $0 < Q = Q^T \in R^{n \times n}$, $0 < Q_4 = Q_4^T \in R^{m \times m}$, $0 < U = U^T \in R^{m \times m}$ and $0 < Q_3 = Q_3^T \in R^{n \times n}$.

Proof. Derivative of the positive-definite Lyapunov function

$$V_3(t) = e_2^T P e_2 + Tr\{\Phi^T Q_4^{-1} \Phi\} + Tr\{\Phi_3^T Q_3^{-1} \Phi_3\}, \tag{56}$$

along (52), upon substituting (53) & (54) completes the proof.

$$\dot{V}_3(t) = -e_2^T Q e_2 - 2m(t) Tr\{\Phi^T U \Phi\} \leq 0. \tag{57}$$

3.2. Robustness Analysis

In this section, we provide the conditions where the solution of the system (49) with adaptation law (54) is bounded.

Theorem 3.2. The differential equation (52) with adaptation law (53) and assumptions in Lemma 3.2. has bounded solution if,

$$x^T(t_0)Px(t_0) < \lambda_{\min}(P)\,[\rho/\tau]^2, \qquad (58\text{-a})$$

$$\sqrt{(V_0)} < \sqrt{\lambda_{\min}(Q_4^{-1})} \times [q_0\rho - 2\|P_b\|\,\xi_r\,\tau]\,/$$
$$[2\,\|P_b\|\,\|\hat{B}^{-1}\|_{\max}\,C_n\,\tau + 2\,\|P_b\|\,\|\hat{A}\|_{\max}\,\rho\,], (58\text{-b})$$

$$\rho = C_0 s_0 \lambda_{\min}(\hat{B}^{-1}\hat{B}^{-T})\lambda_{\min}(\hat{K}_1^{-1}\hat{K}_1^{-T})\lambda_{\min}(P_b P_b^T), \quad (59\text{-a})$$
$$\tau = \|P_b\|\,\|\hat{B}^{-1}\|_{\max}\,\|\hat{K}_1^{-1}\|\,\|S\|\times$$
$$\left|2\,\|P_b\|\,\|\hat{K}_2\|\,\|\hat{A}\|_{\max} - q_0\right|. \qquad (59\text{-b})$$

Here, $q_0 = \lambda_{\min}(Q)$, $s_0 = \lambda_{\min}(S)$, $P_b = PB_r$, C_0 is the minimum component of the vector $C = [C_1,\ldots,C_m]^T$, C_n is the norm-2 of C; $0 < P = P^T \in R^{n\times n}$ such that (55) is held; $0 < Q = Q^T \in R^{n\times n}$, $0 < Q_4 = Q_4^T \in R^{m\times m}$, $0 < S = S^T \in R^{m\times m}$ and V_0 is the initial condition of the Lyapunov function defined in (56).

Proof. From (14), (15) and (38) the plant equation without any uncertainty becomes

$$\dot{x}(t) = A_r x(t) + B_r \Phi_1(t)\hat{B}^{-1}u(t) + B_r \Phi_2(t)\hat{A}(x)x(t)$$
$$+ B_r r(t) + B(x)\Delta u(t). \quad (60)$$

We provide the proof for the boundedness of $x(t)$ by considering the cases when

$$|v_i(t)| \le C_i, \text{ for } all\ i = 1,\ldots,m \text{ and } \Delta u_i = 0, \qquad (61\text{-a})$$
$$|v_i(t)| > C_i, \text{ for } all\ i = 1,\ldots,m, \text{ and } \Delta u_i \ne 0, \qquad (61\text{-b})$$
$$|v_i(t)| > C_i, \text{ for } some\ i = 1,\ldots,m. \qquad (61\text{-c})$$

First, consider (61-a), by defining a positive-definite Lyapunov function $W = \frac{1}{2}x^T Px$, the derivative of W along (60) becomes.

$$\dot{W} = -\frac{1}{2}x^T Qx + x^T P_b \Phi_2 \hat{A}x + x^T P_b r + x^T P_b \Phi_1 \hat{B}^{-1}v \quad (62)$$

Clearly, $\|\Phi\| \le \|\Phi\|_{\max}$ and subsequently, from (50) $\|\Phi_1\| \le \|\Phi\|_{\max}$ and $\|\Phi_2\| \le \|\Phi\|_{\max}$, therefore $\dot{W} \le 0$, if

$$\|x\| > \frac{2\,\|P_b\|\,\|\hat{B}^{-1}\|_{\max}\|\Phi\|_{\max}C_n + 2\|P_b\|\,\xi_r}{q_0 - 2\|P_b\|\,\|\hat{A}\|_{\max}\|\Phi\|_{\max}}. \quad (63)$$

Second, consider (61-b), the derivative of W along (60) when $\Delta u \ne 0$ becomes.

$$\dot{W} = -\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x$$
$$+ \sum_{i=1}^{m}(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}C_i sgn(v_i(t)), \qquad (64)$$

where, $(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}$ is $(1,i)^{th}$-element of $(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1}) \in R^{1\times m}$. Now, consider three cases for the signs of $v_i(t)$.

