Adaptive Dynamic Inversion for Nonaffine-in-Control Systems via Time-Scale Separation: Part II

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Abstract. This paper presents an adaptive dynamic inversion method for uncertain nonaffine-in-control dynamical systems. Online approximation of the adaptive dynamic inversion controller is performed using time-scale separation. The resulting control signal is sought as a solution of a "fast" dynamical equation, which inverts a series parallel model, whose state is shown to track the state of the original nonaffine-in-control system. Numerically verifiable sufficient conditions on the system parameters are given. A simulation example illustrates the theoretical results.

I. INTRODUCTION

In Part I [1], an approximate Dynamic Inversion (DI) methodology is developed for nonaffine-in-control systems using time-scale separation. The methodology invokes fast dynamics to invert the system, and hence relies on time-scale separation property between the system dynamics and the dynamics of the inverting controller. Here we extend the methodology to uncertain systems and develop a direct adaptive counterpart of the method, presented in [1], [2].

In order to illustrate the underlying design idea consider a scalar dynamical system:

$$\dot{x} = f(x,u), \qquad x(0) = x_0, \quad t \geq 0\,, \tag{1}$$

where f is an unknown function of the system state x and the control input u. Assume that $\dfrac{\partial f}{\partial u}$ is bounded away from zero for $(x,u) \in \Omega_x \times \Omega_u \subset \mathbb{R} \times \mathbb{R}$, where Ω_x, Ω_u are compact sets: $\exists\ b_0 > 0,\ \left|\frac{\partial f}{\partial u}\right| > b_0$. The control objective is to find u such that the state of the system (1) tracks a bounded reference input $\mathtt{r}(t) \in \mathcal{C}^1$ from an arbitrary initial condition $x_0 \in \Omega_x$. In [1], [2], we have proven that for *known* f such a controller can be determined via the solution of the fast dynamics:

$$\epsilon \dot{u} = -\mathrm{sign}\left(\frac{\partial f}{\partial u}\right)(f(x,u) + a(x - \mathtt{r}(t)) - \dot{\mathtt{r}}(t)), \quad a > 0, \tag{2}$$

where $\epsilon \ll 1$. We derived sufficient conditions in [1], [2] and showed that they were consistent with the assumptions of Tikhonov's theorem from singular perturbations theory [4].

Here, we extend the design to *uncertain* dynamical systems, i.e. when f is unknown in (1). Towards this end, consider a radial basis function neural network (RBF NN) approximation of the unknown nonlinearity on the compact set of initial conditions $\Omega_x \times \Omega_u$:

$$f(x,u) = W^\top \Phi(x,u) + \varepsilon(x,u), \qquad |\varepsilon(x,u)| < \varepsilon^*\,, \tag{3}$$

where $\Phi(x,u)$ is a vector of Gaussians, while W is a vector of unknown constants (ideal RBF NN weights).

Consider a one-step-ahead state predictor using a series parallel model for the dynamics in (1):

$$\dot{\hat{x}}(t) = \hat{W}^\top(t)\Phi(x(t),u(t)) - a(\hat{x}(t) - x(t)),\ \hat{x}(0) = \hat{x}_0, \tag{4}$$

where $a > 0$. Define the error signals $e(t) = \hat{x}(t) - x(t)$, $\tilde{W}(t) = \hat{W}(t) - W$. Then the prediction error dynamics can be written as:

$$\dot{e}(t) = ae(t) + \tilde{W}^\top(t)\Phi(x(t),u(t)) - \varepsilon(x(t),u(t)) \tag{5}$$

with $e(0) = e_0 (= \hat{x}_0 - x_0)$. Using standard Lyapunov arguments, one can show that the following adaptive law

$$\dot{\hat{W}}(t) = \Gamma \mathrm{Proj}\left(\hat{W}(t), -\Phi(x(t),u(t))e(t)\right)\,,\ \hat{W}(0) = W_0\,, \tag{6}$$

where Γ is a positive definite matrix of adaptation rates. In (6), $\mathrm{Proj}(\cdot,\cdot)$ is the projection operator [8]. Its purpose is to ensure ultimate boundedness of the prediction error $e(t)$ and the parameter estimation error $\tilde{W}(t)$. If $\varepsilon^* = 0$, then invoking Barbalat's lemma one can prove that $e(t) \to 0$ as $t \to \infty$ asymptotically.

