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Vibration Control via Pre-loading

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Abstract— **Control of unwanted vibrations in mechanisms is a ubiquitous and long studied problem. A class of such problems occurs in point-to-point motion common to robotics and hard-disk seek motions. A new approach to this class of problems is described which appears to achieve a faster response time than published alternatives with little or no residual vibration.**

I. INTRODUCTION

Control of vibration is an important topic that has been studied for many years and approached in many ways. The problem arises in almost all situations of control of mechanisms and includes as a sub-problem the control of vibrations in pick-and-place or rest-to-rest motion. The latter problem is especially important in robot control and control of the seek motion of the read-write head of hard disk drives. The classical approach to this problem involves inclusion of notch filters in the closed loop with zeros tuned near the frequency of the resonances corresponding to the unwanted vibration [1]. In 1957, Smith [2] introduced a method he called *Posicast* based on the idea of breaking a step command into two parts so that the vibration started by the first step is exactly cancelled by the out-of-phase vibration induced by the second step. Smith showed that Posicast could be described as the convolution of the original step with the impulse response of a filter comprised of two impulses having carefully selected amplitudes and timing such that the filter placed a zero exactly over the poles corresponding to the unwanted vibration. The response of the Posicast system was known to be quite sensitive to the frequency and damping of the vibration [3]. In his Ph.D. thesis, Singer presented a robust extension to Posicast which he called *Input Shaping*™ [4]. He achieved robustness by increasing the number of impulses and designing their amplitudes and timings so that the filter placed multiple zeros over the nominal location of the vibration poles or in nearby locations selected for a specific measure of residual vibration for an assumed range of system vibration frequencies and damping ratios.

Another theory related to rest-to-rest motion for robots and disk magnetic storage devices is that of minimum time motion in the presence of control limits. The theory, based on the Maximum Principle, is well known and a practical application used in disk drives was studied and formalized as Proximate Time Optimal Servomechanisms (PTOS) by Workman [5]. In this work, the motion was actively controlled along a trajectory in the phase plane derived from the minimum time trajectory of the rigid body portion of the dynamics alone. Vibrations were considered in an Extended PTOS (XPTOS) using LQG design to achieve good performance during the trajectory tracking. The phase plane trajectory was designed with a discount on acceleration to give adequate control authority to damp the vibration modes of the plant dynamics. An alternative to PTOS was proposed by Cooper [6]. The basic concept of his method is to avoid the discontinuity of the steps called for by the minimum time control by effecting a smooth transition in acceleration from one level to the next. Cooper proposed a polynomial spline connecting the optimal acceleration levels with coefficients selected to minimize residual vibration. He also proposed an external model-reference structure that uses incremental feedback to maintain robust control. Pao *et al.* [7] have combined the ideas of command shaping with those of PTOS by adding extra switching curves in the phase plane to require the control to include the multiple steps involved in the command shaping concept. Negative input shapers were shown to improve the rise time for lightly damped system [8]. However, such class of input shapers lead to the overcurrenting of the actuators. In this paper we present another approach for the design of near time-optimal control for faster response without any actuator saturation.

II. THE PRELOADING CONCEPT

A simple model for the systems of interest is that of two masses connected by a spring and viscous damping, as shown in Fig. 1. The equations of motion for this system are given by

$$m_1\ddot{y}_1 + b(\dot{y}_1 - \dot{y}_2) + k(y_1 - y_2) = u \quad (1)$$

$$m_2\ddot{y}_2 - b(\dot{y}_1 - \dot{y}_2) - k(y_1 - y_2) = 0 \quad (2)$$

and the transfer function is

$$\frac{Y_2(s)}{U(s)} = G(s) = \frac{bs+k}{Ms^2(\vec{m}s^2+bs+k)}, \quad (3)$$

where $M = m_1 + m_2$ is the total mass and $\vec{m} = \dfrac{m_1 m_2}{m_1+m_2}$. By suitable parameter definitions, this transfer function may be written as

$$G(s) = K\frac{(2\zeta\omega_n s + \omega_n^2)}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}. \quad (4)$$