Case A, $sgn(v_i) = sgn(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}$, for all $i = 1,\ldots,m$: Here, (64) becomes

$$\dot{W} = -\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x + \sum_{i=1}^{m}C_i\left|x^T P_b \hat{K}_1^{-1}\hat{B}^{-1}\right|_{(1,i)} (65)$$

Since $|v_i| > C_i$, we have

$$\sum_{i=1}^{m}(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}v_i > \sum_{i=1}^{m}\left|(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}\right|C_i, \quad (66\text{-a})$$
then
$$-\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x + \sum_{i=1}^{m}(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}v_i >$$
$$-\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x + \sum_{i=1}^{m}\left|(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}\right|C_i. (66\text{-b})$$

Adding and subtracting terms $x^T P_b K_2 \hat{A}x$ and $x^T P_b r$ to the left-hand side of (66-b) and using (63) yields

$$0 > -\frac{1}{2}x^T Qx + x^T P_b \Phi_2 \hat{A}x + x^T P_b r + x^T P_b \Phi_1 \hat{B}^{-1}v$$
$$> -\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x + \sum_{i=1}^{m}C_i\left|x^T P_b \hat{K}_1^{-1}\hat{B}^{-1}\right|_{(1,i)} (66\text{-c})$$
which, shows $\dot{W} < 0$.

Case B, $sgn(v_i) = -sgn(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}$, for all $i = 1,\ldots,m$: Here, (64) becomes

$$\dot{W} = -\frac{1}{2}x^T Qx + x^T P_b \hat{K}_2 \hat{A}x - \sum_{i=1}^{m}C_i\left|x^T P_b \hat{K}_1^{-1}\hat{B}^{-1}\right|_{(1,i)} (67)$$

which is negative semi-definite if
$$\|x\| < \rho/\tau, \qquad (68)$$
that yields (58-a).

Case C, $sgn(v_i) = sgn(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}$, for some $i = 1,\ldots,m$ and $sgn(v_i) = -sgn(x^T P_b \hat{K}_1^{-1}\hat{B}^{-1})_{(1,i)}$ for others: Using (63) and (68), it can be shown $\dot{W} < 0$ herein.

From (63) and (68) we construct the region A such that $\dot{W} \le 0$, and thus x remains bounded. The region can be written as follows.

$$A \triangleq \left\{\frac{2\|P_b\|\|\hat{B}^{-1}\|_{\max}\|\Phi\|_{\max}C_n + 2\|P_b\|\xi_r}{q_0 - 2\|P_b\|\,\|\hat{A}\|_{\max}\|\Phi\|_{\max}} < \|x\| < \frac{\rho}{\tau}\right\} (69)$$

yielding

$$\|\Phi\|_{\max} < \left[\frac{q_0\rho - 2\|P_b\|\,\xi_r\,\tau}{2\,\|P_b\|\,\|\hat{B}^{-1}\|_{\max}\,C_n\,\tau + 2\,\|P_b\|\,\|\hat{A}\|_{\max}\,\rho}\right]. (70)$$

Therefore we obtain (58-b).

Third, two cases (61-a) and (61-b) cover (61-c), and here we omit the details. Hence, $x(t)$ is bounded due to the conditions in (58). Further e_2, Φ and $u(t)$ are bounded [9].

4. Numerical Example

The results of the preceding sections are now applied to the following nonlinear uncertain system given by [1],

$$\dot{x}_1 = \delta_1 x_1(t) + (1 + \delta_2)x_2(t), \qquad (71\text{-a})$$

$$\dot{x}_2 = -\beta x_1(t) + \varepsilon(\alpha - x_1^2)x_2(t) + \frac{b + \delta_3}{1 + x_1^2 + x_2^2}u(t), \quad (71\text{-b})$$

where, α, β, ε, b, $\delta_i \in R$. are unknown with $\delta_i \in [-1, 1]$. The reference model is given by

$$\dot{x}_r = \begin{bmatrix} 0 & 1 \\ -A_1 & -A_2 \end{bmatrix}x_r + \begin{bmatrix} 0 \\ 1 \end{bmatrix}r(t), \qquad (72)$$

where, $A_1 = 100$ and $A_2 = 10$. We assume that the control input is constrained with $C_1 = 12$, when the maximum of the control input in unconstrained case is $u_{max} = 26$. We choose

