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Asymptotic Adaptive Regulation of Parametric Strict-Feedback Systems with Additive Disturbance

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Abstract: This paper deals with the regulation of uncertain, multi-input/multi-output, nonlinear parametric strict-feedback systems of nth order subjected to additive disturbances. A continuous adaptive control law is proposed using a modified backstepping design that ensures the output is asymptotically driven to zero. Despite the disturbance, the adaptation law does not need a standard leakage term to ensure the aforementioned stability result; rather, we use a dynamic robust control mechanism and a recently-developed projection operator that is arbitrarily many times continuously differentiable. A numerical example illustrates the main result.

1 Introduction

The class of systems to which adaptive control can be applied was vastly broaden with the advent of the integrator backstepping design [7]. This systematic design procedure allows one to adaptively stabilize systems that are in the so-called parametric strict-feedback form [7]. In the presence of bounded external disturbances, it is well known that the performance of adaptive controllers can significantly deteriorate and even lead to instability. In this case, the parameter estimate cannot be proven bounded, leading to the unboundedness of other closed-loop signals. Common approaches for counteracting this problem include adding a robustifying (leakage) term to the adaptation law (e.g., the σ-modification [5] and the e_1-modification [10]), or using a projection operator [5, 7, 13] to confine the parameter estimate to a bounded convex set in the parameter space. Leakage modifications have the major disadvantage of not recovering the disturbance-free stability performance of the unmodified adaptation law if the disturbance disappears after some time. On the other hand, projection operators preserve the ideal properties of the adaptive controller if the disturbance disappears, but require parameter bounds to be known a priori. Generally, projection-based adaptation laws are discontinuous, which violates the Lipschitz condition for existence of classical solutions to differential equations. Furthermore, the discontinuity is not desirable from an implementation standpoint. This shortcoming however was addressed in [12] by introducing a boundary layer around the convex set that resulted in a Lipschitz continuous projection operator. Recently in [1], we proposed a *smoothened* version of the projection operator of [12], which replaces the Lipschitz continuity with the stronger property of arbitrarily many times continuous differentiability while introducing minor or no modifications to the other projection properties. This new projection operator is useful for backstepping-based, robust adaptive controllers that require multiple differentiations of the adaptation law.

Robust adaptive control laws for systems affected by external disturbances can generally be shown to ensure the boundedness of closed-loop signals, but not asymptotic tracking and regulation to zero. For example, [15] proposed a projection-based adaptive backstepping controller for single-input/single-output (SISO), minimum phase linear systems of relative degree two in the presence of input and output disturbances, which guarantees the tracking error is ultimately bounded and small in the mean square sense. In [4], an adaptive backstepping controller with tuning functions for linear systems with output and multiplicative disturbances was redesigned with a switching σ-modification. The controller gives a tracking error proportional to the size of the perturbations. In [9], a class of SISO nonlinear systems affected by unknown, time-varying bounded parameters and additive disturbances was considered, and a robust adaptive tracking controller was presented that achieves boundedness of all signals and arbitrary disturbance attenuation. The work in [11], which studied a class of systems similar to the one in [9], proposed a robust adaptive controller that ensures the $\mathcal{L}_2$ norm of the tracking error is bounded. The tracking control problem for SISO nonlinear systems with unknown control coefficients and time-varying disturbances was recently studied in [3]. The robust adaptive controller proposed in [3] was shown to guarantee the global uniform boundedness of the tracking error.

In this paper, we consider a class of nth-order, multi-input/multi-output (MIMO), nonlinear parametric strict-feedback systems with matched, unknown, time-varying, additive disturbances and unmatched, uncertain, constant parameters. Our goal is to design a $\mathcal{C}^0$ adaptive controller that is insensitive to exogenous disturbances, and asymptotically drives the output to zero while maintaining all closed-loop signals bounded. Under the assumption that the disturbance is $\mathcal{C}^2$ with bounded derivatives, we propose a modified adaptive backstepping design that exploits the new robust control mechanism of [14] and the new sufficiently smooth, projection-based adaptation law of [1]. A Lyapunov-type stability analysis is used to prove semi-global asymptotic output regulation.