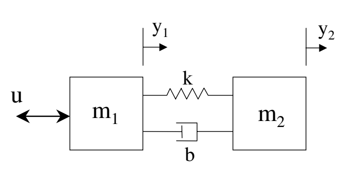

Fig. 1. Two mass example

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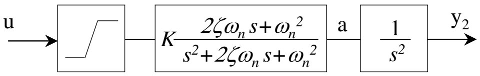

Fig. 2. System Block Diagram

A block diagram of the system with elements in series and control limiting is shown in Fig. 2. The minimum time solution to transfer the output of a simple rigid body with transfer function $1/s^2$ with maximum acceleration L from $y = 0$ to $y = y_f$ is known to be $u = L$ for T sec and $u = -L$ for another T sec, where $T = \sqrt{\frac{y_f}{L}}$. Unfortunately, in Fig. 2 the rigid body acceleration, a, and the control, u, are separated by the flexible mode. To introduce the concept of preloading, consider the situation at the initial time of a command to transfer the output y_2 from 0 to y_f. Rigid body control calls for the signal a to go instantly from 0 to L. The preload approach is to select an optimal time sequence to cause the output of the flexible system to make this transition. Then, when it is time to switch the rigid body acceleration from L to $-L$, we again introduce a rapid sequence of switches to cause the output of the flexible system to make the transition, as, similarly at t_f the acceleration is driven to zero. Because this sequence of switches is not truly the optimal control for the fourth order system, as a final step, it is necessary to modify the times, T to cause the output to reach the actual goal of y_f.

A. Generating the control input

The first step in computing the preload control is to compute the switching times required to transfer the output of a system with lightly damped modes between the levels of $0 \longrightarrow L, L \longrightarrow -L$ and $-L \longrightarrow 0$. The relevant equations are readily obtained from knowing the structure of the control as bang-bang, the requirement that the control place zeros over (or near) the poles of the flexible modes, and the boundary conditions. For a single mode of vibration, the flexible mode of the system has the transfer function given by

$$G_f(s) \quad = \quad \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{5}$$

which has poles at $s = -\alpha \pm j\beta$ where $\alpha = \zeta\omega_n$, $\beta = \omega_n\sqrt{1-\zeta^2}$, ω_n is the undamped natural frequency and ζ is the damping ratio. In input preloading, the control amplitudes are fixed at the actuator limits and only the switching times are to be selected such that the response is vibration free after the last switch. This switching time approach to the problem was introduced by Schmidt [9]. When the maximum possible acceleration and deceleration rates of the system are different, a deceleration factor $\gamma > 0$ can be specified. When $\gamma < 1$, the system has higher acceleration ability, and when $\gamma > 1$, the system has more deceleration ability. For the case of reference input going from $0 \longrightarrow L$, the proposed N step bang-bang input has the transform

$$U_1(s) = L\frac{1-(1+\gamma)e^{-t_1 s}+(1+\gamma)e^{-t_2 s}+\ldots+(1+\gamma)e^{-t_{N-1} s}}{s}. \tag{6}$$

Such an input has $N-1$ switches. The first step is given at time $t_0 = 0$ and the switching times t_i are to be selected so that the output transform has no poles at $-\alpha \pm j\beta$. The equations that assure this are

$$\begin{aligned} \sum_{i=1}^{N-1} (-1)^i\, e^{\alpha t_i} \sin(\beta t_i) &= 0 \\ 1+(1+\gamma)\sum_{i=1}^{N-1} (-1)^i\, e^{\alpha t_i} \cos(\beta t_i) &= 0. \end{aligned} \tag{7}$$

According to the Maximum Principle, for a bound of L, the minimum number of switches is two for a single flexible mode. The solution to these equations for which t_2 is minimum corresponds to the time optimal 3 step control. The problem is greatly simplified by scaling the equations by ω_n for which the poles are $-\zeta \pm j\sqrt{(1-\zeta^2}$. The times computed for these poles can be rescaled to any frequency by dividing them by ω_n. There are many routines for solving such equations. The solutions given in this paper were solved using the optimization toolbox of MATLAB®.