$$\hat{A}(x) = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x_1^2 \end{bmatrix}^T, \quad \hat{B} = 1 + x_1^2 + x_2^2, \qquad (73\text{-a})$$

$$\hat{K}_1 = \frac{1}{b}, \quad \hat{K}_2 = \begin{bmatrix} -\beta + A_1, & \varepsilon\alpha + A_2, & -\varepsilon \end{bmatrix}, \qquad (73\text{-b})$$

$$\Delta B = [0, \frac{\delta_3}{1 + x_1^2 + x_2^2}]^T, \ A_\delta = [1, 0]^T, \ \delta = [\delta_1, \delta_2], \quad (73\text{-c})$$

$$\hat{B}_\Delta(x) = 1/[1 + x_1^2 + x_2^2], \quad \hat{K}_3 = [0, b]^T. \qquad (73\text{-d})$$

Applying the results developed in this paper with the following set of parameters $\alpha = 1$, $\beta = 1$, $\varepsilon = 2$, $b = 3$, $\delta_i = 1$ and initial conditions $x(0) = [0, 0]^T$, $K_1(0) = 0.1$ and $K_2(0) = [0, 0, 0]^T$, we can show that the results reported in the literature are substantially improved. We define $e_1(t) = x_1 - x_{r1}$ and $e_2(t) = x_2 - x_{r2}$ as state error trajectories. The state error trajectories and control signal versus time, in both saturated and unsaturated cases, which, only auxiliary input is used to remove the effect of uncertainties, and the oscillations between plant state and desired state are shown in Fig. 1. The state error trajectories and input control signal versus time for both saturated and unsaturated cases are shown in Fig. 2. Here, we apply $m(t)$ in addition to auxiliary input to the system, so that the fast convergence rate for both saturated and unsaturated cases is obtained, while the maximum amplitude of the controller is reduced. In addition, the error between plant state and desired reference state approaches zero, whether the input is saturated or not.
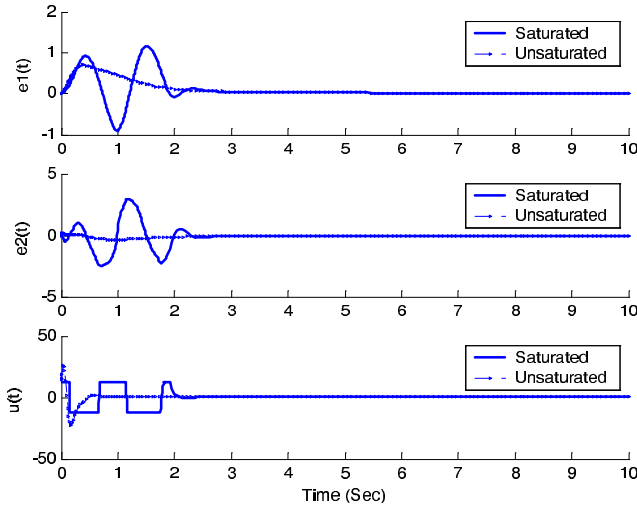

Fig. 1. Error trajectories and control input versus time, $m(t) = 0$.

5. Conclusions

In this paper, we have considered the adaptive control of multivariable nonlinear plants in the presence of uncertainties and disturbance with input constraints. The convergence rate of the adaptive controller is improved and it is shown that the controller in the presence of saturation is locally stable with prior knowledge of the upper bounds for parameter matrices and initial conditions.
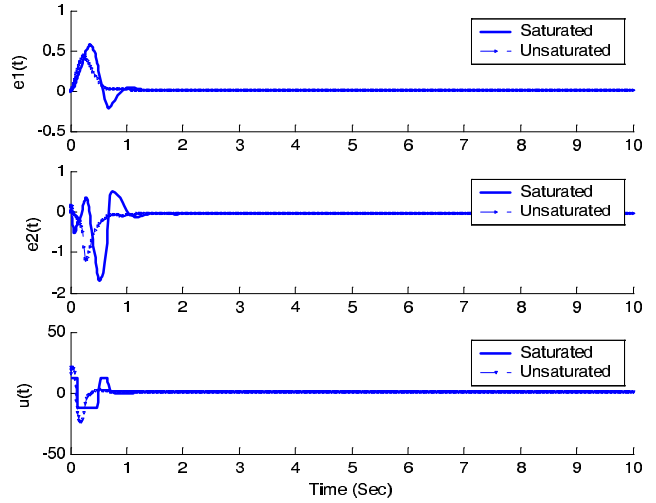

Fig. 2. Error trajectories and control input versus time, $m(t) \neq 0$.

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