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2 Problem Statement

We consider a class of parametric strict-feedback systems of the form

$$\begin{aligned}\dot{x}_1 &= \varphi_1^T(x_1)\,\theta + x_2 &(1)\\ \dot{x}_2 &= \varphi_2^T(x_1,x_2)\,\theta + x_3 &(2)\\ &\vdots \\ \dot{x}_i &= \varphi_i^T(x_1,...,x_i)\,\theta + x_{i+1} &(3)\\ &\vdots \\ \dot{x}_n &= \varphi_n^T(x_1,...,x_n)\,\theta + d + u &(4)\end{aligned}$$

where $x_i(t) \in \mathbb{R}^m$, $i = 1,...,n$ are the measurable states, $\varphi_i(\cdot) \in \mathbb{R}^{p\times m}$, $i = 1,...,n$ are known nonlinearities, $\theta \in \mathbb{R}^p$ is an uncertain constant parameter vector, $d(t) \in \mathbb{R}^m$ is an unknown additive disturbance, $u(t) \in \mathbb{R}^m$ is the control input, and $y = x_1$ is the system output. We make the following assumptions regarding the system model:

A1. $\varphi_i(\cdot)$ is $\mathcal{C}^{n+1-i}$ and $\varphi_i(0,...,0) = 0$.

A2. $d(t)$ is $\mathcal{C}^2$ and $d(t), \dot{d}(t), \ddot{d}(t) \in \mathcal{L}_\infty$.

A3. The parameter vector θ belongs to a compact convex set $\Omega := \{\theta : \|\theta\| \leq \theta_0\}$ where θ_0 is a known positive constant.

The control objective is to ensure $y(t) \to 0$ as $t \to \infty$ and the boundedness of all closed-loop signals. The following notation will be used throughout the paper: $\hat{\theta}_i(t) \in \mathbb{R}^p$, $i = 1,...,n-1$ are parameter estimates;

$$\tilde{\theta}_i(t) := \theta - \hat{\theta}_i(t),\ i = 1,\cdots,n-1 \qquad (5)$$

denote the corresponding parameter estimation errors; $\text{Proj}(\mu_i, \hat{\theta}_i) \in \mathbb{R}^p$, $i = 1,...,n-1$, $\forall \mu_i(t) \in \mathbb{R}^m$ denote $\mathcal{C}^{n-i-1}$ projection operators used to ensure $\hat{\theta}_i(t) \in \mathcal{L}_\infty$ independent of the control (see Appendix A for details); $\Gamma_i \in \mathbb{R}^{p\times p}$, $i = 1,...,n-1$ are constant, diagonal, positive-definite matrices; and c_i, $i = 1,...,n$ are positive constants. To facilitate the readability of the paper, the control design that follows is presented for the case where $m = 1$. Note, however, that *the main result is readily applicable to the MIMO case.*

3 Adaptive Control Design

Step 1

The first $n-1$ steps of the control design follow the standard adaptive backstepping procedure [7]. We begin by defining the following variables

$$\begin{aligned}\eta_1 &: = x_1 &(6)\\ \eta_2 &: = x_2 - \alpha_1 &(7)\end{aligned}$$

where α_1 is a stabilizing function yet to be designed. Let

$$V_1 := \frac{1}{2}\eta_1^2 + \frac{1}{2}\tilde{\theta}_1^T\Gamma_1^{-1}\tilde{\theta}_1, \qquad (8)$$

and differentiate (8) to obtain

$$\dot{V}_1 = \eta_1\left(\varphi_1^T\theta + \eta_2 + \alpha_1\right) - \tilde{\theta}_1^T\Gamma_1^{-1}\,\dot{\hat{\theta}}_1 \qquad (9)$$

where (1) and (5) were used. Based on (9), we design the stabilizing function and parameter update law as follows

$$\begin{aligned}\alpha_1 &= -c_1\eta_1 - \varphi_1^T\hat{\theta}_1 &(10)\\ \dot{\hat{\theta}}_1 &= \Gamma_1\text{Proj}(\mu_1, \hat{\theta}_1), \quad \mu_1 := \varphi_1\eta_1. &(11)\end{aligned}$$

After substituting (10) and (11) into (9), we obtain

$$\dot{V}_1 = -c_1\eta_1^2 + \eta_1\eta_2 + \tilde{\theta}_1^T\left(\mu_1 - \Gamma_1^{-1}\,\dot{\hat{\theta}}_1\right) \leq -c_1\eta_1^2 + \eta_1\eta_2 \qquad (12)$$

where property P2 of the projection operator was used (see Appendix A).