The derivation of (7) follows the linear system theory. A linear and uncoupled vibratory system of any order can be specified as a cascaded set of second-order poles with the step response given as [1]:

$$y(t) = A + A\frac{e^{-\alpha(t-t_0)}}{\sqrt{1-\zeta^2}}\sin\left(\beta(t-t_0) - \cos^{-1}\zeta\right), \tag{8}$$

where A is the amplitude of the step, t is the time, and t_0 is the time of the application of the step input. The overall response for all times greater than the duration of the N step input with fixed amplitude L and $-\gamma L$ is given by:

$$y(t) = L\left(1 + \sum_{i=0}^{N-1} B_i(t)\sin(\beta t - \phi_i)\right) \tag{9}$$

where

$$\begin{aligned} B_i(t) &= \kappa_i\frac{e^{-\alpha(t-t_i)}}{\sqrt{1-\zeta^2}} \qquad \phi_i = \beta t_i + \cos^{-1}\zeta \\ \kappa &= \begin{bmatrix} 1 & -(1+\gamma) & (1+\gamma) & \ldots & (1+\gamma) \end{bmatrix}. \end{aligned} \tag{10}$$

Using the trigonometric relation,

$$\sum_{i=0}^{N-1} B_i(t)\sin(\beta t - \phi_i) = A_v(t)\sin(\beta t - \psi(t)), \tag{11}$$

the amplitude of vibration is given by

$$A_v(t) = \sqrt{\left(\sum_{i=0}^{N-1} B_i(t)\cos\phi_i\right)^2 + \left(\sum_{i=0}^{N-1} B_i(t)\sin\phi_i\right)^2}. \tag{12}$$

Elimination of vibration after the end of input requires that A_v equals zero at the end of last input at t_{N-1}. This is true if each squared term in (12) is independently zero yielding,

$$\sum_{i=0}^{N-1} B_i(t_{N-1}) \cos\phi_i = \sum_{i=0}^{N-1} B_i(t_{N-1}) \sin\phi_i = 0, \qquad (13)$$

which can be simplified to (7) after expanding $B_i(t_{N-1})$ and putting in the values of κ_i and $t_0(=0)$. The solution of this equation is specified as the *Zero Vibration Preload* or ZVP, where the number of switches is equal to the order of the system.

For a plant with a single vibrating mode, the result of an unfiltered step , a Posicast input and a ZVP input to the plant are shown in Fig. 3. As is clear, by input preloading, the flexible mode has been eliminated with the shortest settling time.

III. Robustness

A. Robustness to Errors in Natural Frequency

The bang-bang input cancels vibration only if the system natural frequency and damping ratio are exact. Fig. 4 shows a plot of the vibration error as the function of system's actual natural frequency ω, for the case of a ZVP input designed for the system's estimated natural frequency, ω_n. The first-order bang-bang input is robust for a frequency vibration of less than $\approx \pm$ 3 percent, which is slightly less then the robustness level of the Posicast input [4]. Following the ideas of *Input Shaping*™, the equations for the times are not set up for the actual modes but for a larger number of virtual modes. If each actual mode is replaced by repeated modes in the same place, the solution is called the *Zero Derivative Preloading* or ZDP. This approach can be represented by adding a new constraint to the set of equations, the derivative of (7) with respect to frequency ω set equal to zero. The resulting equations become

$$\sum_{i=1}^{N-1} (-1)^i t_i e^{\alpha t_i} \sin(\beta t_i) = \sum_{i=1}^{N-1} (-1)^i t_i e^{\alpha t_i} \cos(\beta t_i) = 0. \qquad (14)$$

