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Adaptive Controller Design and Disturbance Attenuation for SISO Linear Systems with Zero Relative Degree under Noisy Output Measurements

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***Abstract*— In this paper, we present robust adaptive controller design for SISO linear systems with zero relative degree under noisy output measurements. We formulate the robust adaptive control problem as a nonlinear H^∞-optimal control problem under imperfect state measurements, and then solve it using game theory. By using the *a priori* knowledge of the parameter vector, we apply a soft projection algorithm, which guarantees the robustness property of the closed-loop system without any persistency of excitation assumption on the reference signal. Due to our formulation in state space, we allow the true system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the $j\omega$-axis are uncontrollable from the exogenous disturbance input. This assumption allows the adaptive controller to asymptotically cancel out, at the output, the effect of exogenous sinusoidal disturbance inputs with unknown magnitude, phase, and frequency. These strong robustness properties are illustrated by a numerical example.**

***Index Terms*— Nonlinear H^∞ control; cost-to-come function analysis; adaptive control.**

I. Introduction

The design of adaptive controllers has been an important research topic since 1970s. The classic adaptive control design, based on the certainty equivalence principle [1], [2], is to design the controller as if the system parameters are known and then in implementation to supply the controller with estimates of the parameters, using standard identifiers, as if the estimates are true values. This design method has been proven successful especially for the linear systems with or without stochastic disturbance inputs [3]. This approach leads to structurally simple adaptive controllers. Yet, early designs based on this approach has been shown to be nonrobust [4] when the system is subject to exogenous disturbance inputs and unmodeled dynamics. Then, the stability and the performance of a system under disturbance and/or uncertainty becomes an important issue. This motivates the study of robust adaptive control which has attracted significant research attention since 1980s. Also, this approach fails to generalize to systems with severe nonlinearities. This motivates the study of nonlinear adaptive control in 1990s.

Robust adaptive control has been an important research topic in late 1980s and early 1990s. Various adaptive controllers were modified to render the closed-loop systems robust [5]. Despite their successes, they fell short of directly addressing the disturbance attenuation property of the closed-loop system.

The topic of nonlinear adaptive control has been widely studied in the last decade after the celebrated characterization of feedback linearizable or partially feedback linearizable systems [6]. The introduction of the integrator backstepping methodology [7] allows us to design adaptive controllers for parametric strict-feedback and parametric pure-feedback nonlinear systems systematically. Since then, many results flourished using this approach, and a complete list of references can be found in the book [8]. More recently, systems with unknown sign of the high frequency gain have been studied using this approach [9]. Moreover, this approach has been applied to linear systems to compare performance with the certainty equivalence approach. As to be expected, a systematically designed nonlinear adaptive control law leads to better closed-loop performance than that for the certainty equivalence based design when the system is free of disturbance. However, this approach has also been shown to to nonrobust when the system is subject to exogenous disturbance inputs.

H^∞-optimal control has been proposed as a solution to the robust control problem. It achieves the objectives of robust control, namely, improving transient response, accommodating unmodeled dynamics, and rejecting exogenous disturbance inputs, by studying only the disturbance attenuation property for the closed-loop system. The game-theoretic approach to H^∞-optimal control [10] developed for the linear quadratic problems, offers the most promising tool to generalize the results to nonlinear systems [11], [12]. The worst-case analysis approach to adaptive control was proposed to address the disturbance attenuation property of the closed-loop system directly. In this approach, the robust adaptive control problem is formulated as a nonlinear H^∞-optimal control problem under imperfect state measurements. Using *cost-to-come* function analysis, it can be converted into a problem under full information measurements. This full information measurement problem is then solved for a suboptimal solution using the integrator backstepping methodology. This design paradigm has been applied to worst-case parameter identification problems, which has led to new classes of parametrized identifiers for linear and nonlinear systems. It has also been applied to adaptive control problems [13], [14], [15], which has led to new classes of parametrized robust adaptive controllers for linear and nonlinear systems. In [13], adaptive control for strict-feedback nonlinear systems was considered under noiseless output measurements. In [14] and [15], linear systems were considered under noisy output measurements. [15] generalizes the results of [14], by assuming part of the disturbance inputs are measured, which then lead to disturbance feedforword structure in the adaptive controller.

In this paper, we study the adaptive control design for SISO linear systems with zero relative degree under noisy output measurements using a similar approach as that of [14]. We assume that the linear system admits a known upper bound for its dynamic

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order, is observable, has a strictly minimum phase transfer function with relative degree 0. The linear system may be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and all uncontrollable modes on the $j\omega$-axis are uncontrollable from the disturbance input. Under these assumptions, the system may be transformed into the design model, which is linear in all of the unknown quantities. We formulate the robust adaptive control problem as a nonlinear H^∞-optimal control problem under imperfect state measurements, where the objectives of transient performance, asymptotic tracking, and disturbance attenuation are incorporated into a single game theoretic cost function. To avoid singularity is the estimation step, we assume that the measurement is noisy. Then, we can apply *cost-to-come* function methodology to derive the estimator, which has a finite-dimensional structure. To relieve the persistency of excitation condition for the closed-loop system, we apply a soft projection algorithm for the estimator. Then, the closed-loop system is robust with or without the persistently exciting signals. After the estimator is determined, the original problem becomes a nonlinear H^∞-optimal control problem under full-information measurement, and the controller can be obtained directly based on the cost function at that step. The closed-loop system admits a guaranteed disturbance attenuation level with respect to the exogenous disturbance inputs, where the ultimate attenuation lower bound for the achievable performance level is equal to the noise intensity in the measurement channel. All closed-loop signals are bounded when the exogenous disturbance and the reference trajectory are bounded. Furthermore, it achieves asymptotic tracking of uniformly continuous and bounded reference trajectories for all bounded disturbance inputs that are of finite energy. This result has significant impact on active noise cancellation problems. That is, when the true system is subject to disturbances generated by an unknown exogenous linear system, we can extend our system model to include the states of the exogenous system as part of the model, and then asymptotically cancel out the effect of the noise at the output. This feature is illustrated by an example in the paper.