Step 2

Let

$$\eta_3 := x_3 - \alpha_2 \qquad (13)$$

where α_2 is the second stabilizing function, and use (7) to rewrite (2) as

$$\dot{\eta}_2 = \varphi_2^T\theta + \eta_3 + \alpha_2 - \frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial\alpha_1}{\partial\hat{\theta}_1}\,\dot{\hat{\theta}}_1\,. \qquad (14)$$

Define

$$V_2 := V_1 + \frac{1}{2}\eta_2^2 + \frac{1}{2}\tilde{\theta}_2^T\Gamma_2^{-1}\tilde{\theta}_2, \qquad (15)$$

and differentiate (15) to obtain

$$\begin{aligned}\dot{V}_2 \leq\ & -c_1\eta_1^2 + \eta_1\eta_2 - \tilde{\theta}_2^T\Gamma_2^{-1}\,\dot{\hat{\theta}}_2 \\ & +\eta_2\left(\left(\varphi_2^T - \frac{\partial\alpha_1}{\partial x_1}\varphi_1^T\right)\theta + \eta_3 + \alpha_2 \right.\\ & \left. -\frac{\partial\alpha_1}{\partial\hat{\theta}_1}\,\dot{\hat{\theta}}_1 - \frac{\partial\alpha_1}{\partial x_1}x_2\right) \qquad (16)\end{aligned}$$

where (12), (5), and (1) were used.

Based on (16), we design the second stabilizing function and parameter update law as follows

$$\begin{aligned}\alpha_2 =\ & -c_2\eta_2 - \eta_1 - \left(\varphi_2^T - \frac{\partial\alpha_1}{\partial x_1}\varphi_1^T\right)\hat{\theta}_2 \\ & +\frac{\partial\alpha_1}{\partial\hat{\theta}_1}\,\dot{\hat{\theta}}_1 + \frac{\partial\alpha_1}{\partial x_1}x_2 \qquad (17)\end{aligned}$$

$$\dot{\hat{\theta}}_2 = \Gamma_2\text{Proj}(\mu_2, \hat{\theta}_2), \quad \mu_2 := \left(\varphi_2 - \varphi_1\frac{\partial\alpha_1}{\partial x_1}\right)\eta_2. \qquad (18)$$

Note that $\alpha_2 = \alpha_2(x_1, x_2, \hat{\theta}_1, \hat{\theta}_2)$ due to (11), (6), (10), and (7). Substituting (17) and (18) into (16) and making use of property P2 of the projection operator yields

$$\dot{V}_2 \leq -c_1\eta_1^2 - c_2\eta_2^2 + \eta_2\eta_3. \qquad (19)$$

Step i $(3 \leq i \leq n-1)$

Let

$$\eta_{i+1} := x_{i+1} - \alpha_i, \qquad (20)$$

and rewrite (3) as

$$\dot{\eta}_i = \dot{x}_i - \dot{\alpha}_{i-1} = \varphi_i^T\theta + \eta_{i+1} + \alpha_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j . \quad (21)$$

Define

$$V_i := V_{i-1} + \frac{1}{2}\eta_i^2 + \frac{1}{2}\tilde{\theta}_i^T\Gamma_i^{-1}\tilde{\theta}_i \quad (22)$$

whose derivative is

$$\begin{aligned}\dot{V}_i \leq\ & \eta_i\left(\left(\varphi_i^T - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j^T\right)\theta - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1}\right.\\ &\left. - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j + \eta_{i+1} + \alpha_i\right)\\ & - \sum_{j=1}^{i-1}c_j\eta_j^2 + \eta_{i-1}\eta_i - \tilde{\theta}_i^T\Gamma_i^{-1}\dot{\hat{\theta}}_i . \end{aligned} \quad (23)$$

Based on (23), design

$$\begin{aligned}\alpha_i =\ & -c_i\eta_i - \eta_{i-1} - \left(\varphi_i^T - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j^T\right)\hat{\theta}_i\\ & + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j\end{aligned} \quad (24)$$

$$\dot{\hat{\theta}}_i = \Gamma_i\mathrm{Proj}(\mu_i, \hat{\theta}_i), \quad \mu_i := \left(\varphi_i - \sum_{j=1}^{i-1}\varphi_j\frac{\partial\alpha_{i-1}}{\partial x_j}\right)\eta_i. \quad (25)$$