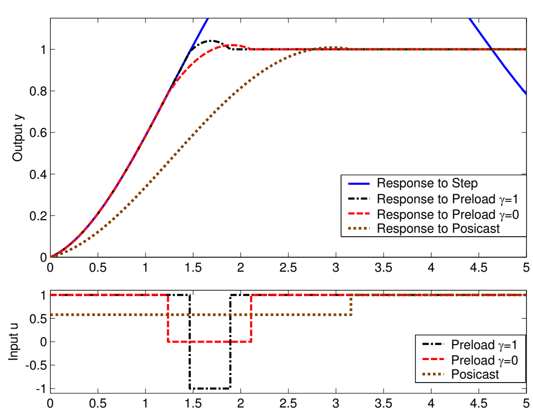

Fig. 3. (a) Response of a flexible system with $\omega_n = 1$ and $\zeta = 0.1$ to a step, preload and posicast control input. (b) Preload control input.

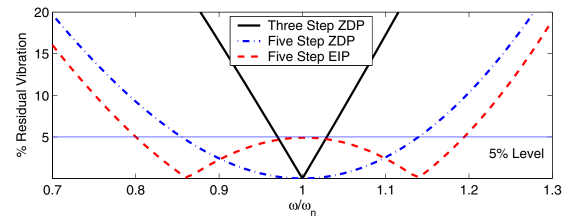

Fig. 4. Vibration error versus system natural frequency curve for a system excited by the preload input. Damping ratio is taken as 0.1

Equations (7) and (14) can be solved simultaneously to get near optimal switching times. For a second order system, this leads to a five step input and the corresponding vibration error curve depicts robustness to $\pm$ 15 percent variation in plant frequency. If the virtual modes are placed to either side of the actual modes at a distance to give a desired degree of robustness, the solution is called the *Extra Insensitive Preloading* or EIP. The input is designed for $\omega_n + \varepsilon_1$ and $\omega_n - \varepsilon_2$, such that the maximum vibration at ω_n is the desired acceptable limit of 5%. Equation (7) is solved for $\omega_n \pm \varepsilon_i$ along with the added constraint that the residual vibration, $A_v(\omega_n)$, is 5%. The solution which yields the smallest ε values give the optimal switching times. A simple observation of the Fig. 4 depicts that EIP is more robust then the ZDP.

B. Robustness to Errors in Damping

For robustness to uncertainty in damping ratio, we consider derivatives of (7) with respect to ζ to be zero which leads to the same set of equations as (14). Hence robustness to damping is already achieved by the addition of robustness to errors in frequency. Vibration error curves for the variations in damping ratio for ZVP and ZDP is shown in Fig. 5.

Fig. 6 shows the normalized solution of (7), scaled by ω_n, for different values of ζ. For any value of γ, the optimal time solution lies between the curves corresponding to $\gamma = 1$ and $\gamma = 0$, which corresponds to the system's ability to only provide positive acceleration values. A quick reference table can be made out of these solutions after choosing the desired robustness level. As is clear from the Fig. 5, ZVP scheme requires a table of at least 20 entries for the resulting vibration to be below the acceptable 5% limit. For the ZDP case, the control input is more robust for a wide range of ζ

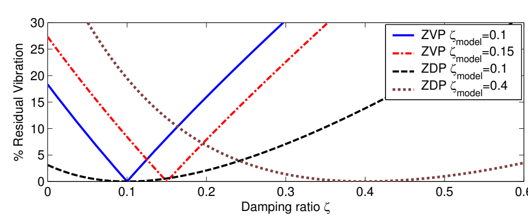

Fig. 5. Vibration error versus damping ratio for system excited by the bang-bang input