At this point, we apply the control design method from [1], [2] to make the state of the one-step-ahead predictor (4) track the reference input $\mathtt{r}(t)$. We introduce fast dynamics to invert the series parallel model in (4):

$$\begin{aligned}\epsilon\dot{u}(t) &= -\mathrm{sign}\left(\hat{W}^\top(t)\frac{\partial \Phi(x,u)}{\partial u}\right)(\hat{W}^\top(t)\Phi(x(t),u(t)) \\ &- ae(t) + k(\hat{x}(t) - \mathtt{r}(t)) - \dot{\mathtt{r}}(t))\,,\end{aligned}$$

where $k > 0$ is the control design gain. As a result, the state of the one-step-ahead predictor (4) tracks the reference input asymptotically. At the same time, the the state of the predictor (4) tracks the state of the original system (1) with bounded errors. We therefore conclude that the state of the original system (1) tracks the reference input with bounded errors. If $\varepsilon^* = 0$, then this tracking performance is of order $O(\epsilon)$.

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The paper is organized as follows. In Section II, we recall Tikhonov's theorem from singular perturbation theory, which is the key result used to prove our main theorem. We give our main result on tracking a given reference signal for single input systems in Section III. A relevant simulation example is given in Section IV.

II. Preliminaries on singular perturbations

For proving our main result we will use Tikhonov's theorem on singular perturbations, which we recall below (see for instance Theorem 11.2 on page 439 of [4]).

Consider the problem of solving the system

$$\Sigma_0 : \left\{ \begin{array}{ll} \dot{x}(t) = f(t, x(t), u(t), \epsilon), & x(0) = \xi(\epsilon) \\ \epsilon\dot{u}(t) = g(t, x(t), u(t), \epsilon), & u(0) = \eta(\epsilon) \end{array} \right\}, \quad (7)$$

where $\xi : \epsilon \mapsto \xi(\epsilon)$ and $\eta : \epsilon \mapsto \eta(\epsilon)$ are smooth. Assume that f and g are continuously differentiable in their arguments for $(t, x, u, \epsilon) \in [0, \infty] \times D_x \times D_u \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_u \subset \mathbb{R}^m$ are domains, $\epsilon_0 > 0$. In addition, let Σ_0 be in *standard form*, that is,

$$0 = g(t, x, u, 0) \quad (8)$$

has $k \geq 1$ isolated real roots $u = h_i(t, x), \;\; i \in \{1, \dots, k\}$ for each $(t, x) \in [0, \infty] \times D_x$. We choose one particular i, which is fixed. We drop the subscript i henceforth. Let $v(t, x) = u - h(t, x)$. In singular perturbation theory, the system given by

$$\Sigma_{00} : \;\; \dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \;\; x(0) = \xi(0), \quad (9)$$

is called the *reduced system*, and the system given by

$$\begin{array}{rcl} \Sigma_b : \dfrac{dv}{d\tau} & = & g(t, x, v + h(t, x), 0) \\ v(0) & = & \eta_0 - h(0, \xi_0) \end{array} \quad (10)$$

is called the *boundary layer system*, where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$, $(t, x) \in [0, \infty) \times D_x$ are treated as fixed parameters. The new time scale τ is related to the original time t via the relationship $\tau = \dfrac{t}{\epsilon}$. The following result is due to Tikhonov.