Note that $\alpha_i = \alpha_i(x_1, \cdots, x_i, \hat{\theta}_1, \cdots, \hat{\theta}_i)$. Substituting (24) and (25) into (23) gives

$$\dot{V}_i \leq -\sum_{j=1}^{i}c_j\eta_j^2 + \eta_i\eta_{i+1} \quad (26)$$

Remark 1 *Notice that in step 3 the term $\partial\alpha_2/\partial x_1$ in (24) is a function of $\partial Proj(\mu_1, \hat{\theta}_1)/\partial\mu_1$ and $\partial Proj(\mu_1, \hat{\theta}_1)/\partial\hat{\theta}_1$, which requires $Proj(\mu_1, \hat{\theta}_1)$ to be $\mathcal{C}^1$. In the subsequent steps, the differentiability order of the projection operator will increase by one after each step. For example, in step 4, the stabilizing function α_4 will require $Proj(\mu_1, \hat{\theta}_1)$ to be $\mathcal{C}^2$ and $Proj(\mu_2, \hat{\theta}_2)$ to be $\mathcal{C}^1$. At step n, the control input u will require $Proj(\mu_i, \hat{\theta}_i)$, $i = 1, ..., n-1$ to be $\mathcal{C}^{n-i-1}$. As a result, we require a projection operator that is sufficiently smooth such as the one in [1].*

Step n

In the last step, we introduce a modified backstepping procedure based on the new control mechanism of [14] to deal with the additive disturbance in (4). Let the auxiliary variable $r(t)$ be defined as

$$r := \dot{\eta}_n + \eta_n \quad (27)$$

where

$$\eta_n := x_n - \alpha_{n-1}. \quad (28)$$

After differentiating (27), we obtain

$$\dot{r} = \ddot{\eta}_n + r - \eta_n. \quad (29)$$

Differentiating η_n twice produces

$$\ddot{\eta}_n = \dot{\varphi}_n^T\theta + \dot{d} + \dot{u} - \ddot{\alpha}_{n-1} \quad (30)$$

where (4) was used.

We first concentrate on the calculation of the term $\ddot{\alpha}_{n-1}$ in (30). To that end, we have

$$\begin{aligned}\dot{\alpha}_{n-1} &= \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j\\ &= \underbrace{\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j}_{\Phi(x_1, \cdots, x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1})}\\ &\quad + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j^T\theta.\end{aligned} \quad (31)$$

Note that $\Phi(\cdot)$ is *known*. Differentiating (31) gives

$$\begin{aligned}\ddot{\alpha}_{n-1} =\ & \dot{\Phi}(\cdot)\\ & + \underbrace{\sum_{j=1}^{n-1}\left(\sum_{k=1}^{n-1}\left[\frac{\partial^2\alpha_{n-1}}{\partial x_j\partial x_k}\varphi_j^T + \frac{\partial\alpha_{n-1}}{\partial x_j}\frac{\partial\varphi_j^T}{\partial x_k}\right]\times\left(\varphi_k^T\theta + x_{k+1}\right) + \frac{\partial^2\alpha_{n-1}}{\partial x_j\partial\hat{\theta}_k}\dot{\hat{\theta}}_k\,\varphi_j^T\right)\theta.}_{g_1(x_1, \cdots x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1})}\end{aligned} \quad (32)$$

We now turn our attention to the calculation of the term $\dot{\varphi}_n^T\theta$ in (30). To that end, we have

$$\begin{aligned}\dot{\varphi}_n^T\theta &= \sum_{j=1}^{n-1}\frac{\partial\varphi_n^T}{\partial x_j}\dot{x}_j\theta + \frac{\partial\varphi_n^T}{\partial x_n}\dot{x}_n\theta\\ &= \sum_{j=1}^{n-1}\frac{\partial\varphi_n^T}{\partial x_j}\left(\varphi_j^T\theta + x_{j+1}\right)\theta\\ &\quad + \frac{\partial\varphi_n^T}{\partial x_n}\left(r - \eta_n + \dot{\alpha}_{n-1}\right)\theta\\ &= \sum_{j=1}^{n-1}\frac{\partial\varphi_n^T}{\partial x_j}\left(\varphi_j^T\theta + x_{j+1}\right)\theta\\ &\quad + \frac{\partial\varphi_n^T}{\partial x_n}\left(r - \eta_n + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j^T\theta + \Phi\right)\theta\\ &:= g_2(x_1, \cdots x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1}, r)\end{aligned} \quad (33)$$

where (27), (28), and (31) were used.