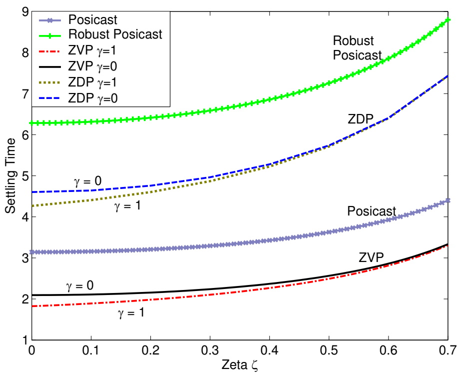

Fig. 6. Normalized settling time for various control techniques.

variations and hence requires just 3 entries. A polynomial curve fitting can also be performed on the solution set. Such an efficient tabulation scheme is useful in an adaptive control scenario where the system parameters are estimated online and the controller is changed accordingly.

C. Higher Flexibility Modes

For higher modes of flexibility, the same procedure can be repeated and the time sequence can be determined by solving (7) and (14) for each flexible mode of the system. For the ZVP case, 2n+1 steps are required for eliminating vibration of a system with n flexible modes. In the disk drive like systems, the preload control input would be designed for a maximum of three flexible modes, and a simple notch filter may be used for higher flexibilities.

IV. Different reference step input

The analysis done in the previous sections was based on acceleration going from $0 \rightarrow L$. For the acceleration being $L \rightarrow -\gamma L$ or $-\gamma L \rightarrow 0$, the same approach is applicable, which leads to the set of equations which look very similar to (7). The exact form is given below for each type of desired input step.

$\mathbf{u = L \rightarrow -\gamma L}$

The bang-bang input has the transfer function given by

$$U_2(s) = -L(1+\gamma)\,\frac{1-e^{-t_1 s}+e^{-t_2 s}+\ldots+e^{-t_{N-1} s}}{s}. \qquad (15)$$

The constraint equations can be represented as

$$\begin{aligned} \sum_{i=1}^{N-1} (-1)^i\, e^{\alpha t_i} \sin(\beta t_i) &= 0 \\ 1+\sum_{i=1}^{N-1} (-1)^i\, e^{\alpha t_i} \cos(\beta t_i) &= 0. \end{aligned} \qquad (16)$$

$\mathbf{u = -\gamma L \rightarrow 0}$

The transfer function is given by

$$U_3(s) = L\frac{(1+\gamma)-(1+\gamma)e^{-t_1 s}+(1+\gamma)e^{-t_2 s}+..+(\gamma)e^{-t_{N-1} s}}{s}. \qquad (17)$$

The set of equations which assures flexible pole cancellation is given as

$$(1+\gamma)\sum_{i=1}^{N-2}(-1)^i e^{\alpha t_i}\sin(\beta t_i)+\gamma e^{\alpha t_{N-1}}\sin(\beta t_{N-1}) = 0$$

$$(1+\gamma)\left(1+\sum_{i=1}^{N-2}(-1)^i e^{\alpha t_i}\cos(\beta t_i)\right)+\gamma e^{\alpha t_{N-1}}\cos(\beta t_{N-1}) = 0. \qquad (18)$$

V. Rest to Rest motion

A servomechanism consisting of a rigid body and the one flexible mode has the transfer function

$$G(s) \;=\; \frac{2\zeta\omega_n s+\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}. \qquad (19)$$

Optimal control theory for the minimum time motion of the rigid mode part of this model requires three switches of the control from 0 to $\pm L$, from $\pm L$ to $\mp\gamma L$, and from $\mp\gamma L$ to 0 again. For the vibration elimination, a preload control is applied at each transition switch, which can be ZVP, ZDP or EIP depending upon the desired robustness value. Let T_i be the rigid body switching times, and t_{ij} be the solution of (7), (16) and (18) for the $N-1$ switches at each transition. Here $i = 1,2,3$ denotes the rigid body transitions, and $j = 1,..,N-1$ the preload switches. Considering only the case where the final target position is positive, the transfer function of the control is given by