The balance of the paper is organized as follows. In Section II, we list the notations to be used in this paper. In Section III, we formulate adaptive control problem and discuss the general solution methodology. In Section IV, we present the estimation and control design using cost-to-come function methodology. In Section V, we present the main result of the paper which states the robustness properties of the closed-loop system. The theoretical results are illustrated by one numerical example in Section VI. The paper ends with some concluding remarks in Section VII. Due to page limitation, some details, for example, the detailed proof of the main result, some derivations, simulation details of the example, are omitted in this shortened version. For interested readers, please contact us for a copy of the full version of the paper.

II. Notations

We denote $\mathbb{R}$ to be the real line; $\mathbb{N}$ to be the set of natural numbers; $\mathbb{C}$ to be the set of complex numbers. For a function f, we say that it belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is k-times continuously (partial) differentiable. For any matrix A, A' denotes its transpose. For any $b \in \mathbb{R}$, $\mathrm{sgn}(b) = \begin{cases} -1 & b<0 \\ 0 & b=0 \\ 1 & b>0 \end{cases}$. For any vector $z \in \mathbb{R}^n$, where $n \in \mathbb{N}$, and any $n \times n$-dimensional symmetric matrix M, $|z| = (z'z)^{1/2}$ and $|z|_M^2 = z'Mz$. For any matrix M, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix M, $\overleftarrow{M}$ denotes the vector formed by stacking up the column vector of the lower triangular part of M. For $n \times n$-dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. For $n \in \mathbb{N} \cup \{0\}$, I_n denotes the $n \times n$-dimensional identity matrix. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2$ denotes the set of square integrable functions and $\mathcal{L}_\infty$ denotes the set of bounded functions.

III. Problem Formulation

We consider the adaptive control problem for single-input and single-output (SISO) linear time-invariant systems.

Assumption 1: The linear system is known to be at most n dimensional, $n \in \mathbb{N}$. ◇

We consider the following true system dynamics:

$$\dot{\grave{x}} = \grave{A}\grave{x} + \grave{B}u + \grave{D}\grave{w}; \quad \grave{x}(0) = \grave{x}_0 \tag{1a}$$

$$y = \grave{C}\grave{x} + b_0 u + \grave{E}\grave{w} \tag{1b}$$

where $\grave{x}$ is the $\grave{n}$-dimensional state vector, $\grave{n} \in \mathbb{N} \cup \{0\}$; u is the scalar control input; $b_0 \in \mathbb{R}$ and $b_0 \neq 0$; y is the scalar measurement output; $\grave{w}$ is the $\grave{q}$-dimensional unmeasured disturbance input vector, $\grave{q} \in \mathbb{N}$; and all input and output signals y, u, and $\grave{w}$ are continuous; the matrices $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$, and $\grave{E}$ are of the appropriate dimensions, generally unknown or partially unknown. The transfer function from u to y is $H(s) = \grave{C}(sI_{\grave{n}} - \grave{A})^{-1}\grave{B} + b_0$.

Assumption 2: [1] The pair $(\grave{A}, \grave{C})$ is observable. The transfer function $H(s)$ is known to have relative degree 0, and is strictly minimum phase. Moreover, the uncontrollable part (with respect to u) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}$ on the $j\omega$-axis is uncontrollable from $\grave{w}$. ◇

Remark 1: If the true system is of order $\grave{n} < n$, we can add $(n - \grave{n})$-dimensional dynamics as outlined in the full version of the paper, such that the expanded system is of order n and satisfies the Assumption 2. Hence, without loss of generality, we will assume that the true system (1) is of order n.

Since $(\grave{A}, \grave{C})$ is observable, there always exists an unknown state diffeomorphism $\grave{x} = \grave{T}x$, and an unknown disturbance transformation $w = \grave{M}\grave{w}$, where w is $q \in \mathbb{N}$ dimensional, such that the system (1) can be transformed into the following form

$$\dot{x} = Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw; \; x(0) = x_0 \tag{2a}$$

$$y = Cx + u\bar{C}_1\theta + b_{p0}u + Ew \tag{2b}$$

where θ is the σ-dimensional vector of unknown parameters of the system, $\sigma \in \mathbb{N}$; the matrices A, $\bar{A}_{211}$, $\bar{A}_{212}$, B, D, C, E, and $\bar{C}_1$ are of appropriate dimensions and completely known, and $b_{p0} \in \mathbb{R}$ is also known. In addition, the high frequency gain of the transfer function $H(s)$, b_0, is equal to $b_{p0} + \bar{C}_1\theta$. The system (2) is called the design model.

We have the following assumptions about the design model.

Assumption 3: Define $\zeta := (EE')^{-1/2} > 0$ and $L := DE'$. ◇

Remark 2: The above transformation matrix $\grave{T}$ always exists. One may choose the state diffeomorphism $\grave{x} = \grave{T}x$ to be the one which transforms the pair $(\grave{A}, \grave{C})$ into its observer canonical form. Next, we can choose a unknown matrix $\grave{M}$ such that the matrices D and E are completely known.

[1] When $\grave{n} = 0$, Assumption 2 is considered satisfied.

Assumption 4: The sign of the high-frequency gain b_0 is known. There exists a known smooth nonnegative radially-unbounded strictly convex function $P : \mathbb{R}^\sigma \to \mathbb{R}$, such that the true value of θ belongs to the set $\Theta := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) \leq 1\}$. Furthermore, for any $\bar{\theta} \in \Theta$, we have $\operatorname{sgn}(b_0)(b_{p0} + \bar{C}_1\bar{\theta}) > 0$. ⋄

We make the following assumption about the reference signal.

Assumption 5: The reference trajectory, y_d, is continuous, and available for the control design. ⋄

The control law is generated by $u(t) = \mu(y_{[0,t]}, y_{d[0,t]})$. Furthermore, it must satisfy the following condition. For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \grave{\mathcal{W}} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathcal{C}$, these exists a unique solution $\grave{x}_{[0,\infty)}$ for the closed-loop system, which result in a continuous control input waveform $u_{[0,\infty)}$. We denote the class of these admissible controllers by $\mathcal{M}_u$.