Theorem 1: Consider the singular perturbation system Σ_0 given in (7) and let $u = h(t, x)$ be an isolated root of (8). Assume that the following conditions are satisfied for all $[t, x, u - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_v \times [0, \epsilon_0]$ for some domains $D_x \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}^m$, which contain their respective origins:

A1. On any compact subset of $D_x \times D_v$, the functions f, g, their first partial derivatives with respect to (x, u, ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t, x)$ and $\left[\frac{\partial g}{\partial u}(t, x, u, 0)\right]$ have bounded first derivatives with respect to their arguments, $\left[\frac{\partial f}{\partial x}(t, x, h(t, x))\right]$ is Lipschitz in x, uniformly in t, and the initial data given by ξ and η are smooth functions of ϵ.

A2. The origin is an exponentially stable equilibrium point of the reduced system Σ_{00} given by equation (9). There exists a Lyapunov function $V : [0, \infty) \times D_x \to [0, \infty)$ that satisfies

$$\begin{array}{c} W_1(x) \leq V(t, x) \leq W_2(x) \\ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, h(t, x), 0) \leq -W_3(x) \end{array}$$

for all $(t, x) \in [0, \infty) \times D_x$, where W_1, W_2, W_3 are continuous positive definite functions on D_x, and let c be a nonnegative number such that $\{x \in D_x \mid W_1(x) \leq c\}$ is a compact subset of D_x.

A3. The origin is an equilibrium point of the boundary layer system Σ_b given by equation (10) which is exponentially stable uniformly in (t, x).

Let $R_v \subset D_v$ denote the region of attraction of the autonomous system $\frac{dv}{d\tau} = g(0, \xi_0, v + h(0, \xi_0), 0)$, and let Ω_v be a compact subset of R_v. Then for each compact set $\Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c, \; 0 < \rho < 1\}$, there exists a positive constant ϵ_* such that for all $t \geq 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(0, \xi_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, Σ_0 has a unique solution x_ϵ on $[0, \infty)$ and

$$x_\epsilon(t) - x_{00}(t) = O(\epsilon)$$

holds uniformly for $t \in [0, \infty)$, where $x_{00}(t)$ denotes the solution of the reduced system Σ_{00} in (9).

The following Remark will be useful in the sequel.

Remark 1: Verification of Assumption A3 can be done via a Lyapunov argument: if there is a Lyapunov function $V(t, x, v)$ that satisfies

$$\begin{array}{l} c_1\|v\|^2 \leq V(t, x, v) \leq c_2\|v\|^2 \\ \dfrac{\partial V}{\partial v} g(t, x, v + h(t, x), 0) \leq -c_3\|v\|^2, \end{array}$$

for all $(t, x, v) \in [0, \infty) \times D_x \times D_v$, then Assumption A3 is satisfied. Alternately, Assumption A3 can be *locally* verified by linearization. Let φ denote the map $v \mapsto g(t, \xi, v + h(t, \xi), \epsilon)$. It can be shown that if there exists $\omega_0 > 0$ such that the Jacobian matrix $\left[\frac{\partial \varphi}{\partial v}\right]$ satisfies the eigenvalue condition

$$\text{Re}\left(\lambda\left[\frac{\partial \varphi}{\partial v}(t, x, h(t, x), 0)\right]\right) \leq -\omega_0 < 0,$$

for all $(t, x) \in [0, \infty) \times D_x$, then Assumption A3 is satisfied.

III. Tracking design for single input systems

Consider the following nonlinear single-input system in normal form:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bf(x(t), z(t), u(t)) \\ \dot{z}(t) & = & \zeta(x(t), z(t), u(t)) \end{array} \quad (11)$$

with $x(0) = x_0$, $z(0) = z_0$, for $(x, z, u) \in D_x \times D_z \times D_u$, where $D_x \subset \mathbb{R}^r$, $D_z \subset \mathbb{R}^{n-r}$ and $D_u \subset \mathbb{R}$ are domains containing their respective origins, while A and B correspond to the controllable canonical normal form representation of the nonlinear system dynamics, i.e.

$$A = \left[\begin{array}{ccccc} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array}\right].$$