$$U(s) = U_1(s)+e^{-T_1 s}\,U_2(s)+e^{-T_2 s}\,U_3(s), \qquad (20)$$

where $U_1(s)$, $U_2(s)$ and $U_3(s)$ are the preload control inputs as defined in (6), (15) and (17) respectively. The system should come to rest at the end of control input and the final value of output is to be the desired y_f. Application of the boundary conditions and the final value theorem on the transform function of velocity V and output Y gives the following quadratic solution of the rigid body switching times:

$$\begin{aligned} T_1 &= \frac{-\rho+\sqrt{\rho^2-4\sigma}}{2} \\ T_2 &= -\frac{A_c}{\gamma}-\frac{(1+\gamma)\rho}{2\gamma}+\frac{(1+\gamma)\sqrt{\rho^2-4\sigma}}{2\gamma}, \end{aligned} \qquad (21)$$

where the different parameters are given as

$$\begin{aligned} \rho &= 2\sum_{j=1}^{N-1}(-1)^{j+1}\left[(1+\gamma)t_{1j}-t_{2j}\right] \\ \sigma &= \frac{A_c^2}{1+\gamma}-2\frac{A_c}{1+\gamma}\left[(1+\gamma)\sum_{j=1}^{N-2}(-1)^j t_{3j}+\gamma t_{3N-1}\right] \\ &\quad +B_c\left(\frac{\gamma}{1+\gamma}\right)-\frac{2y_f}{L}\left(\frac{\gamma}{1+\gamma}\right) \\ A_c &= (1+\gamma)\left[\sum_{j=1}^{N-1}(-1)^j(t_{1j}-t_{2j})+\sum_{j=1}^{N-2}(-1)^j t_{3j}\right]+\gamma t_{3N-1} \\ B_c &= (1+\gamma)\left[\sum_{j=1}^{N-1}(-1)^j(t_{1j}^2-t_{2j}^2)+\sum_{j=1}^{N-2}(-1)^j t_{3j}^2\right]+\gamma t_{3N-1}^2. \end{aligned} \qquad (22)$$

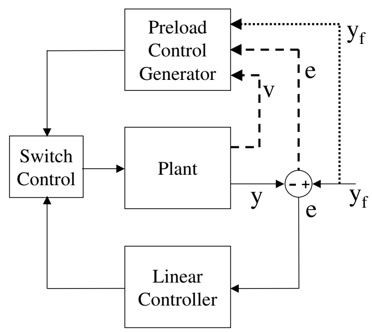

Fig. 7. Proposed Control Structure, (a) (......) Open loop seek with feedforward preloading (b) (_ _ _ _) Closed loop phase plane approach

Once the switching times are tabulated, the preloading scheme can be implemented using the control structure shown in Fig. 7. The preload controller can be either an open loop control, or the switch times can be computed from the rigid body parameter in a pseudo feedback scheme based on the two dimensional phase-plane [7]. Both the control schemes are ultimately combined with a linear controller in the same way as PTOS [5], such that when the error is small, the controller is switched to linear and the system seamlessly enters a linear zone where the response decays quickly and gracefully to zero.

The feedforward approach is the open-loop seek and closed loop tracking approach. The switching to the linear controller is done at the end of the last switch. This approach is beneficial for very "short" maneuver distances. The closed loop preloading scheme is based upon an analytical expression of the switching curves easily obtained from the preloading switching times and the system parameters. The transition between the actuator limits of L and $-\gamma L$ is done based on the location in the projected phase plane. For example, the phase plane trajectories for the ZVP control are shown in Fig. 8 with the corresponding control input in each region.