The objectives of our control design are to make the output of the system, $Cx + b_0u$, to asymptotically track the reference trajectory y_d, and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \grave{\mathcal{W}}$. For design purposes, instead of attenuating the effect of $\grave{w}$, we attenuate the effect of w. We take the uncertainty $(x_0, \theta, w_{[0,\infty)}, y_{d[0,\infty)})$ to belong to the set $\mathcal{W} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathcal{C}$. All of these objectives can be captured by the optimization of a single game-theoretic cost function, defined as follows.

Definition 1: A controller $\mu \in \mathcal{M}_u$ is said to achieve *disturbance attenuation level* γ if there exist a nonnegative function $l(t, \theta, x, y_{[0,t]}, y_{d[0,t]})$ such that

$$\sup_{(x_0,\theta,\grave{w}_{[0,\infty)},y_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \leq 0; \quad \forall t_f \geq 0 \tag{3}$$

where

$$\begin{aligned}J_{\gamma t_f} &:= \int_0^{t_f} \Big(\big(Cx(\tau) + u(\tau)\bar{C}_1\theta + b_{p0}u(\tau) - y_d(\tau)\big)^2 \\ &+ l(\tau, \theta, x(\tau), y_{[0,\tau]}, y_{d[0,\tau]}) - \gamma^2|w(\tau)|^2 \Big)\, d\tau \\ &- \gamma^2 \left| \begin{bmatrix} \theta' - \breve{\theta}_0' & x_0' - \breve{x}_0' \end{bmatrix}' \right|^2_{\bar{Q}_0}\end{aligned} \tag{4}$$

$\breve{\theta}_0 \in \Theta$ is the initial guess of θ; $\breve{x}_0$ is the initial guess of x_0; $\bar{Q}_0$ is the quadratic weighting matrix, quantifying the level of confidence in the estimate $\begin{bmatrix} \breve{\theta}_0' & \breve{x}_0' \end{bmatrix}'$; $\bar{Q}_0^{-1}$ admits the structure $\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi_0' \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi_0' \end{bmatrix}$, where Q_0 and Π_0 are $\sigma\times\sigma$- and $n \times n$-dimensional positive definite matrices, respectively.

Clearly, when the inequality (3) is achieved, the squared $\mathcal{L}_2$ norm of the output tracking error $Cx + u\bar{C}_1\theta + b_{p0}u - y_d$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of the transformed disturbance input w plus some constant. When the $\mathcal{L}_2$ norm of $\grave{w}$ is finite, the squared $\mathcal{L}_2$ norm of $Cx+u\bar{C}_1\theta+b_{p0}u-y_d$ is also finite, which implies $\lim_{t\to\infty}(Cx(t) + u(t)\bar{C}_1\theta + b_{p0}u(t) - y_d(t)) = 0$, under additional assumptions.

The following notation will be used throughout this paper. Let $\breve{x}$ denote the estimate of x, $\tilde{x}$ denote $x - \breve{x}$, $\breve{\theta}$ denote the estimate of θ, $\tilde{\theta}$ denote $\theta - \breve{\theta}$.

To formulate this robust adaptive control problem as an H^∞ control problem with imperfect state measurements, we expand the state space to include the parameter θ as part of the state. Let ξ denote the expanded state vector $\xi = [\theta' \; x']'$. We have

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ y\bar{A}_{211} + u\bar{A}_{212} & A \end{bmatrix}\xi + \begin{bmatrix} \mathbf{0}_{\sigma\times 1} \\ B \end{bmatrix} u \\ &\quad + \begin{bmatrix} \mathbf{0}_{\sigma\times q} \\ D \end{bmatrix} w =: \bar{A}(u,y)\xi + \bar{B}u + \bar{D}w \\ y &= \begin{bmatrix} u\bar{C}_1 & C \end{bmatrix}\xi + b_{p0}u + Ew =: \bar{C}(u)\xi + b_{p0}u + Ew\end{aligned}$$

The worst-case optimization of the cost function (4) can be carried out in two steps as depicted in the following inequality.

$$\begin{aligned}&\sup_{(x_0,\theta,\grave{w}_{[0,\infty)},y_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \\ &\leq \sup_{y_{[0,\infty)}\in\mathcal{C},y_{d[0,\infty)}\in\mathcal{C}} \; \sup_{(x_0,\theta,w_{[0,\infty)},y_{d[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y_{d[0,\infty)}} J_{\gamma t_f}\end{aligned} \tag{6}$$

The inner supremum operator will be carried out first. It is the estimation design step, which will be presented in Section IV. We will calculate the maximum cost over all uncertainties that is consistent with the given measurement waveform.

The outer supremum operator will be carried out second. It is the control design step, which will be discussed after the estimation design. In this step we design the control input u, which guarantees the robustness of the closed-loop system.

This completes the formulation of the problem. Next, we turn to the estimation and control design in the next section.

IV. Estimation and Control Design

In this section, we present the estimation and control design for the adaptive control problem formulated. The first step is estimation design. In this step, the measurement waveform $y_{[0,\infty)}$ and the reference trajectory $y_{d[0,\infty)}$ are assumed to be known. Since the control input is a causal function of y and y_d, then it is also known. We apply the cost-to-come function methodology. Set function l in (4) to be $|\xi - \hat{\xi}|^2_{\bar{Q}}$, where $\hat{\xi}$ is the worst-case estimate for ξ, $\hat{\xi} = [\hat{\theta}' \; \hat{x}']'$, and $\bar{Q}$ is a matrix-valued weighting function to be introduced later. The cost function becomes

$$\begin{aligned}J_{\gamma t_f} &= \int_0^{t_f} \Big(|Cx(\tau) - y_d(\tau) + u(\tau)\bar{C}_1\theta + b_{p0}u(\tau)|^2 \\ &\quad + |\xi(\tau) - \hat{\xi}(\tau)|^2_{\bar{Q}(\tau,y_{[0,\tau]},y_{d[0,\tau]})} - \gamma^2|w(\tau)|^2 \Big)\, d\tau \\ &\quad - \gamma^2 \left| \begin{bmatrix} \theta' - \breve{\theta}_0' & x_0' - \breve{x}_0' \end{bmatrix}' \right|^2_{\bar{Q}_0}\end{aligned} \tag{7}$$