After substituting (30), (32), and (33) into (29), we get

$$\dot{r} = g - \dot{\Phi} + \dot{d} + \dot{u} + r - \eta_n \qquad (34)$$

where $g := g_2 - g_1$,

$$\left\| g(x_1, \cdots x_n, \hat{\theta}_1, \cdots, \hat{\theta}_{n-1}, r) \right\| \leq \rho_g(\|z\|)\,\|z\|, \qquad (35)$$

$\rho_g(\cdot) \in \mathbb{R}_{\geq 0}$ is some globally invertible, nondecreasing function, and $z(t)$ is defined as follows

$$\eta := [\eta_1 \; \eta_2 \; ... \; \eta_n]^T \qquad z := \left[\eta^T \; r\right]^T. \qquad (36)$$

(See [2] for proof of (35)).

Based on (34), we design $\dot{u}(t)$ as [14]

$$\dot{u} = \dot{\Phi} - \beta \text{sgn}(\eta_n) - (c_n + 2)r \qquad (37)$$

where β is a positive constant. The actual $\mathcal{C}^0$ control input can be written from (37) as follows

$$\begin{aligned} u(t) = \;& \Phi(t) - \Phi(0) - (c_n + 2)(\eta_n(t) - \eta_n(0)) \\ & - \int_0^t \left[(c_n + 2)\eta_n(\tau) + \beta \text{sgn}(\eta_n(\tau))\right] d\tau \end{aligned} \qquad (38)$$

where $u(0) = 0$. After substituting (37) into (34), we obtain the closed-loop system

$$\dot{r} = -r - \eta_n - c_n r + g + \left(\dot{d} - \beta \text{sgn}(\eta_n)\right). \qquad (39)$$

4 Stability Analysis

Before presenting the main result, we state the following two lemmas which will be invoked later.

Lemma 1 *(See [14] for proof) Let the auxiliary function $L(t)$ be defined as follows*

$$L := r\left(\dot{d} - \beta sgn(\eta_n)\right). \qquad (40)$$

If the control gain β is selected to satisfy the following sufficient condition

$$\beta > \left\|\dot{d}(t)\right\|_{\mathcal{L}_\infty} + \left\|\ddot{d}(t)\right\|_{\mathcal{L}_\infty}, \qquad (41)$$

then

$$\int_0^t L(\tau)d\tau \leq \zeta_b \qquad (42)$$

where the positive constant ζ_b is defined as

$$\zeta_b = \beta\,|\eta_n(0)| - \eta_n(0)\dot{d}(0). \qquad (43)$$

Lemma 2 *(See proof of Theorem 8.4 in [6]) Consider the system $\dot{\xi} = f(\xi, t)$ where $f\colon \mathbb{R}^q \times \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ for which a solution exists. Let the region $\mathcal{D}$ be defined as follows $\mathcal{D} := \{\xi \in \mathbb{R}^q : \|\xi\| < \varepsilon_s\}$ where ε_s is some positive constant and let $V\colon \mathcal{D} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a $\mathcal{C}^1$ function such that*

$$W_1(\xi) \leq V(\xi, t) \leq W_2(\xi) \quad \text{and} \quad \dot{V}(\xi, t) \leq -W(\xi) \qquad (44)$$

$\forall t \geq 0$ and $\forall \xi \in \mathcal{D}$, where $W_1(\xi), W_2(\xi)$ are $\mathcal{C}^0$ positive definite functions and $W(\xi)$ is a differentiable, positive semi-definite function. If $\xi(0) \in \mathcal{S}$, where

$$\mathcal{S} := \{\xi \in \mathcal{D} : W_2(\xi) \leq \delta\}, \quad \delta < \min_{\|\xi\| = \varepsilon_s} W_1(\xi) \qquad (45)$$

with δ being some positive constant, then $\xi(t)$ is bounded. Furthermore, if $W(\xi)$ is uniformly continuous, then

$$W(\xi(t)) \to 0 \;\; as \;\; t \to \infty. \qquad (46)$$

We now state the main result for the proposed adaptive controller.

