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Observers for Systems with Nonlinearities Satisfying an Incremental Quadratic Inequality

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***Abstract*— We consider the problem of designing observers to asymptotically estimate the state of a system whose nonlinear time-varying terms satisfy an incremental quadratic inequality that is parameterized by a set of multiplier matrices. Observer design is reduced to solving linear matrix inequalities for the observer gain matrices. The proposed observers guarantee exponential convergence of the state estimation error to zero. In addition to considering a larger class of nonlinearities than previously considered, this paper unifies some earlier results in the literature. The results are illustrated by application to a model of an underwater vehicle.**

I. INTRODUCTION

A fundamental problem in system analysis and control design is that of determining the state of a system from its measured input and output. Many solutions to this problem use an asymptotic observer (or state estimator) which produces an estimate of the system state that asymptotically approaches the actual system state. Typical observers for linear systems consist of a copy of the system dynamics along with a linear correction term based on the output error, that is, the difference between the measured output and its estimate based on the estimated state [20], [9].

References [15], [16], [22], [4], [7], [6] consider systems with globally Lipschitz nonlinearities and nonlinearities in unbounded sectors. Reference [13] extends these results to multivariable nonlinearities satisfying a monotonicity condition, as well as relaxing the observer feasibility conditions via a multiplier by exploiting the decoupled nature of the multivariable nonlinearity. They present asymptotic observers that consist of a copy of the system dynamics and two correction terms based on the output error; one term is the usual linear correction term while the other term (called the nonlinear injection term) enters the copy of the nonlinear element in the observer. Reference [24] gives a general description of a nonlinear observer with an "output injection" form, that is also analyzed by [3] in an incremental stability framework. Additional results on observers for nonlinear systems can also be found in [18], [10], [23], [25].

In this paper, we consider systems whose state space description consists of a linear time-invariant part and nonlinear/time-varying parts. We characterize the state dependent nonlinear/time-varying terms by a set of symmetric matrices which we call multiplier matrices. More specifically, this nonlinear term satisfies an incremental quadratic constraint (δQC) that is parameterized by its associated multiplier matrices; see inequality (3). The nonlinearities considered here include many commonly encountered nonlinearities including those considered in [15], [16], [22], [4], [7], [6], [13]. Consequently, this paper unifies earlier results by characterizing nonlinearities via an incremental quadratic constraint. Beyond the unification and generalization of previous observer results, we consider two other general classes of nonlinearities described by polytopic and conic parameterizations. These additional characterizations can provide less conservative feasibility results for globally Lipschitz multivariable nonlinearities and multivariable nonlinearities in unbounded sectors by further exploiting their structure. We also consider the case of multiple nonlinearities with different characterizations for each portion of the nonlinearity, and provide corresponding multiplier matrices. For the systems under consideration, we present observers whose structure is inspired by [7]. These observers are characterized by two gain matrices: the gain matrix L for the linear correction term and the gain matrix L_n for the nonlinear injection term. Initially, we consider L_n fixed and convert the problem of determining L into that of solving linear matrix inequalities. Such inequalities can be readily treated using, for example, the LMI toolbox in MATLAB [14].

We also consider the problem of simultaneously computing L and L_n. By imposing a specific condition on the set of multiplier matrices describing the nonlinearities, we convert the problem of determining L and L_n into that of solving linear matrix inequalities. All of these results are based on the analysis of the state estimation error dynamics using quadratic Lyapunov functions.

To illustrate our results, we apply the nonlinear observer design technique to an underwater vehicle from [19].

Observer based output feedback controller design (such as given in [7], [5], [21]) is out of the scope of this paper, and it will be the subject of a separate paper [1].

II. SYSTEM DESCRIPTION AND INCREMENTAL QUADRATIC INEQUALITIES

We consider nonlinear/time-varying systems described by

$$\begin{aligned} \dot{x} &= Ax + B_p p(t,x,u) + B(t,u,y) \\ y &= Cx + D_p p(t,x,u) + D(t,u,y) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is a known input, $y(t) \in \mathbb{R}^l$ is the measured output and $t \in \mathbb{R}$ is the time variable. All the nonlinear/time-varying elements

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in the system are lumped into the terms B, D and p. We suppose that $p(t,x,u) \in \mathbb{R}^{l_p}$ and

$$p(t,x,u) = \psi(t,z) \quad \text{where} \quad z = C_q x + D_q(t,u,y) \quad (2)$$

and ψ is a piecewise continuous function of t and a continuous function of $z \in \mathbb{R}^{l_q}$. The matrices A, B_p, C, D_p and C_q are constant and of appropriate dimensions.

Our characterization of ψ is based on a set $\mathcal{M}$ of symmetric matrices which we refer to as multiplier matrices. Specifically, for all $M \in \mathcal{M}$, the following incremental quadratic constraint (δQC) holds for all $t \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^{l_q}$:

$$\begin{pmatrix} q(t,z_2) - q(t,z_1) \\ \psi(t,z_2) - \psi(t,z_1) \end{pmatrix}^T M \begin{pmatrix} q(t,z_2) - q(t,z_1) \\ \psi(t,z_2) - \psi(t,z_1) \end{pmatrix} \geq 0 \quad (3)$$

where

$$q(t,z) = z + D_{qp}\psi(t,z) \,. \quad (4)$$

Basically, the constant matrix D_{qp} and $\mathcal{M}$ provide a characterization of ψ in an incremental sense. When $D_{qp} = 0$, the above inequality reduces to

$$\begin{pmatrix} z_2 - z_1 \\ \psi(t,z_2) - \psi(t,z_1) \end{pmatrix}^T M \begin{pmatrix} z_2 - z_1 \\ \psi(t,z_2) - \psi(t,z_1) \end{pmatrix} \geq 0 \quad (5)$$

Section V exhibits some of the nonlinearities under consideration along with their multiplier matrices.