By the cost-to-come function analysis of [14], we have

$$\begin{aligned}\dot{\bar{\Sigma}} &= (\bar{A}(u,y) - \zeta^2\bar{L}\bar{C}(u))\bar{\Sigma} + \bar{\Sigma}\,(\bar{A}(u,y) - \zeta^2\bar{L}\bar{C}(u))' \\ &\quad + \gamma^{-2}\bar{D}\bar{D}' - \gamma^{-2}\zeta^2\bar{L}\bar{L}' - \bar{\Sigma}\left(\gamma^2\zeta^2\,(\bar{C}(u))'\bar{C}(u) - (\bar{C}(u))'\right. \\ &\quad \left.\cdot\bar{C}(u) - \bar{Q}(t, y_{[0,t]}, y_{d[0,t]})\right)\bar{\Sigma};\;\; \bar{\Sigma}(0) = \gamma^{-2}\bar{Q}_0^{-1}\end{aligned} \tag{8a}$$

$$\begin{aligned}\dot{\breve{\xi}} &= \bar{A}(u,y)\breve{\xi} - \bar{\Sigma}\bar{Q}(t, y_{[0,t]}, y_{d[0,t]})\xi_c + \bar{\Sigma}\,(\bar{C}(u))'(\bar{C}(u)\breve{\xi} \\ &\quad - (y_d - b_{p0}u)) + \bar{B}u + \zeta^2\left(\gamma^2\bar{\Sigma}\,(\bar{C}(u))' + \bar{L}\right)(y - b_{p0}u \\ &\quad - \bar{C}(u)\breve{\xi});\;\; \breve{\xi}(0) = \begin{bmatrix} \breve{\theta}_0' & \breve{x}_0' \end{bmatrix}'\end{aligned} \tag{8b}$$

where $\bar{L} = \begin{bmatrix} \mathbf{0}_{1\times\sigma} & L' \end{bmatrix}'$ and $\xi_c := \hat{\xi} - \breve{\xi}$.

Then, the cost function (7) can be equivalently written as, when $\bar{\Sigma}$ exists on $[0, t_f]$ and $\bar{\Sigma}(t)$ is positive definite $\forall t \in [0, t_f]$.

$$\begin{aligned}J_{\gamma t_f} &= -|\xi(t_f) - \breve{\xi}(t_f)|^2_{(\bar{\Sigma}(t_f))^{-1}} + \int_0^{t_f} \Big((\bar{C}(u(\tau))\breve{\xi}(\tau) \\ &\quad + b_{p0}u(\tau) - y_d(\tau))^2 - \gamma^2\zeta^2(y(\tau) - b_{p0}u(\tau) \\ &\quad - \bar{C}(u(\tau))\breve{\xi}(\tau))^2 + |\hat{\xi}(\tau) - \breve{\xi}(\tau)|^2_{\bar{Q}(\tau,y_{[0,\tau]},y_{d[0,\tau]})} \\ &\quad - \gamma^2|w(\tau) - w_*(\xi(\tau), \breve{\xi}(\tau), \bar{\Sigma}(\tau), u(\tau), w(\tau))|^2 \Big)\, d\tau\end{aligned} \tag{9}$$

where $w_* : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \times \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is the worst-case disturbance for estimation step, given by $w_*(\xi, \check{\xi}, \bar{\Sigma}, u, w) = \zeta^2 E'(y - b_{p0}u - \bar{C}(u)\xi) + \gamma^{-2}(I_q - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1}(\xi - \check{\xi})$.

The following steps closely resembles that in [14]. Partition $\bar{\Sigma}(t)$ as $\begin{bmatrix} \Sigma(t) & \bar{\Sigma}_{12}(t) \\ \bar{\Sigma}_{21}(t) & \bar{\Sigma}_{22}(t) \end{bmatrix}$, where $\Sigma(t)$ is $\sigma \times \sigma$-dimensional, and introduce $\Phi(t) := \bar{\Sigma}_{21}(t)(\Sigma(t))^{-1}$ and $\Pi(t) := \gamma^2(\bar{\Sigma}_{22}(t) - \bar{\Sigma}_{21}(t)(\Sigma(t))^{-1}\bar{\Sigma}_{12}(t))$. Also partition $\check{\xi}$ compatibly as $\begin{bmatrix} \check{\theta}' & \check{x}' \end{bmatrix}'$.

For the boundedness of Σ, the weighting matrix $\bar{Q}$ in (7) admits the following structure

$$\bar{Q}(t, y_{[0,t]}, y_{d[0,t]}) = \begin{bmatrix} -\Phi' \\ I_n \end{bmatrix} \gamma^4 \Pi^{-1}\Delta\Pi^{-1} \begin{bmatrix} -\Phi' \\ I_n \end{bmatrix}' + \begin{bmatrix} \epsilon(\bar{C}_1 u + C\Phi)'(\gamma^2\zeta^2 - 1)(\bar{C}_1 u + C\Phi) & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix}$$

where $\Delta(t) = \gamma^{-2}\beta_\Delta\Pi(t) + \Delta_1$, with $\beta_\Delta \geq 0$ being a constant and $\Delta_1 > 0$ being a constant matrix, ϵ is defined by

$$\epsilon(t) := K_c^{-1}s_\Sigma(t) := \mathrm{Tr}((\Sigma(t))^{-1})/K_c; t \in [0, \infty) \quad (10a)$$
$$\text{or } \epsilon(t) := 1 \quad (10b)$$

and $K_c \geq \gamma^2\mathrm{Tr}(Q_0)$ is a constant. We will later treat $\bar{Q}$ as a function $\bar{Q} : \mathbb{R}^{n\times\sigma} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{(n+\sigma)\times(n+\sigma)}$, $\bar{Q}(\Phi, u, s_\Sigma)$.