Theorem 1 *The control law of (38) ensures that all system signals are bounded and $y(t) \to 0$ as $t \to \infty$, provided the control gain β is adjusted according to (41), and the control gains c_n is selected sufficiently large relative to the system initial conditions.*

Proof: Let the auxiliary function $P(t)$ be defined as follows

$$P(t) := \zeta_b - \int_0^t L(\tau)d\tau \qquad (47)$$

where ζ_b and $L(t)$ were defined in Lemma 1. It is easy to see that the use of Lemma 1 ensures $P(t) \geq 0$. We now define the function V as follows

$$V := V_{n-1} + \frac{1}{2}\eta_n^2 + \frac{1}{2}r^2 + P. \qquad (48)$$

Note that (48) can be bounded as follows

$$\lambda_1 \|\xi\|^2 \leq V \leq \lambda_2 \|\xi\|^2 \qquad (49)$$

where

$$\begin{aligned} & \xi = [\eta^T \; \tilde{\theta}^T \; r \; \sqrt{P}]^T, \quad \tilde{\theta} = [\tilde{\theta}_1^T \; ... \; \tilde{\theta}_{n-1}^T]^T, \\ & \lambda_1 = \frac{1}{2}\min\left\{1, \lambda_{\min}\left(\Gamma_i^{-1}\right)\right\}, \\ & \lambda_2 = \max\left\{\frac{1}{2}\lambda_{\max}\left(\Gamma_i^{-1}\right), 1\right\}. \end{aligned} \qquad (50)$$

Remark 2 *Let $f(\xi, t)$ denote the right-hand side of the closed-loop system for which the stability analysis is being performed. Notice from (39) and (40) that $f(\xi, t)$ has a discontinuity on the set of Lebesgue measure zero $\{(\xi, t) : \eta_n = 0\}$. Since Lemma 2 requires that a solution exist for $\dot{\xi} = f(\xi, t)$, see [14] for a discussion on the existence of solutions.*

After taking the time derivative of (48) and substituting from (27), (39), and (47), we obtain

$$\dot{V} = \dot{V}_{n-1} - r^2 - c_n r^2 + rg - \eta_n^2 \qquad (51)$$

upon use of (40). Substituting now from (26) for $i = n-1$ and (35) gives

$$\begin{aligned} \dot{V} \;\leq\; & -\sum_{j=1}^{n-2} c_j \eta_j^2 + \left[\eta_{n-1}\eta_n - k\eta_{n-1}^2\right] - \eta_{n-1}^2 \\ & + \left[|r|\,\rho_g(\|z\|)\,\|z\| - c_n r^2\right] - \eta_n^2 - r^2 \end{aligned} \qquad (52)$$

where $c_{n-1} = k + 1$ and k is positive constant. After completing the squares on the bracketed terms of (52), we obtain

$$\dot{V} \leq -\sum_{j=1}^{n-2} c_j \eta_j^2 - \left(1 - \frac{1}{4k}\right) \eta_{n-1}^2 - \eta_n^2 - r^2 + \frac{\rho_g^2(\|z\|)\|z\|^2}{4c_n} \quad (53)$$

If $k > 1/4$, then (53) can be rewritten as

$$\dot{V} \leq -\left(\sigma - \frac{\rho_g^2(\|z\|)}{4c_n}\right)\|z\|^2 . \quad (54)$$

where $\sigma := \min\{c_1, ..., c_{n-2}, 1 - 1/4k\}$. It follows from (54) that

$$\dot{V} \leq -\gamma \|z\|^2 \quad \text{for} \quad c_n > \frac{\rho_g^2(\|z\|)}{4\sigma} \quad (55)$$

where the constant γ is such that $0 < \gamma < 1$.

We now apply Lemma 2 by first determining from (49) and (55) that

$$W_1(\xi) = \lambda_1 \|\xi\| \quad W_2(\xi) = \lambda_2 \|\xi\|^2 \quad W(\xi) = \gamma \|z\|^2 . \quad (56)$$

From (55), we define the sets $\mathcal{D}$ and $\mathcal{S}$ as

$$\mathcal{D} := \left\{\xi : \|\xi\| < \rho_g^{-1}(2\sqrt{\sigma c_n})\right\}, \quad (57)$$

$$\mathcal{S} := \left\{\xi : W_2(\xi(t)) < \lambda_1 \left(\rho_g^{-1}(2\sqrt{\sigma c_n})\right)^2\right\}. \quad (58)$$