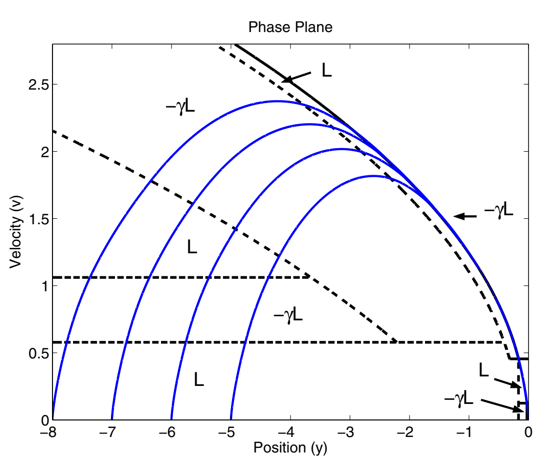

Fig. 8. Phase-plane trajectories for ZVP with $\omega_n = 1$, $\zeta = 0.1$ and $\gamma = 0.8$

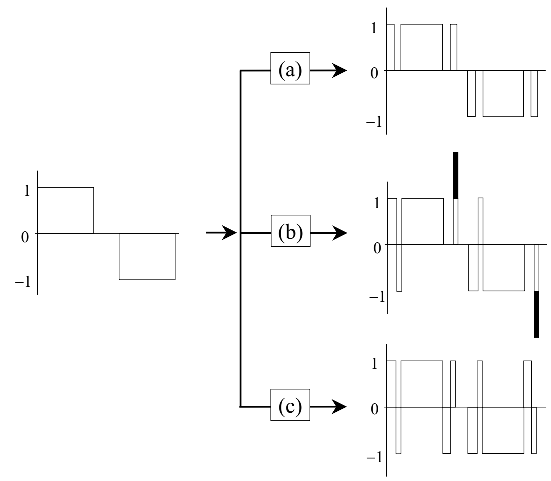

Fig. 9. Comparison of various control schemes for trapeziodal velocity profile, (a) Unity-Magnitude (UM) Input, (b) Partial-Sum (PS) Input, overcurrenting shown by shaded area (c) Preloading Control

VI. Discussion

According to the Pontryagin's maximum principle, the optimal time control for the rest to rest motion of lightly damped system is bang-bang with a finite number of switches. The boundary conditions of zero final velocity and desired final position impose the following constraints on the switching times:

$$\sum_{i=1}^{N-1} \kappa_i t_i = 0\,, \qquad \sum_{i=1}^{N-1} \kappa_i t_i^2 = \frac{2y_f}{L} \tag{23}$$

Time optimal control switches satisfy the zero vibration constraints (7), the boundary constraints (23), and the necessary conditions specified by the maximum principle. The optimal number of switches varies with the maneuver distance and the various system parameters. For short maneuver distances, 4 switches are required for the lightly and heavily damped systems, while for moderate damping, it is 6 [10]. The implementation of such a controller is very complicated and requires recalculation of these non-linear equations for slight change in the equation parameters. The switch timings are dependent on the desired move distance in a non-linear way. Thus a generalized formulation of the phase-plane switching curves for such scheme is extremely difficult and the implementation is very inefficient owing to huge online computational demand.

The Posicast scheme gives an analytical expression for the switching times [2], but incurs a time penalty. The proposed input preloading approach reduces this time penalty, while keeping the online computation minimal. Preload switching times are a near time optimal solution of the above mentioned boundary constraints, transition switches satisfy zero vibration constraints, and the rigid body times (21) satisfy (23). The scaling of the nonlinear constraint equations by ω_n makes the numerical computation much simpler, and makes it possible to do all the computation offline and tabulting the results in a quick reference memory