Detailed analysis leads to Σ and Φ satisfy (13b), (13e), and Π satisfies (13a) with proper initialization.

The matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. The choice of $\bar{Q}$ guarantees that Σ is bounded from above and bounded from below away from 0 as depicted in the following Lemma, whose proof is given in [14].

Lemma 1: Consider the dynamic equation (13b) for the covariance matrix Σ. Let $K_c \geq \gamma^2\mathrm{Tr}(Q_0)$, $Q_0 > 0$, and $\gamma \geq \zeta^{-1}$. Then, the matrix Σ is upper and lower bounded as follows: $K_c^{-1}I_\sigma \leq \Sigma(t) \leq \Sigma(0) = \gamma^{-2}Q_0^{-1}$; $\gamma^2\mathrm{Tr}(Q_0) \leq \mathrm{Tr}\big((\Sigma(t))^{-1}\big) \leq K_c$; $\forall t \in [0, t_f]$, for either choice of $\epsilon(t)$ as in (10), whenever Σ is defined on $[0, t_f]$, and Φ and u are continuous on $[0, t_f]$.

To avoid the inversion of Σ on-line, we define $s_\Sigma(t) := \mathrm{Tr}\big((\Sigma(t))^{-1}\big)$. It satisfies (13c). Then, $\epsilon(t) = K_c^{-1}s_\Sigma(t)$, which does not require the inversion of $\Sigma(t)$, when ϵ is defined by (10a).

Based on Lemma 1, we note that the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system, using the design method proposed in this paper.

Assumption 6: If the matrix $A - \zeta^2 LC$ is Hurwitz, then the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. In case $\gamma = \zeta^{-1}$, choose $\beta_\Delta \geq 0$ such that $A - \zeta^2 LC + \beta_\Delta/2I_n$ is Hurwitz. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then the desired disturbance attenuation level $\gamma > \zeta^{-1}$. ◇

Assumption 7: The matrix Π_0 is chosen as the unique positive definite solution to the algebraic Riccati equation (13a). ◇

Then, Π is constant, and the matrix A_f of (13d) is Hurwitz.

To guarantee the boundness of estimated parameters without persistently exciting signals, we introduce soft projection design on the parameter estimate, which is based on the Assumption 4.

Define $\rho := \inf\{P(\bar{\theta}) \mid \bar{\theta} \in \mathbb{R}^\sigma \text{ and } b_{p0} + \bar{C}_1\bar{\theta} = 0\}$. By Assumption 4, we have $1 < \rho \leq \infty$. Fix any $\rho_o \in (1, \rho)$, and define the open set $\Theta_o := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) < \rho_o\}$. Our control design will guarantee the estimate $\check{\theta}$ lies in Θ_o, which immediately implies $\check{b}_0 := b_{p0} + \bar{C}_1\check{\theta} \geq c_0 > 0$. Moreover, the convexity of P implies the following inequality $\frac{\partial P}{\partial\theta}(\check{\theta})(\theta - \check{\theta}) < 0$, $\forall\check{\theta} \in \mathbb{R}^\sigma\backslash\Theta$. Add the term $-\bar{\Sigma}\begin{bmatrix} \big(P_r(\check{\theta})\big)' & \mathbf{0}_{1\times n} \end{bmatrix}'$ to the right-hand-side of the dynamics (8b), where

$$P_r(\check{\theta}) := \begin{cases} \frac{\exp\left(\frac{1}{1-P(\check{\theta})}\right)}{\left(\rho_o - P(\check{\theta})\right)^3}\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)' & \forall\check{\theta} \in \Theta_o\backslash\Theta \\ \mathbf{0}_{\sigma\times 1} & \forall\check{\theta} \in \Theta \end{cases} \quad (11)$$
$$=: p_r(\check{\theta})\left(\frac{\partial P}{\partial\theta}(\check{\theta})\right)' \quad (12)$$

and $P_r(\check{\theta})$ and $p_r(\check{\theta})$ are smooth functions on Θ_o.

Detailed equations for the estimator are

$$(A - \zeta^2 LC + \beta_\Delta/2I_n)\Pi + \Pi(A - \zeta^2 LC + \beta_\Delta/2I_n)' - \Pi C'(\zeta^2 - \gamma^{-2})C\Pi + DD' - \zeta^2 LL' + \gamma^2\Delta_1 = \mathbf{0} \quad (13a)$$
$$\dot{\Sigma} = (\epsilon - 1)\Sigma(\bar{C}_1 u + C\Phi)'(\gamma^2\zeta^2 - 1)(\bar{C}_1 u + C\Phi)\Sigma; \quad \Sigma(0) = \gamma^{-2}Q_0^{-1} \quad (13b)$$
$$\dot{s}_\Sigma = (\gamma^2\zeta^2 - 1)(1 - \epsilon)(\bar{C}_1 u + C\Phi)(\bar{C}_1 u + C\Phi)'; \quad s_\Sigma(0) = \gamma^2\mathrm{Tr}(Q_0); \quad (13c)$$
$$A_f = A - \zeta^2 LC - \Pi C'(\zeta^2 - \gamma^{-2})C \quad (13d)$$
$$\dot{\Phi} = A_f\Phi + y\bar{A}_{211} + u(\bar{A}_{212} - \zeta^2 L\bar{C}_1 - \Pi C'(\zeta^2 - \gamma^{-2})\bar{C}_1); \quad \Phi(0) = \Phi_0 \quad (13e)$$
$$\dot{\check{\theta}} = -\Sigma P_r(\check{\theta}) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix}\bar{Q}\xi_c - (\Sigma\bar{C}_1' u + \Sigma\Phi' C')(y_d - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}) + \gamma^2\zeta^2(\Sigma\bar{C}_1' u + \Sigma\Phi' C')(y - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}); \quad \check{\theta}(0) = \check{\theta}_0 \quad (13f)$$
$$\dot{\check{x}} = -\Phi\Sigma P_r(\check{\theta}) + A\check{x} + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} + Bu - \begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix}\bar{Q}\xi_c + \zeta^2(\gamma^2(\Phi\Sigma\bar{C}_1' u + \gamma^{-2}\Pi C' + \Phi\Sigma\Phi' C') + L)(y - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}) - (\Phi\Sigma\bar{C}_1' u + \gamma^{-2}\Pi C' + \Phi\Sigma\Phi' C') \cdot (y_d - b_{p0}u - \bar{C}_1\check{\theta}u - C\check{x}); \quad \check{x}(0) = \check{x}_0 \quad (13g)$$