III. Observers

We propose the following observers to provide an estimate $\hat{x}$ of the state x of a system described in the previous section:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_p\hat{p} + B(t,u,y) + L(\hat{y} - y) \quad (6) \\ \hat{y} &= C\hat{x} + D_p\hat{p} + D(t,u,y) \end{aligned}$$

where

$$\hat{p} = \psi(t,\ \hat{z} + L_n(\hat{y} - y)) \quad \text{and} \quad \hat{z} = C_q\hat{x} + D_q(t,u,y) \,. \quad (7)$$

Here, L is the gain for the linear output error term and L_n is the gain for the nonlinear injection term. The nonlinear injection term results in additional flexibility in the design.

In the observer description above, we have

$$\hat{p} = \psi(t,\ \hat{z} + L_n(C\hat{x} + Du - y) + L_n D_p \hat{p}) \,. \quad (8)$$

When $L_n D_p \neq 0$, this is an implicit equation for $\hat{p}$. So, we assume that there is a continuous function ϕ such that for all t and η, the equation

$$\hat{p} = \psi(t,\ \eta + L_n D_p \hat{p}) \quad (9)$$

is uniquely solved by $\hat{p} = \phi(t,\eta)$. Then, $\hat{p}$ is uniquely given by

$$\hat{p} = \phi(t, \hat{z} + L_n(C\hat{x} + Du - y)) \,. \quad (10)$$

Using (7), we obtain that

$$\begin{aligned} \hat{p} &= \psi(t,\ z + (C_q + L_nC)e + L_nD_p(\hat{p} - p)) \\ &= \phi(t,\ z + (C_q + L_nC)e - L_nD_p p) \quad (11) \end{aligned}$$

where $e := \hat{x} - x$ is the state estimation error. The error dynamics are described by

$$\dot{e} = (A + LC)e + (B_p + LD_p)\delta p(t,e) \quad (12)$$

where

$$\delta p(t,e) = \phi\,(t, z(t) + (C_q + L_nC)e - L_nD_p\psi(t,z(t))) - \psi(t,z(t)).$$

It follows from (11) that δp satisfies

$$\delta p = \psi(t, z + (C_q + L_nC)e + L_nD_p\delta p) - \psi(t,z) \,.$$

Let

$$\delta q(t,e) = (C_q + L_nC)e + (D_{qp} + L_nD_p)\delta p(t,e) \,. \quad (13)$$

Then, using (3) with $z_1 = z$ and $z_2 = z + (C_q + L_nC)e + L_nD_p\delta p$ we obtain that for all t and e,

$$\begin{pmatrix} \delta q(t,e) \\ \delta p(t,e) \end{pmatrix}^T M \begin{pmatrix} \delta q(t,e) \\ \delta p(t,e) \end{pmatrix} \geq 0 \quad \text{for all} \quad M \in \mathcal{M} \,. \quad (14)$$

The following result yields conditions for observer gain matrices which result in exponentially decaying estimation errors.

Theorem 1: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices. Suppose that there exist matrices $P = P^T > 0$, L, L_n and $M \in \mathcal{M}$ and a scalar $\alpha > 0$ such that the matrix inequality (15) (on the next page) is satisfied. Also suppose that there is a continuous function ϕ such that $\hat{p} = \phi(t,\eta)$ uniquely solves equation (9). Then, given any input $u(\cdot)$ and initial condition $x(t_0) = x_0$ such that system (1) has a solution for all $t \geq t_0$, the state estimation error $e := \hat{x} - x$ corresponding to observer (6) decays exponentially to zero with rate α.

Proof: Introducing

$$A_c = A + LC,\ B_c = B_p + LD_p,\ C_c = C_q + L_nC,\ D_c = D_{qp} + L_nD_p, \quad (16)$$

inequality (15) simplifies to

$$\begin{pmatrix} A_c^TP + PA_c + 2\alpha P & PB_c \\ B_c^TP & 0 \end{pmatrix} + \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix} \leq 0. \quad (17)$$

Also, the error dynamics, given by (12), can be described by

$$\dot{e} = A_c e + B_c \delta p(t,e)$$

where, for all t, e, and $M \in \mathcal{M}$, the term δp satisfies inequality (14) with

$$\delta q(t,e) = C_c e + D_c \delta p(t,e) \,,$$

Pre- and post-multiplying both sides of inequality (17) by $[e^T\ \delta p^T]$ and its transpose and using condition (14) we obtain

$$e^TP\dot{e} \leq -\alpha e^TPe$$

for all t, e. This shows that the error dynamics are quadratically stable about zero with rate α; (see [11], [2] and/or [8] for a definition of quadratic stability). This implies that the error decays exponentially to zero with rate α. ■

$$\begin{pmatrix} PA+A^TP+PLC+C^TL^TP+2\alpha P & PB_p+PLD_p \\ B_p^TP+D_p^TL^TP & 0 \end{pmatrix} + \begin{pmatrix} C_q+L_nC & D_{qp}+L_nD_p \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} C_q+L_nC & D_{qp}+L_nD_p \\ 0 & I \end{pmatrix} \leq 0 \quad (15)$$

$$\begin{pmatrix} PA + A^T P + RC + C^T R^T + 2\alpha P & PB_p + RD_p \\ B_p^T P + D_p^T R^T & 0 \end{pmatrix} + \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix} \leq 0 \quad (18)$$

The following corollary yields an observer design procedure for a given L_n.

Corollary 1: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices. For a given matrix L_n and scalar $\alpha > 0$, suppose that there exist matrices $P = P^T > 0$, R and $M \in \mathcal{M}$ such that matrix inequality (18) above holds where C_c and D_c are given in (16), and let

$$L = P^{-1}R\,. \quad (19)$$

Also suppose that there is a continuous function ϕ such that $\hat{p} = \phi(t,\eta)$ solves equation (9). Then, given any input $u(\cdot)$ and initial condition $x(t_0) = x_0$ such that system (1) has a solution for all $t \geq t_0$, the state estimation error for the observer (6) decays exponentially to zero with a rate α.

Remark 1: Note that, for a fixed α and L_n, inequality (18) is an LMI (linear matrix inequality) in the variables P, R, and M. Using the LMI toolbox in MATLAB [14], the feasibility of such an inequality can readily be determined and a solution to a feasible inequality can be obtained.