We can now invoke Lemma 2 to state that $\xi(t) \in \mathcal{L}_\infty$. From (6), we know $x_1(t) \in \mathcal{L}_\infty$. From (10), (11), and property P3 of the projection operator, we then know $\alpha_1(t), \dot{\hat{\theta}}_1(t) \in \mathcal{L}_\infty$. We can now use (7) to show $x_2(t) \in \mathcal{L}_\infty$. From (1), we know $\dot{x}_1(t) \in \mathcal{L}_\infty$. Continuing with this procedure, we can show $\alpha_i(t), x_{i+1}(t), \in \mathcal{L}_\infty$, $i = 2, ..., n-1$. We can now state $\dot{\eta}_i(t), \dot{\alpha}_{i-1}(t) \in \mathcal{L}_\infty$, $i = 1, ..., n-1$ by using (21). Using (39) and assumption A2, we can show $\dot{r}(t) \in \mathcal{L}_\infty$. From (27) and (31), we know $\dot{\eta}_n(t), \dot{\alpha}_{n-1}(t) \in \mathcal{L}_\infty$; hence, from the time derivative of (28), we can conclude $\dot{x}_n(t) \in \mathcal{L}_\infty$. Finally, we can use (4) to show that $u(t) \in \mathcal{L}_\infty$.

Using the above boundedness statements, it is clear from (56) that $\dot{W}(\xi(t)) \in \mathcal{L}_\infty$, which is a sufficient condition for $W(\xi(t))$ being uniformly continuous. It then follows from Lemma 2 that $\gamma \|z(t)\|^2 \to 0$ as $t \to \infty$ $\forall \xi(0) \in \mathcal{S}$. From (36) and (6), we then know that $y(t) \to 0$ as $t \to \infty$ $\forall \xi(0) \in \mathcal{S}$.

Note that the set (58) can be made arbitrarily large to include any initial conditions by increasing the control gains c_n (i.e., a semi-global stability result). Specifically, we can use the second equation in (56) and (58) to calculate the set as follows

$$\|\xi(0)\| < \sqrt{\frac{\lambda_1}{\lambda_2}} \rho_g^{-1}(2\sqrt{\sigma c_n}), \quad (59)$$

or

$$c_n > \frac{1}{4\sigma} \rho_g^2 \left(\sqrt{\frac{\lambda_1}{\lambda_2}} \|\xi(0)\|\right) \quad (60)$$

where

$$\|\xi(0)\| = \sqrt{\sum_{j=0}^{n} \eta_j^2(0) + r^2(0) + \sum_{j=0}^{n-1} \tilde{\theta}_j^T(0)\tilde{\theta}_j(0) + P(0)} \quad (61)$$

Note that $r(0)$ is not a function of c_n since $u(0) = 0$. ■

5 Simulation Results

For simulation purposes, we considered the parametric strict-feedback system of (1)-(4) with $m = 1$ and $n = 4$ and the following model

$$\begin{aligned} \varphi_1 &= \begin{bmatrix} -x_1^2 \\ \frac{1}{4}x_1^3 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} -x_1^2 \\ \frac{1}{4}x_2^3 \end{bmatrix}, \\ \varphi_3 &= \begin{bmatrix} -x_2^2 \\ \frac{1}{4}x_1 x_3^2 \end{bmatrix}, \quad \varphi_4 = \begin{bmatrix} -x_2^2 \\ \frac{1}{4}x_2 x_4^2 \end{bmatrix}, \\ \theta &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad d(t) = 2\sin(t). \end{aligned} \quad (62)$$

The initial conditions of the system were set to $x_1(0) = 0.1$, $x_2(0) = x_3(0) = x_4(0) = 0$, and $\hat{\theta}_1(0) = \hat{\theta}_2(0) = \hat{\theta}_3(0) = [0\ \ 0]^T$. The controller parameters were selected by trial-and-error until good regulation performance was obtained. This tuning process resulted in the following values for the parameters: $c_1 = c_2 = c_3 = 5$, $c_4 = 30$, $\beta = 60$, $\Gamma_1 = \Gamma_2 = \Gamma_3 = \text{diag}(10, 10)$, $\theta_0 = 3$, $\varepsilon = 1$, and $\delta = 1$. The simulation results are shown in Figures 1 and 2. Figure 1 shows the output $y(t)$ and the control input $u(t)$ while the parameter estimates $\hat{\theta}_i(t)$, $i = 1, 2, 3$ are shown in Figure 2.

6 Conclusion

This paper considered the regulation problem for nth-order MIMO nonlinear parametric strict-feedback systems in the presence of matched additive disturbances and unmatched parametric uncertainties. A new continuous adaptive controller was proposed whose formulation is founded on the fusion of a modified backstepping design, a a dynamic robust control mechanism, and an arbitrarily many times continuously differentiable projection operator. The resulting adaptive controller, which exploits the two times continuous differentiability of the disturbance, asymptotically regulates the output to zero without the need for a leakage term in the adaptation law.