To simplify the controller structure, the dynamics for Φ can be implemented with $3n$ integrators instead of the σn integrators. First, we observe that the pair (A_f, C) is observable. Then we introduce the matrix $M_f := \begin{bmatrix} A_f^{n-1}p_n & \cdots & A_f p_n & p_n \end{bmatrix}$, where p_n is an n-dimensional vector such that the pair (A_f, p_n) is controllable, which implies that M_f is invertible. Then the following $3n$-dimensional prefiltering system for y and u generates Φ online: $\dot{\eta} = A_f\eta + p_n y$, $\eta(0) = \eta_0 \in \mathbb{R}^n$, $\dot{\lambda} = A_f\lambda + p_n u$, $\lambda(0) = \lambda_0 \in \mathbb{R}^n$, $\dot{\lambda}_o = A_f\lambda_o$, $\lambda_o(0) = p_n$,

$$\Phi = \begin{bmatrix} A_f^{n-1}\eta & \cdots & A_f\eta & \eta \end{bmatrix} M_f^{-1}\bar{A}_{211} + \begin{bmatrix} A_f^{n-1}\lambda_o & \cdots & A_f\lambda_o & \lambda_o \end{bmatrix} M_f^{-1}\Phi_{o0} + \begin{bmatrix} A_f^{n-1}\lambda & \cdots & A_f\lambda & \lambda \end{bmatrix} M_f^{-1}(\bar{A}_{212} - \zeta^2 L\bar{C}_1 - \Pi C'(\zeta^2 - \gamma^{-2})\bar{C}_1); \quad \Phi_{o0} \in \mathbb{R}^{n\times\sigma} \quad (14)$$

where η_0, λ_0, and Φ_{o0} are such that (14) holds at $t = 0$.

Associated with the above identifier, introduce the value function, $W : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \to \mathbb{R}$,

$$W(\xi, \check{\xi}, \bar{\Sigma}) = |\theta - \check{\theta}|^2_{\Sigma^{-1}} + \gamma^2|x - \check{x} - \Phi(\theta - \check{\theta})|^2_{\Pi^{-1}} \quad (15)$$

whose time derivative along the system dynamics is given by

$$\dot{W}(\xi, \check{\xi}, \bar{\Sigma}, s_\Sigma, y_d, \hat{\xi}, u, w) = -|Cx + \bar{C}_1\theta u - (y_d - b_{p0}u)|^2 + |C\check{x} + \bar{C}_1\check{\theta}u - (y_d - b_{p0}u)|^2 - |\xi - \hat{\xi}|^2_{\bar{Q}} + |\check{\xi} - \hat{\xi}|^2_{\bar{Q}} + \gamma^2|w|^2 - \gamma^2|w - w_*|^2 - \gamma^2\zeta^2|y - b_{p0}u - C\check{x} - \bar{C}_1\check{\theta}u|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) \quad (16)$$

which holds as long as $\Sigma > 0$, $\check{\theta} \in \Theta_o$. We note that the last term in $\dot{W}$ is nonpositive, zero on the set Θ and approaches $-\infty$ as

$\breve{\theta}$ approaches the boundary of the set Θ_o, which guarantees the boundness of $\breve{\theta}$.

Then the cost function (7) can be equivalently written as, assuming $\Sigma(t) > 0$ and $\breve{\theta}(t) \in \Theta_o$, $\forall t \in [0, t_f]$,

$$\begin{aligned} J_{\gamma t_f} &= \int_0^{t_f} \Big((C\breve{x}(\tau) + b_{p0}u(\tau) + \bar{C}_1\breve{\theta}(\tau)u(\tau) - y_d(\tau))^2 \\ &\quad -\gamma^2\zeta^2 (y(\tau) - C\breve{x}(\tau) - b_{p0}u(\tau) - \bar{C}_1\breve{\theta}(\tau)u(\tau))^2 \\ &\quad +|\xi_c(\tau)|^2_{\bar{Q}(\Phi(\tau),u(\tau),s_\Sigma(\tau))} + 2(\theta - \breve{\theta}(\tau))'P_r(\breve{\theta}(\tau)) \\ &\quad -\gamma^2|w(\tau) - w_*(\xi(\tau), \breve{\xi}(\tau), \bar{\Sigma}(\tau), u(\tau), w(\tau))|^2 \Big) d\tau \\ &\quad -|\xi(t_f) - \breve{\xi}(t_f)|^2_{(\bar{\Sigma}(t_f))^{-1}} \end{aligned} \tag{17}$$

This completes the estimation design step.

Now, we present the control design step. Based on the inequality (6), the controller design is to guarantee that the following supremum is less than or equal to zero

$$\begin{aligned} \sup_{(x_0,\theta,\grave{w}_{[0,\infty)},y_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} &\le \sup_{y_{[0,\infty)}\in\mathcal{C}, y_{d[0,\infty)}\in\mathcal{C}} \Big\{ \int_0^{t_f} \Big((C\breve{x}(\tau) \\ &+(b_{p0} + \bar{C}_1\breve{\theta}(\tau))u(\tau) - y_d(\tau))^2 + |\xi_c(\tau)|^2_{\bar{Q}(\Phi(\tau),u(\tau),s_\Sigma(\tau))} \\ &-\gamma^2\zeta^2 (y(\tau) - C\breve{x}(\tau) - (b_{p0} + \bar{C}_1\breve{\theta}(\tau))u(\tau))^2 \Big) d\tau \Big\} \end{aligned} \tag{18}$$

By inequality (18), we observe that the cost function is expressed in terms of signals that we can measure or construct. This is then a nonlinear H^∞-optimal control problem under full information measurements. Instead of considering y as the maximizing variable, we can equivalently deal with the transformed variable: $v := \zeta (y - C\breve{x} - (b_{p0} + \bar{C}_1\breve{\theta})u)$.