Remark 2: If $L_nD_p \neq 0$, we need to be able to solve equation (9) for $\hat{p}$ to implement the observer. This equation defines an implicit relation for $\hat{p}$ in terms of t and η; here $\eta = \hat{z}+L_n(C\hat{x}+Du-y)$. We provide a sufficient condition in [1] wich guarantees that, for each t and η, equation (9) has a solution $\hat{p} = \phi(t,\eta)$, where ϕ is continuous.

IV. Simultaneous Design of L and L_n via LMIs

The previous section contains an observer design procedure where the observer gain L is designed for a fixed L_n. However, the simultaneous design of L and L_n is not addressed. To obtain tractable conditions that permit the simultaneous design of L and L_n, we consider multiplier matrices M that are parameterized by two matrices X and Y of lower dimensions, and that satisfy the following condition.

Condition 1: There exist a nonsingular matrix T and a set $\mathcal{N}$ of matrix pairs (X,Y) with $Y \in \mathbb{R}^{m_p\times m_p}$ such that $X = X^T > 0$, $Y = Y^T \geq 0$, and the matrix

$$M = T^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} T \quad (20)$$

is in $\mathcal{M}$. In addition, $T_{22} + T_{21}D_{qp}$ is nonsingular where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad (21)$$

and $T_{22} \in \mathbb{R}^{m_p\times m_p}$.

To demonstrate the significance of the above condition, we introduce a transformation that transforms the nonlinear term p to a new nonlinear term $\tilde{p}$.

A. A Transformation

Suppose Condition 1 holds and note that, with $q = z + D_{qp}p$,

$$T\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} T_{11}z + \Gamma_{12}p \\ T_{21}z + \Gamma_{22}p \end{pmatrix}$$

where $\Gamma_{12} = T_{12}+T_{11}D_{qp}$, and $\Gamma_{22} = T_{22}+T_{21}D_{qp}$. Now introduce the transformed nonlinear term $\tilde{p}$ defined by

$$\tilde{p} := T_{21}z + \Gamma_{22}p\,. \quad (22)$$

Since by assumption Γ_{22} is nonsingular, we have

$$p = -\Gamma_{22}^{-1}T_{21}z + \Gamma_{22}^{-1}\tilde{p}\,; \quad (23)$$

hence $T_{11}z + \Gamma_{12}p = \tilde{z} + \tilde{D}_{qp}\tilde{p}$ where

$$\tilde{z} = \Sigma z\,,\ \Sigma = T_{11} - \Gamma_{12}\Gamma_{22}^{-1}T_{21}\,,\ \tilde{D}_{qp} = \Gamma_{12}\Gamma_{22}^{-1}\,. \quad (24)$$

We now show that that Σ is invertible. Note that

$$\begin{pmatrix} T_{11} & \Gamma_{12} \\ T_{21} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12}+T_{11}D_{qp} \\ T_{21} & T_{22}+T_{21}D_{qp} \end{pmatrix} = T\begin{pmatrix} I & D_{qp} \\ 0 & I \end{pmatrix}.$$

Since the two matrices on the righthandside of the second equality are invertible, the matrix on the lefthandside of the first equality is invertible. Since Γ_{22} is assumed to be invertible, by using the matrix inversion lemma [17], the first matrix above is invertible if and only if the following Schur complement of the matrix is invertible:

$$T_{11} - \Gamma_{12}\Gamma_{22}^{-1}T_{21} = \Sigma\,.$$

This implies that Σ is invertible. Consequently, $z = \Sigma^{-1}\tilde{z}$ and

$$\tilde{p}(t,x,u) = \tilde{\psi}(t,\tilde{z}) := T_{21}\Sigma^{-1}\tilde{z} + \Gamma_{22}\psi(t,\Sigma^{-1}\tilde{z})\,. \quad (25)$$

Letting

$$\tilde{q}(t,\tilde{z}) = \tilde{z} + \tilde{D}_{qp}\tilde{\psi}(t,\tilde{z})$$

we obtain that

$$T\begin{pmatrix} q(t,z_2) - q(t,z_1) \\ \psi(t,z_2) - \psi(t,z_1) \end{pmatrix} = \begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2) - \tilde{\psi}(t,\tilde{z}_1) \end{pmatrix}.$$

Hence satisfaction of inequality (3) by ψ implies that the transformed nonlinear function $\tilde{\psi}$ satisfies

$$\begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2)-\tilde{\psi}(t,\tilde{z}_1) \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2)-\tilde{\psi}(t,\tilde{z}_1) \end{pmatrix} \geq 0. \quad (26)$$

Now, using the transformed term $\tilde{\psi}$, system (1) is described by

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}_p\tilde{\psi}(t,\tilde{z}) + \tilde{B}(t,u,y) \\ y &= \tilde{C}x + \tilde{D}_p\tilde{\psi}(t,\tilde{z}) + \tilde{D}(t,u,y) \\ \tilde{z} &= \tilde{C}_q x + \tilde{D}_q(t,u,y) \end{aligned} \tag{27}$$

where $\tilde{\psi}$ satisfies (26),

$$\begin{array}{ll} \tilde{A} = A - \tilde{B}_p T_{21} C_q\,, & \tilde{B}_p = B_p \Gamma_{22}^{-1}\,, \\ \tilde{C} = C - \tilde{D}_p T_{21} C_q\,, & \tilde{D}_p = D_p \Gamma_{22}^{-1}\,, \\ \tilde{C}_q = \Sigma C_q\,, & \end{array} \tag{28}$$

and

$$\tilde{B} = B - \tilde{B}_p T_{21} D_q \quad \tilde{D} = D - \tilde{D}_p T_{21} D_q \quad \tilde{D}_q = \Sigma D_q\,.$$

B. Observer Based on the Transformed System

Inspired by the previous section, we propose the following observers for a system described by (1)-(2); these observers are based on the transformed system (27) and are described by

$$\begin{aligned} \dot{\hat{x}} &= \tilde{A}\hat{x} + \tilde{B}_p\hat{p} + \tilde{B}(t,u,y) + L(\hat{y} - y) \qquad (29) \\ \hat{y} &= \tilde{C}\hat{x} + \tilde{D}_p\hat{p} + \tilde{D}(t,u,y) \\ \hat{p} &= \tilde{\psi}(t,\ \hat{z}+L_n(\hat{y}-y)) \\ \hat{z} &= \tilde{C}_q\hat{x} + \tilde{D}_q(t,u,y) \end{aligned}$$