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A Projection Operator

We use the following smoothened version of the projection operator introduced in [12], which was recently proposed in [1]:

$$\mathrm{Proj}(\mu_i, \hat{\theta}_i) = \mu_i - \frac{\pi_1 \pi_2 \nabla p(\hat{\theta}_i)}{4\left(\varepsilon^2 + 2\varepsilon\theta_0\right)^{n-i} \theta_0^2}, \quad i = 1, ..., n-1 \tag{63}$$

where

$$p(\hat{\theta}_i) = \hat{\theta}_i^T \hat{\theta}_i - \theta_0^2, \tag{64}$$

$$\pi_1 = \begin{cases} p^{n-i}(\hat{\theta}_i) & \text{if } p(\hat{\theta}_i) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{65}$$

$$\pi_2 = \frac{1}{2}\nabla p(\hat{\theta}_i)\mu_i + \sqrt{\left(\frac{1}{2}\nabla p(\hat{\theta}_i)\mu_i\right)^2 + \delta^2}, \tag{66}$$

μ_i was defined in (25), ∇ is the gradient operator, ε, δ are arbitrary positive constants, and θ_0 was defined in assumption A3. It can be proven that the above projection operator has the following properties [1]: If $\hat{\theta}_i(0) \in \Omega$, then

P1. $\left\|\hat{\theta}_i(t)\right\| \leq \theta_0 + \varepsilon \ \forall t \geq 0$;

P2. $\tilde{\theta}_i^T \mathrm{Proj}(\mu_i, \hat{\theta}_i) \geq \tilde{\theta}_i^T \mu_i$;

P3. $\left\|\mathrm{Proj}(\mu_i, \hat{\theta}_i)\right\| \leq \|\mu_i\| \left[1 + \left(\frac{\theta_0 + \varepsilon}{\theta_0}\right)^2\right] + \frac{\theta_0 + \varepsilon}{2\theta_0^2}\delta$;

P4. $\mathrm{Proj}(\mu_i, \hat{\theta}_i)$ is $\mathcal{C}^{n-i-1}$.

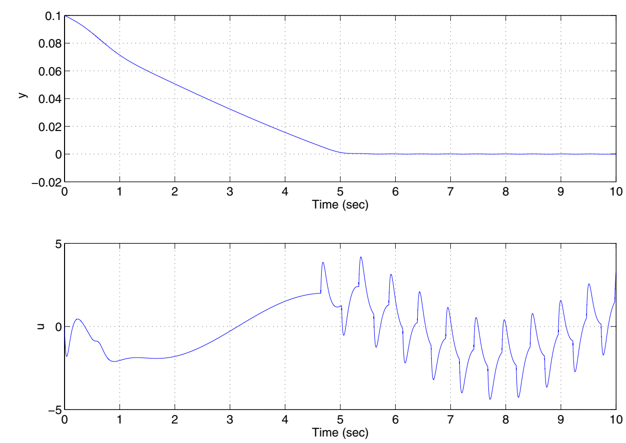

Figure 1: Output $y(t)$ (top plot) and control input $u(t)$ (bottom plot).

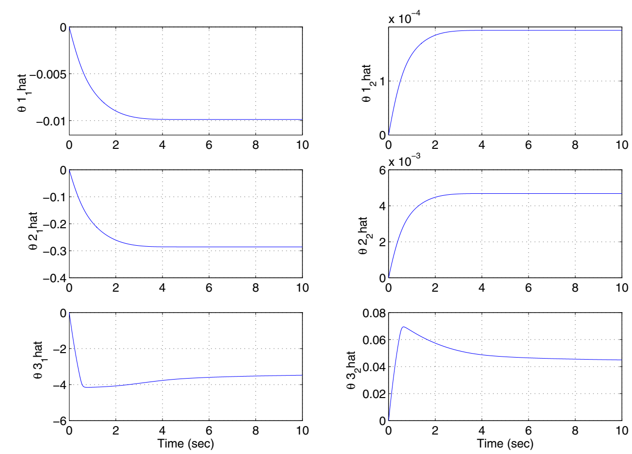

Figure 2: Parameters estimates $\hat{\theta}_1(t)$ (top plots), $\hat{\theta}_2(t)$ (middle plots), and $\hat{\theta}_3(t)$ (bottom plots).