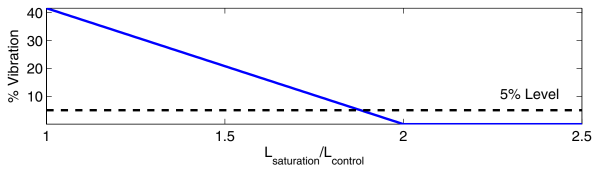

Fig. 10. Sensitivity curve for overcurrenting

table. A comparison of the various negative input shapers is shown in Fig. 9. Although the *Partial-Sum* (PS) shaper leads to faster rise time than the *Unity-Magnitude* (UM), it always causes overcurrenting for a bang-bang profile [8]. Fig. 10 illustrates the effect of saturation on the residual vibrations, which shows that vibration reduction is very sensitive to the overcurrenting. The inputs are always fixed at the actuator limits in the preloading approach, which makes the response faster than UM and PS shapers and without any actuator saturation. It is interesting to observe that UM shaper timings actually corresponds to the Preloading solution for the $0 \rightarrow L$ transition with γ set to 0, even for the allowable actuator limits of $\pm L$. Thus a comparison of system response for UM and Preloading can be made from Figs. 3 and 6. The response of the system (19) for both ZVP and ZDP control is shown in Fig. 11. The complete elimination of vibrations is quite evident from the acceleration and the output plots. Fig. 12 compares the performance of the ZVP approach with the time optimal and Posicast scheme, which shows the relative improvement in the settling time for ZVP as compared to Posicast.

For vibration reduction of the digitally controlled flexible plants, the design and implementation of a discreet version of the preload controller will be discussed in the future publications. Ongoing work will also address the robustness issues related to the usage of an adaptive scheme as compared to the virtual mode approach mentioned here.

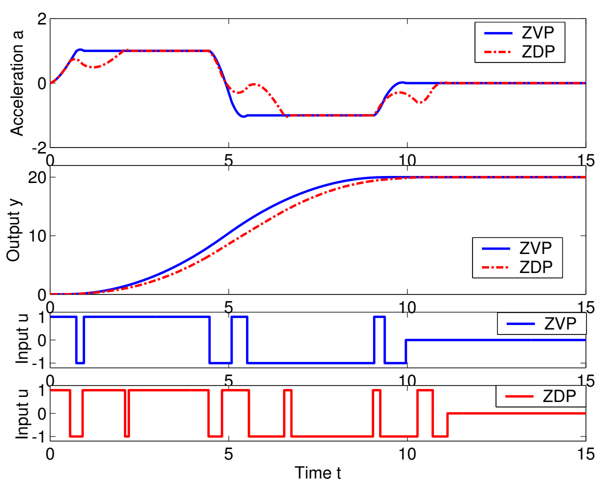

Fig. 11. Acceleration, output and control input for the rest-to-rest motion of (19) in response to ZVP and ZDP inputs. $\omega_n = 2$, $\zeta = 0.1$ and $y_f = 20$.

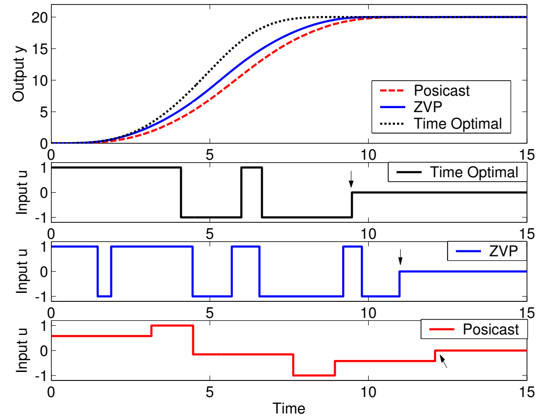

Fig. 12. Output response and control input for the rest-to-rest motion of (19) with $\omega_n = 1$, $\zeta = 0.1$ and $y_f = 20$. Arrows indicate last switch.

VII. Conclusions

In this paper a new method for achieving rest-to-rest motion of lightly damped systems in near minimum time with negligible residual vibrations is described. The method begins with the bang-bang control of the rigid body dynamics alone. Next the controls needed to *preload* the flexible modes so as to deliver the bang-bang acceleration to the rigid body are computed. Finally, the switch times for the rigid body are tweeked to deliver the system to the final position at rest. Using the ideas introduced in connection with *Input Shaping* the designs are made robust to changes in the actual mode frequencies and damping factors. Additionally, a phase-plane formulation is provided for close loop implementation of Preloading. Issues of implementation are discussed and simulation responses are plotted illustrating the performance of the proposed method.

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