In the observer description we have,

$$\hat{p} = \tilde{\psi}(t,\ \hat{z}+L_n(\tilde{C}\hat{x}+\tilde{D}u-y) + L_n\tilde{D}_p\hat{p})\,.$$

So, when $L_n\tilde{D}_p \neq 0$, we assume that there is a continuous function $\tilde{\phi}$ such that for all t and η, the equation

$$\hat{p} = \tilde{\psi}(t,\ \eta + L_n\tilde{D}_p\hat{p}) \tag{30}$$

is uniquely solved by $\hat{p} = \tilde{\phi}(t,\eta)$. Then,

$$\hat{p} = \tilde{\phi}(t,\hat{z}+L_n(\tilde{C}\hat{x}+\tilde{D}u-y))\,. \tag{31}$$

Now, we can present the main result of this section, which is a corollary to Theorem 1.

Corollary 2: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices that satisfy Condition 1. Suppose that, for some scalar $\alpha > 0$, there exist matrices $P = P^T > 0$, R_1, R_2 and $(X,\ Y) \in \mathcal{N}$ such that matrix inequality (32) (on the next page) is satisfied, and let

$$L = P^{-1}R_1\,, \qquad L_n = X^{-1}R_2\,. \tag{33}$$

Also suppose that there is a continuous function $\tilde{\phi}$ such that $\hat{p} = \tilde{\phi}(t,\eta)$ solves equation (30). Then, given any input $u(\cdot)$ and initial condition $x(t_0) = x_0$ such that system (1) has a well defined solution for all $t \geq t_0$, the state estimation error, $e := \hat{x} - x$, decays exponentially to zero with a rate of α.

Proof: Substitute (33) into inequality (32) and apply a Schur complement result ([8]) to obtain

$$\begin{pmatrix} P\tilde{A}+\tilde{A}^TP+PL\tilde{C}+\tilde{C}^TL^TP+2\alpha P & P\tilde{B}_p+PL\tilde{D}_p \\ \tilde{B}_p^TP+\tilde{D}_p^TL^TP & 0 \end{pmatrix} + \begin{pmatrix} \tilde{C}_c & \tilde{D}_c \\ 0 & I \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \tilde{C}_c & \tilde{D}_c \\ 0 & I \end{pmatrix} \leq 0.$$

where $\tilde{C}_c := \tilde{C}_q + L_n\tilde{C}$ and $\tilde{D}_c := \tilde{D}_{qp} + L_n\tilde{D}_p$. The result now follows by applying Theorem 1 to the transformed system. ■

Remark 3: Note that, for a fixed α, inequality (32) is an LMI (linear matrix inequality) in the variables P, R_1, R_2, X and Y.

V. Examples of Nonlinearities Satisfying an Incremental Quadratic Constraint

In this section, we discuss some typical nonlinearities satisfying the incremental quadratic constraint (3). We also present additional conditions under which these nonlinearities satisfy Condition 1. The first two classes include globally Lipschitz nonlinearities, monotonic nonlinearities, and nonlinearities in bounded and unbounded sectors, which are also studied in [7], [5], and [6]. Then, we consider nonlinearities which can be parameterized by sets of matrices; in particular we consider polytopic and conic sets. These parameterizations are useful for fully exploiting the structure of a nonlinear term.

A. Incrementally Sector Bounded Nonlinearities

Here we consider nonlinearities that, for all t, and z_1, z_2, satisfy

$$(\delta\psi - K_1\delta q)^T X(K_2\delta q - \delta\psi) \geq 0 \quad \text{for all } X \in \mathcal{X}\,, \tag{34}$$

where

$$\delta\psi := \psi(t,z_2) - \psi(t,z_1),\ \ \delta q := \delta z + D_{qp}\delta\psi,\ \ \delta z := z_2 - z_1, \tag{35}$$

while $\mathcal{X}$ is a set of symmetric positive definite matrices and K_1, K_2 are fixed matrices. Here, without loss of generality, we assume that the set $\mathcal{X}$ is invariant under multiplication by a positive number. It readily follows from (34) that a set $\mathcal{M}$ of multiplier matrices for the nonlinearities under consideration is given by

$$\mathcal{M} = \left\{ \begin{pmatrix} -K_1^TXK_2 - K_2^TXK_1 & (K_1+K_2)^TX \\ X(K_1+K_2) & -2X \end{pmatrix} :\ X \in \mathcal{X} \right\}.$$

To satisfy Condition 1, suppose that there exists a positive scalar σ such that $S_1 - \sigma S_2$ is nonsingular where $S_1 := K_2D_{qp} - I$ and $S_2 := I - K_1D_{qp}$. One can verify by substitution that the following equality holds

$$2\begin{pmatrix} -K_1^TXK_2 - K_2^TXK_1 & (K_1+K_2)^TX \\ X(K_1+K_2) & -2X \end{pmatrix} = T^T \begin{pmatrix} X & 0 \\ 0 & -\sigma^{-1}X \end{pmatrix} T,$$

where

$$T = \begin{pmatrix} \frac{1}{\sqrt{\sigma}}K_2 - \sqrt{\sigma}K_1 & -\frac{1}{\sqrt{\sigma}}I + \sqrt{\sigma}I \\ K_2 + \sigma K_1 & -I - \sigma I \end{pmatrix}.$$

Here $\Gamma_{22} = S_1 - \sigma S_2$ is nonsingular. Therefore, Condition 1 is satisfied with the matrix T defined above and

$$\mathcal{N} = \left\{ (\,X\,,\ \sigma^{-1}X\,) \ :\ X \in \mathcal{X} \right\}.$$