Set the control input u and the worst-case estimate $\hat{\xi}$ as

$$u := \bar{\mu}(\breve{\theta}, \breve{x}, y_d) = \frac{y_d - C\breve{x}}{\bar{C}_1\breve{\theta} + b_{p0}}; \quad \text{and} \quad \hat{\xi} = \breve{\xi} \tag{19}$$

where $\bar{\mu} : \Theta_o \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is smooth. The value function for the closed-loop system is simply W, whose time derivative along solutions of the dynamics for ξ, $\breve{\xi}$, and $\bar{\Sigma}$ is

$$\begin{aligned} \dot{W} &= -(Cx + \bar{C}_1\theta u + b_{p0}u - y_d)^2 - |\xi - \hat{\xi}|^2_{\bar{Q}} + \gamma^2|w|^2 \\ &\quad -\gamma^2|w - w_{opt}(\xi, \breve{\xi}, y_d, \bar{\Sigma})|^2 + 2(\theta - \breve{\theta})'P_r(\breve{\theta}) \end{aligned} \tag{20}$$

where the worst-case disturbance with respect to the value function W is given by, $w_{opt} : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathbb{R} \times \mathcal{S}_{+(n+\sigma)} \to \mathbb{R}^q$,

$$\begin{aligned} w_{opt}(\xi, \breve{\xi}, y_d, \bar{\Sigma}) &= -\zeta^2 E' \begin{bmatrix} \bar{\mu}(\breve{\theta}, \breve{x}, y_d)\bar{C}_1 & C \end{bmatrix} (\xi - \breve{\xi}) \\ &\quad +\gamma^{-2} (I_q - \zeta^2 E'E)\bar{D}'\bar{\Sigma}^{-1} (\xi - \breve{\xi}) \end{aligned} \tag{21}$$

which holds as long as $\Sigma > 0$ and $\breve{\theta} \in \Theta_o$. Clearly, the closed-loop system is dissipative with storage function W and supply rate $-(Cx + \bar{C}_1\theta u + b_{p0}u - y_d)^2 + \gamma^2|w|^2$.

This completes the control design step. We will turn to present the main results in the next section.

V. Main Result

With the estimation and control design of the previous section, the state of the closed-loop system is given by $X := \begin{bmatrix} \theta' & x' & \overrightarrow{\Sigma}' & s_\Sigma & \breve{\theta}' & \breve{x}' & \overrightarrow{\Phi}' \end{bmatrix}'$, which belongs to the open set $\mathcal{D} := \{X \mid \Sigma > 0, s_\Sigma > 0, \breve{\theta} \in \Theta_o\}$

The dynamics for X are, with $x(0) = x_0$,

$$\dot{X} = F(X, y_d) + G(X, y_d)w = F(X, y_d) + G(X, y_d)\grave{M}\grave{w} \tag{22}$$

where F and G are smooth mappings of $\mathcal{D} \times \mathbb{R}$; and the initial condition $X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} \mid \theta \in \Theta, \breve{\theta}_0 \in \Theta, \Sigma(0) = \gamma^{-2}Q_0^{-1} > 0, s_\Sigma(0) = \gamma^2\mathrm{Tr}(Q_0) \le K_c\}$. Since (20) holds, by Lemma 7 in [14], the value function W satisfies a Hamilton-Jacobi-Isaacs equation.

$$\begin{aligned} &\frac{\partial W}{\partial X}(X)F(X, y_d) + \frac{1}{4\gamma^2}\frac{\partial W}{\partial X}(X)G(X, y_d)(G(X, y_d))' \\ &\quad \cdot \Big(\frac{\partial W}{\partial X}(X)\Big)' + Q(X, y_d) = 0; \quad \forall X \in \mathcal{D}, \forall y_d \in \mathbb{R} \end{aligned}$$

where $Q : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is smooth and given by

$$\begin{aligned} Q(X, y_d) &= |Cx + (b_{p0} + \bar{C}_1\theta)\mu(\breve{\theta}, \breve{x}, y_d) - y_d|^2 \\ &\quad + \big|\xi - \breve{\xi}\big|^2_{\bar{Q}(\Phi,\mu(\breve{\theta},\breve{x},y_d),s_\Sigma)} - 2\,(\theta - \breve{\theta})'P_r(\breve{\theta}) \end{aligned}$$

The closed-loop adaptive system possesses a strong robustness property, which will be stated precisely in the following theorem.

Theorem 1: Consider the robust adaptive control problem formulated in Section III with Assumptions 1–7 holding. The robust adaptive controller μ defined by (19), with the optimal choice (19) for $\hat{\xi}$, achieves the following strong robustness properties.