When q and p are scalars, one can always choose a positive scalar σ such that $S_1 - \sigma S_2$ is nonzero. To prove this claim, note that if $S_1 - \sigma S_2$ is zero for all $\sigma > 0$ then, $S_1 = S_2 = 0$. In this case, $K_1 = K_2 = 1/D_{qp}$ and $\delta\psi = K\delta q$ where $K := K_1 = K_2$. Using $\delta q = \delta z + D_{qp}\delta\psi$

$$\begin{pmatrix} \tilde{A}^T P + P\tilde{A} + R_1\tilde{C} + \tilde{C}^T R_1^T + 2\alpha P & P\tilde{B}_p + R_1\tilde{D}_p & \tilde{C}_q^T X + \tilde{C}^T R_2^T \\ \tilde{B}_p^T P + \tilde{D}_p^T R_1^T & -Y & \tilde{D}_{qp}^T X + \tilde{D}_p^T R_2^T \\ X\tilde{C}_q + R_2\tilde{C} & X\tilde{D}_{qp} + R_2\tilde{D}_p & -X \end{pmatrix} \le 0 \,. \tag{32}$$

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and $\delta\psi = K\delta q$, we have $\delta z = (1 - D_{qp}K)\delta q = 0$. However, δz should be arbitrary; hence we cannot have $S_1 = S_2 = 0$. Consequently, Condition 1 is always satisfied by $\mathcal{M}$ in the scalar case.

As a specific example of a nonlinearity under consideration, consider a globally Lipschitz nonlinearity which satisfies $\|\delta\psi\| \le \gamma\|\delta q\|$ for some $\gamma > 0$. In this case, inequality (34) holds with $K_1 = -\gamma I$, $K_2 = \gamma I$, and $\mathcal{X} = \{\lambda I : \lambda > 0\}$.

B. Incrementally Positive Real Nonlinearities

This class of nonlinearities is described by a set $\mathcal{X}$ of symmetric positive definite matrices X such that for all t and z_1, z_2,

$$\delta q^T X \delta\psi \ge 0 \qquad \text{for all } X \in \mathcal{X}\,, \tag{36}$$

where δq and $\delta\psi$ are as defined in (35). It is clear from (36) that, without loss of generality, we can assume that the set $\mathcal{X}$ of matrices is invariant under multiplication by a positive scalar. Note that nondecreasing nonlinearities satisfy (36) with $\mathcal{X} = \{\lambda I : \lambda > 0\}$. It readily follows from (36) that a set $\mathcal{M}$ of multiplier matrices for the nonlinearities under consideration is given by

$$\mathcal{M} = \left\{ \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \;:\; X \in \mathcal{X} \right\}.$$

To satisfy Condition 1 choose any scalar $\sigma > 0$ such that $D_{qp} - \sigma I$ is nonsingular. Then, we can readily show that

$$2\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\sigma}}I & \sqrt{\sigma}I \\ I & -\sigma I \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -\sigma^{-1}X \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sigma}}I & \sqrt{\sigma}I \\ I & -\sigma I \end{pmatrix}.$$

Consequently, if we let

$$T = \begin{pmatrix} \frac{1}{\sqrt{\sigma}}I & \sqrt{\sigma}I \\ I & -\sigma I \end{pmatrix}, \quad \mathcal{N} = \{\, (X\,,\ \sigma^{-1}X) : X \in \mathcal{X}\},$$

then $\Gamma_{22} = D_{qp} - \sigma I$ is nonsingular and Condition 1 holds.

C. Nonlinearities with Matrix Parameterizations

In this section, we consider nonlinear uncertain terms that are characterized by some known set Ω of matrices. Specifically, we assume that there is a known set Ω of real $l_p \times l_q$ matrices with the following property. For each t, z_1 and z_2, there is a matrix in $\Theta \in \Omega$ such that

$$\delta\psi = \Theta\delta q \tag{37}$$

where $\delta\psi$ and δq are defined in (35).

For example, suppose that

$$\psi(t,z) = g(t,q) \qquad \text{with} \qquad q = z + D_{qp}\psi(t,z) \tag{38}$$

where g is a function which is continuously differentiable with respect to its second argument, and for each t and q the derivative $\frac{\partial g}{\partial q}(t,q)$ lies in some known closed convex set Ω, that is,

$$\frac{\partial g}{\partial q}(t,q) \in \Omega \qquad \text{for all } t \text{ and } q\,. \tag{39}$$

Then, it follows from Lemma 3.5.1 in [12] that for each t, q and $\tilde{q}$, there exists a matrix $\Theta \in \Omega$ such that

$$g(t,q) - g(t,\tilde{q}) = \Theta(q - \tilde{q})\,.$$

It now follows that for every t, z_1 and z_2, there is a matrix Θ in Ω such that (37) holds.

Since $\delta\psi = \Theta\delta q$ for some Θ in Ω, it follows that a symmetric matrix M satisfies the multiplier condition (3) if

$$\begin{pmatrix} I \\ \Theta \end{pmatrix}^T M \begin{pmatrix} I \\ \Theta \end{pmatrix} \ge 0 \quad \text{for all} \quad \Theta \in \Omega\,.$$

Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$$

where partitioning is in accordance with $(\delta q,\, \delta\psi)$. Then the above inequalities can be expressed as

$$M_{11} + M_{12}\Theta + \Theta^T M_{12}^T + \Theta^T M_{22}\Theta \ge 0 \quad \text{for all } \Theta \in \Omega.$$

We restrict consideration to those matrices M that satisfy $M_{22} \le 0$. When $D_{qp} + L_n D_p = 0$ this leads to no loss of generality; this is a consequence of inequality (15). With $M_{22} \le 0$, the above inequalities are equivalent to:

$$\begin{pmatrix} M_{11} + \Theta^T M_{12}^T + M_{12}\Theta & \Theta^T M_{22} \\ M_{22}\Theta & -M_{22} \end{pmatrix} \ge 0, \ \forall\, \Theta \in \Omega\,. \tag{40}$$

Thus any symmetric matrix M that satisfies (40) is a multiplier matrix.

1) Polytopic case: Here we consider the case in which

$$\Omega = Co\{\Theta_1\,, \ldots, \Theta_\nu\}\ ,$$

that is, Ω is the set of matrices Θ that are given by $\Theta = \sum_{k=1}^{\nu} \lambda_k \Theta_k$ where $\lambda_k \ge 0$, $k = 1, \ldots, \nu$, and $\sum_{k=1}^{\nu} \lambda_k = 1$. In this case, condition (40) is satisfied if and only if, for $k = 1, \cdots, \nu$,

$$\begin{pmatrix} M_{11} + \Theta_k^T M_{12}^T + M_{12}\Theta_k & \Theta_k^T M_{22} \\ M_{22}\Theta_k & -M_{22} \end{pmatrix} \ge 0\,. \tag{41}$$

Since $M_{22} \le 0$, the above inequalities are equivalent to $M_{22} \le 0$ and

$$M_{11} + M_{12}\Theta_k + \Theta_k^T M_{12}^T + \Theta_k^T M_{22}\Theta_k \ge 0 \quad \text{for } k = 1, 2 \cdots, \nu.$$

Thus, the set $\mathcal{M}$ of symmetric matrices M that satisfy

$$\begin{pmatrix} I \\ \Theta_k \end{pmatrix}^T M \begin{pmatrix} I \\ \Theta_k \end{pmatrix} \ge 0, \ \text{for } k = 1, \ldots, \nu, \text{ and } M_{22} \le 0\,, \tag{42}$$

is a set of multipliers matrices.

To obtain a set of multiplier matrices satisfying Condition 1, choose any nonsingular matrix T and consider multiplier matrices of the form given in (20) where $X^T = X > 0$ and $Y^T = Y \geq 0$. A matrix M of this structure satisfies inequalities (42) if and only if X and Y satisfy

$$\begin{aligned} \begin{pmatrix} I \\ \Theta_k \end{pmatrix}^T T^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} T \begin{pmatrix} I \\ \Theta_k \end{pmatrix} &\geq 0, \text{ for } k=1,\ldots,\nu \\ T_{12}^T X T_{12} - T_{22}^T Y T_{22} &\leq 0 . \end{aligned} \quad (43)$$

Then, provided $T_{22} + T_{21}D_{qp}$ is invertible, Condition 1 is satisfied with

$$\mathcal{N} = \left\{(X,Y) : X^T{=}X>0 \text{ and } Y^T{=}Y \geq 0 \text{ satisfy (43)}\right\}.$$

Once T is chosen, (43) is a set of linear matrix inequalities in X and Y. However, the choice of T to yield a large subset of multipliers in some sense is not clear. Therefore, T is treated as a design parameter at this point. For example, the simple choice of $T = I$ satisfies Condition 1 with $\mathcal{N}$ defined by

$$\mathcal{N} = \{(X,Y) : X^T{=}X>0 \text{ and } Y^T{=}Y \geq 0 \text{ satisfy (44)}\}$$

where

$$X - \Theta_k^T Y \Theta_k \geq 0 \quad \text{for} \quad k = 1, \ldots, \nu . \quad (44)$$

As an example of a nonlinearity treated in this section, consider

$$\psi(t,z) = \begin{pmatrix} \sin z_1 & \sin z_2 \end{pmatrix}^T .$$

Here $\psi(t,z) = g(t,q)$ where $q = z$,

$$g(t,q) = \begin{pmatrix} \sin q_1 \\ \sin q_2 \end{pmatrix} \text{ and } \frac{\partial g}{\partial q}(t,q) = \begin{pmatrix} \cos q_1 & 0 \\ 0 & \cos q_2 \end{pmatrix}.$$

Hence Ω is the polytope defined by the four matrices

$$\Theta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Theta_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \Theta_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \Theta_4 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

As another example, consider

$$\psi(t,z) = \sin z_1 \sin z_2$$

Here $\psi(t,z) = g(t,q)$ where $q = z$,

$$g(t,q) = \sin q_1 \sin q_2 \text{ and } \frac{\partial g}{\partial q}(t,q) = \begin{pmatrix} \cos q_1 \sin q_2 & \sin q_1 \cos q_2 \end{pmatrix}.$$

Hence Ω is the polytope defined by the four matrices

$$\Theta_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}, \; \Theta_2 = \begin{pmatrix} -1 & 1 \end{pmatrix}, \; \Theta_3 = \begin{pmatrix} 1 & -1 \end{pmatrix}, \; \Theta_4 = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

2) Conic Case: In this case, Ω is a closed convex set defined by

$$\Omega = Cone\{\Theta_1, \ldots, \Theta_\nu\},$$

that is, Ω is the set of matrices Θ that satisfy $\Theta = \sum_{k=1}^{\nu} \lambda_k \Theta_k$ where $\lambda_k \geq 0$, $k = 1, \ldots, \nu$.

As in the previous section, we only consider multiplier matrices with $M_{22} \leq 0$. In this case, condition (40) is satisfied if and only if

$$\begin{pmatrix} M_{11} + \Theta^T M_{12}^T + M_{12}\Theta & \Theta^T M_{22} \\ M_{22}\Theta & -M_{22} \end{pmatrix} \geq 0, \; \forall \Theta \in Cone\{\Theta_1, \ldots, \Theta_\nu\}. \quad (45)$$

Consider any matrix Θ_k. For any $\lambda \geq 0$, the matrix $\lambda\Theta_k$ is also in $Cone\{\Theta_1, \ldots, \Theta_\nu\}$; hence

$$\begin{pmatrix} M_{11} + \lambda\Theta_k^T M_{12}^T + \lambda M_{12}\Theta_k & \lambda\Theta_k^T M_{22} \\ \lambda M_{22}\Theta_k & -M_{22} \end{pmatrix} \geq 0 .$$

Considering $\lambda = 0$, we obtain that

$$\begin{pmatrix} M_{11} & 0 \\ 0 & -M_{22} \end{pmatrix} \geq 0 ,$$

that is, $M_{11} \geq 0$ and $M_{22} \leq 0$. Considering $\lambda > 0$, we obtain

$$\begin{pmatrix} \lambda^{-1} M_{11} + \Theta_k^T M_{12}^T + M_{12}\Theta_k & \Theta_k^T M_{22} \\ M_{22}\Theta_k & -\lambda^{-1} M_{22} \end{pmatrix} \geq 0 .$$