1) Given $c_w \ge 0$, and $c_d \ge 0$, there exists a constant $c_c \ge 0$ and a compact set $\Theta_c \subset \Theta_o$, such that for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $|x_0| \le c_w$, $|y_d(t)| \le c_d$, $|\grave{w}(t)| \le c_w$, $\forall t \in [0, \infty)$, all closed-loop state variables x, $\breve{x}$, $\breve{\theta}$, Σ, s_Σ, and Φ are bounded as follows, $\forall t \in [0, \infty)$, $|x(t)| \le c_c$; $|\breve{x}(t)| \le c_c$; $\breve{\theta}(t) \in \Theta_c$; $|\overrightarrow{\Phi(t)}| \le c_c$; $K_c^{-1}I_\sigma \le \Sigma(t) \le \gamma^{-2}Q_0^{-1}$; $\gamma^2\mathrm{Tr}(Q_0) \le s_\Sigma(t) \le K_c$. Therefore, there is a compact set $S \subseteq \mathcal{D}$ such that $X(t) \in S$ $\forall t \in [0, \infty)$. Hence, there exists a constant $c_u \ge 0$ such that $|u(t)| \le c_u$, $|\hat{\xi}(t)| \le c_u$, $|\eta(t)| \le c_u$, $|\lambda(t)| \le c_u$, and $|\lambda_o(t)| \le c_u$.
2) The controller $\mu \in \mathcal{M}_u$ achieves disturbance attenuation level γ for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \grave{\mathcal{W}}$.
3) For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, y_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $\grave{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $y_{d[0,\infty)} \in \mathcal{L}_\infty$, and $y_{d[0,\infty)}$ being uniformly continuous, we have

$$\lim_{t\to+\infty} (Cx(t) + (\bar{C}_1\theta + b_{p0})u(t) - y_d(t)) = 0$$

VI. Example

In this section, we present one example to illustrate the main results of this paper. The design was carried out using MATLAB symbolic computation tools, and the closed-loop system was simulated using SIMULINK.

Consider the following circuit problem in Figure 1(a), where v_i is the input voltage source; v_o is the measured output; v_e is an unknown sinusoidal voltage source; v_w is an unmeasured exogenous voltage source; b_0 is the ratio of v_t to v_i. The objective is to achieve asymptotic tracking of $v_o - v_w$ to the reference trajectory y_d.

Using the robust adaptive control design method formulated in this paper, we present the following simulation results, in Figure 1(b)-(d), to illustrate the regulatory behaviour of the adaptive controller. We observe that the parameter estimates converge to their true values, and the tracking error converges to 0. The transient of the system is well-behaved, and the control magnitude is upper bounded by 2.2. The effect of v_e is asymptotically cancelled at the output.

VII. Conclusions

In this paper, we studied the adaptive control design for tracking and disturbance attenuation for SISO linear systems with zero relative degree under noisy output measurements. We assume that

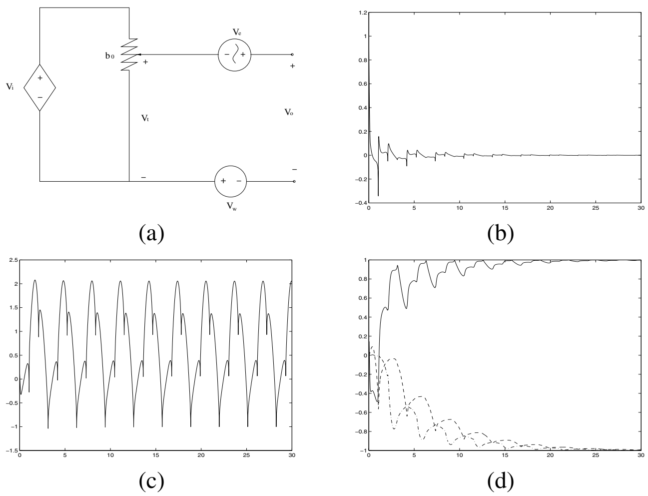

Fig. 1. Circuit and system response under $y_d(t) = \sqrt{|sin(3t)|}$ and $\grave{w}(t) = 0$.
(a) Diagram of Circuit (b) Tracking error; (c) Control input; (d) Parameter estimates.

the linear system has a known upper bound of the dynamic order, is observable, has a strictly minimum phase transfer function with known relative degree 0. We allow the system to be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov and those uncontrollable modes on the $j\omega$-axis are uncontrollable from the disturbance input. Under these assumptions, the system may be transformed into the design model, which is linear in all of the unknown quantities. The objectives of the control design are to make the noiseless output of the system to asymptotically track the reference trajectory, and guarantee the boundedness of all closed-loop signals, while rejecting the uncertainties in the system. We use H^∞-optimal control formulation and game theoretic approach to derive the robust adaptive controller. We treat the unknown parameter vector as part of the expanded state vector, and formulate this adaptive control problem as a nonlinear H^∞-optimal control problem with imperfect state measurements. For the design model, we assume that the measurement channel is noisy, such that the estimation step is a nonsingular optimization problem. We further assume that the unknown parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function $P(\bar{\theta})$. Furthermore, for any parameter vector belonging to the set, the corresponding high frequency gain is never zero. Then, the *cost-to-come* function analysis is applied to derive the worst-case identifier and state estimator, which have a finite-dimensional structure. Using *a priori* information on the parameter vector, a smooth soft projection algorithm is applied in the estimation step, which relieves the persistency of excitation condition for the closed-loop system. Then, the closed-loop system is robust with or without the persistently exciting signals. After the estimation step is completed, the original problem becomes a nonlinear H^∞-optimal control problem under full-information measurements. Then, the controller can be obtained directly from the cost function in one step. The controller then achieve the desired disturbance attenuation level, with the ultimate lower bound of the attenuation level being the noise intensity in the measurement channel. It guarantees the boundedness of all closed-loop signals and achieves asymptotic tracking of uniformly continuous bounded reference trajectories when the disturbance is of finite energy and bounded. Because of the assumptions we made on the unknown system, the adaptive controller can asymptotically cancel out the effect of exogenous sinusoidal inputs with unknown magnitudes, phases, and frequencies, as long as we extend our system model to incorporate the knowledge of the existence of such sinusoidal inputs. This property of our adaptive controller has significant impact on active noise cancellation applications. This feature is illustrated by a numerical example, which corroborates all of our theoretical findings.

Future research directions that are of interest are described as follows. One direction lies in the generalization of the results to nonlinear systems. Another fruitful direction lies in the extension of the results to multiple-input and multiple-output systems. The class of MIMO systems under study involves two subsystems, $\mathbf{S_1}$ and $\mathbf{S_2}$, interconnected to each other, where the connection is serial with feedback. Preliminary results have been obtained for this class of MIMO systems.

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