Since λ can be arbitrary large, we must have

$$\begin{pmatrix} \Theta_k^T M_{12}^T + M_{12}\Theta_k & \Theta_k^T M_{22} \\ M_{22}\Theta_k & 0 \end{pmatrix} \geq 0 .$$

Hence, satisfaction of (45) implies that

$$\left. \begin{array}{c} M_{11} \geq 0 , \qquad M_{22} \leq 0 \\ M_{12}\Theta_k + \Theta_k^T M_{12}^T \geq 0 \\ M_{22}\Theta_k = 0 \end{array} \right. \quad \text{for} \quad k = 1, \cdots, \nu. \quad (46)$$

Clearly, satisfaction of condition (46) implies (45). Thus any symmetric matrix M which satisfies (46) is a multiplier matrix for this case.

To obtain a set of multiplier matrices satisfying Condition 1, one could choose any nonsingular matrix T and consider multiplier matrices of the form given in (20). Once T is chosen, (46) defines a set of linear matrix inequalities in X and Y. However, the choice of T to yield a large subset of multipliers in some sense is not clear. Therefore, T is treated as a design parameter at this point. For example, the following is a simple choice

$$T = \begin{pmatrix} I & F \\ F^T & -I \end{pmatrix}$$

where F is a full rank matrix of appropriate dimensions satisfying $FF^T = I$ or $F^T F = I$ as appropriate. With this choice,

$$\begin{aligned} M_{11} &= X - FYF^T \\ M_{12} &= XF + FY \\ M_{22} &= F^T XF - Y . \end{aligned}$$

Consider the case $FF^T = I$ and let $Y = F^T XF$. Then $M_{22} = 0$, $M_{11} = 0$, and $M_{12} = 2XF$. Hence, Condition 1 is satisfied with

$$\mathcal{N} = \left\{(X, F^T XF) : X^T{=}X > 0 \text{ and (47) is satisfied}\right\},$$

where

$$\Theta_k^T F^T X + XF\Theta_k \geq 0, \text{ for } k = 1,\ldots,\nu. \quad (47)$$

Consider now the case $F^T F = I$ and let $Y = F^T XF$. Then $M_{22} = 0$, $M_{11} = X - FF^T XFF^T$, and $M_{12} = (I + FF^T)XF$. Hence, Condition 1 is satisfied with

$$\mathcal{N} = \left\{(X, F^T XF) : X^T{=}X>0, \; X - FF^T XFF^T \geq 0, \text{ (48) holds}\right\},$$

$$(I + FF^T)XF\Theta_k + \Theta_k^T F^T X(I + FF^T) \geq 0, \text{ for } k = 1,\ldots,\nu . \quad (48)$$

As an example of a nonlinearity treated in this section, consider

$$\psi(t,z) = \begin{pmatrix} z^3 & z^5 \end{pmatrix}^T .$$

Here $\psi(t,z) = g(t,q)$ where $q = z$,

$$g(t,q) = \begin{pmatrix} q^3 & q^5 \end{pmatrix}^T \quad \text{and} \quad \frac{\partial g}{\partial q}(t,q) = \begin{pmatrix} 3q^2 & 5q^4 \end{pmatrix}^T .$$

Hence Ω is the cone defined by the two matrices

$$\Theta_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T , \quad \Theta_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T .$$

As another example, consider

$$\psi(t,z) = e^{(z_1+z_2^3)}$$

Here $\psi(t,z) = g(t,q)$ where $q = z$,

$$g(t,q) = e^{(q_1+q_2^3)} \text{ and } \frac{\partial g}{\partial q}(t,q) = \begin{pmatrix} e^{(q_1+q_2^3)} & 3q_2^2 e^{(q_1+q_2^3)} \end{pmatrix}.$$

Hence Ω is the cone defined by the two matrices

$$\Theta_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} , \quad \Theta_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} .$$

D. Multiple Nonlinearities

In this subsection, we consider multiple nonlinearities that may have different characterizations for each nonlinearity, that is

$$p(t,x,u) = \begin{pmatrix} p_1(t,x,u), & \ldots, & p_\mu(t,x,u) \end{pmatrix} ,$$

where $p_k(t,x,u) = \psi_k(t,z_k)$ with $z_k = C_{q,k}x + D_{q,k}(t,u,y)$ for $k = 1,\ldots,\mu$. For each k, there is a set $\mathcal{M}_k$ of multiplier matrices such that for all $M_k \in \mathcal{M}_k$ and all $t, z_k, \tilde{z}_k$, we have

$$\begin{pmatrix} q_k(t,z_k) - q_k(t,\tilde{z}_k) \\ \psi_k(t,z_k) - \psi_k(t,\tilde{z}_k) \end{pmatrix}^T M_k \begin{pmatrix} q_k(t,z_k) - q_k(t,\tilde{z}_k) \\ \psi_k(t,z_k) - \psi_k(t,\tilde{z}_k) \end{pmatrix} \geq 0 \tag{49}$$

where

$$q_k(t,z_k) = z_k + D_{qp,k}\psi_k(t,z_k) . \tag{50}$$

The results of this section also contain the feasibility relaxations obtained for strictly positive real conditions for multivariable monotone nonlinearities presented in [13]. If we define $z = (z_1,\ldots,z_\mu)$, $\psi(t,z) = (\psi_1(t,z_1),\ldots,\psi_\mu(t,z_\mu))$ and $q = (q_1,\ldots,q_\mu)$, we can easily show that the nonlinearity $p(t,x,u) = \psi(t,z)$ where $z = C_q x + D_q(t,u,y)$

$$C_q = \text{diag}(C_{q,1},\ldots,C_{q,\mu}) , \quad D_q = (D_{q,1},\ldots,D_{q,\mu}) ,$$

satisfies (3) with $\mathcal{M}$, where for each $M \in \mathcal{M}$ we have

$$M_{ij} = \text{diag}(M_{1,ij},\ldots,M_{\mu,ij}); \; i,j = 1,2, \; \begin{pmatrix} M_{k,11} & M_{k,12} \\ M_{k,12}^T & M_{k,22} \end{pmatrix} = M_k .$$

Note that, for any set of matrices $Q_1,\ldots,Q_\mu$, $\text{diag}(Q_1,\ldots,Q_\mu)$ is a matrix of appropriate dimensions with matrices $Q_1,\ldots,Q_\mu$ on the diagonal, and with zero off-diagonal blocks.

Now, suppose that Condition 1 is satisfied by a set $\mathcal{M}_k$ of multiplier matrices by each component p_k of p with some T_k and set of pairs $(X_k,Y_k) \in \mathcal{N}_k$, $k = 1,\ldots,\mu$. Then Condition 1 is also satisfied with matrix pairs $(X,Y) \in \mathcal{N}$ and transformation T where

$$X = \text{diag}(X_1,\ldots,X_\mu) , \quad Y = \text{diag}(Y_1,\ldots,Y_\mu) ,$$

and

$$T_{ij} = \text{diag}(T_{1,ij},\ldots,T_{\mu,ij}); \; i,j = 1,2, \; \begin{pmatrix} T_{k,11} & T_{k,12} \\ T_{k,21} & T_{k,22} \end{pmatrix} = T_k .$$

VI. An Example: Underwater Vehicle

In this section we consider a simple model of an underwater vehicle with thruster dynamics. This example is taken from [19] where a similar objective of designing observers is considered in a different framework. The simplified dynamics of the vehicle is given by

$$\begin{aligned} \ddot{\phi}_1 &= -3\dot{\phi}_1|\dot{\phi}_1| + u \\ \ddot{\phi}_2 &= \dot{\phi}_1|\dot{\phi}_1| - 10\dot{\phi}_2|\dot{\phi}_2| , \end{aligned}$$

where ϕ_1 is propeller angle, ϕ_2 is vehicle position and u is the torque input to the propeller. It is assumed that we can only measure ϕ_1 and ϕ_2; the angular velocity $\dot{\phi}_1$ of the propeller and the speed $\dot{\phi}_2$ of the vehicle will be estimated using an observer. In this model, $\dot{\phi}_1|\dot{\phi}_1|$ represents the propeller thrust and $10\dot{\phi}_2|\dot{\phi}_2|$ represents the hydraulic drag on the vehicle.

Introducing the state $x = (\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$, and the output $y = (\phi_1, \phi_2)$, and letting $p = (\dot{\phi}_1|\dot{\phi}_1|, \dot{\phi}_2|\dot{\phi}_2|)$, we can write this system in state space form (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad B_p = \begin{pmatrix} 0 & 0 \\ -3 & 0 \\ 0 & 0 \\ 1 & -10 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad D_p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

$B(t,u,y) = (0,u,0,0)$ and $D = 0$. With $z = (x_2, x_4)$, the nonlinear term is described by (2) where

$$\psi(t,z) = \begin{pmatrix} z_1|z_1| \\ z_2|z_2| \end{pmatrix} , \; C_q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; D_q = 0 .$$

Note that the nonlinear function given by $f(\nu) = \nu|\nu|$ is a nondecreasing function. Considering $D_{qp} = 0$, the nonlinear term here is an incrementally positive real nonlinearity satisfying (36) with $\mathcal{X}$ being the set of matrices X of the form

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where λ_1 and λ_2 are any positive scalars.

Therefore, we can design an observer using the results in Corollary 2. This is done by using the LMI toolbox in MATLAB [14]. The observer gains obtained for $\alpha = 4$ are

$$L = \begin{pmatrix} -9.4678 & -0.0134 \\ -21.6510 & 0.3072 \\ -0.0039 & -19.0395 \\ -0.2699 & -211.0569 \end{pmatrix} , \; L_n = \begin{pmatrix} -4.4758 & 0.0189 \\ -0.3196 & -13.0741 \end{pmatrix} .$$

A two second simulation was carried out with initial state $x(0) = (0, 0, 0, 5)$, initial state estimate, $\hat{x}(0) = (0, 4, 0, -10)$, and control input

$$u(t) = \begin{cases} 5 & \text{for } 0 \leq t < 1 \\ -10 & \text{for } 1 \leq t < 2 \end{cases} .$$

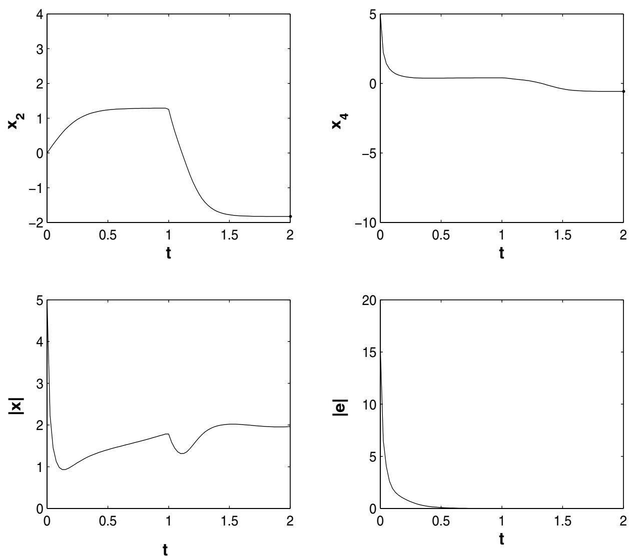

Fig. 1. Estimating the state of an underwater vehicle

In these simulations, dotted lines represent the state estimate, which converged to the vehicle state in less than 0.5 seconds.

VII. Conclusions

We considered the problem of state estimation for systems whose nonlinear time-varying terms satisfy an incremental quadratic constraint which is parameterized by a set of multiplier matrices. We also demonstrate that many common nonlinear/time-varying terms satisfy such an inequality. We present observers which guarantee that the resulting state estimation error exponentially converges to zero. Observer design involves solving linear matrix inequalities (LMIs) for the observer gain matrices. The results of this paper will be useful in obtaining observer based output feedback controllers for systems with nonlinear/time-varying terms satisfying an incremental quadratic inequality.

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