Here $[x^\top(t)\ z^\top(t)]^\top$ denotes the state vector of the system, $x(t) = [x_1(t)\ \cdots\ x_r(t)]^\top \in \mathbb{R}^r$, $u(t)$ is the control input, r is the relative degree of the system, and $f : D_x \times D_z \times D_u \to \mathbb{R}$, $\zeta : D_x \times D_z \times D_u \to \mathbb{R}^{n-r}$ are continuously differentiable *unknown* functions of their arguments. Furthermore, assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for $(x, z, u) \in \Omega_{x,z,u} \subset D_x \times D_z \times D_u$, where $\Omega_{x,z,u}$ is a compact set of possible initial conditions; i.e. there exists $b_0 > 0$ such that $\left|\frac{\partial f}{\partial u}\right| > b_0$. Consider a RBF NN approximation of $f(x, z, u)$ over the compact set $\Omega_{x,z,u} \in D_x \times D_z \times D_u$ as:

$$f(x, z, u) = W^\top \Phi(x, z, u) + \varepsilon(x, z, u), \quad |\varepsilon(x, z, u)| < \varepsilon^*, \tag{12}$$

where W is a vector of unknown constants, while Φ is a vector of known basis functions, and $\varepsilon(x, z, u)$ is the uniformly bounded approximation error.

Let the reference model dynamics be given by:

$$\dot{x}_{\mathtt{r}}(t) = A_{\mathtt{r}} x_{\mathtt{r}}(t) + B_{\mathtt{r}} \mathtt{r}(t), \quad x_{\mathtt{r}}(0) = x_{\mathtt{r},0},$$

where $\mathtt{r}(t)$ is a continuously differentiable reference input signal, $x_{\mathtt{r}}(t) = [x_{\mathtt{r},1}(t)\ \cdots\ x_{\mathtt{r},r}(t)]^\top \in \mathbb{R}^r$ is the state of the reference model, and the Hurwitz matrix $A_{\mathtt{r}}$ and the column vector $B_{\mathtt{r}}$ have the following structure:

$$A_{\mathtt{r}} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_1 & -a_2 & \dots & -a_r \end{bmatrix}, \quad B_{\mathtt{r}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b \end{bmatrix}.$$

The control *objective* is to design a tracking control law to ensure that $x(t) \to x_{\mathtt{r}}(t)$ as $t \to \infty$, while all other error signals remain bounded.

Consider the following one-step-ahead state predictor using the series parallel model for the dynamics in (11):

$$\dot{\hat{x}}(t) = Ax(t) + B\hat{W}^\top(t)\Phi(x(t), z(t), u(t)) + A_s e_s(t), \tag{13}$$

with $\hat{x}(0) = \hat{x}_0$, where $e_s(t) = \hat{x}(t) - x(t)$ is the prediction error signal, A_s is a Hurwitz matrix of the same structure as $A_{\mathtt{r}}$ with coefficients $-a_{s,i}$, while $\hat{W}(t)$ is an adaptive parameter for estimating the unknown constant vector W. Then the prediction error dynamics for the series parallel model in (13) will be:

$$\begin{aligned} \dot{e}_s(t) &= A_s e_s(t) + B(\tilde{W}^\top(t)\Phi(x(t), z(t), u(t)) \\ &\quad - \varepsilon(x(t), z(t), u(t))), \qquad (14) \\ \dot{z}(t) &= \zeta(\hat{x}(t) + e_s(t), z(t), u(t)) \qquad (15) \end{aligned}$$

with $e_s(0) = \hat{x}_0 - x_0$, $z(0) = z_0$, $\tilde{W}(t) = \hat{W}(t) - W$.

Theorem 2: The adaptive law

$$\dot{\hat{W}}(t) = \Gamma \text{Proj}(\hat{W}(t), -\Phi(x(t), z(t), u(t))\ e_s^\top(t)PB) \tag{16}$$

with $\hat{W}(0) = W_0$, where $\text{Proj}(\cdot, \cdot)$ denotes the Projection operator [8], $P = P^\top > 0$ solves the Lyapunov equation $A_s^\top P + PA_s = -Q$ for arbitrary $Q > 0$, $\Gamma > 0$ is the adaptation gain matrix, ensures that the prediction error dynamics (14), (15) is ultimately bounded with respect to $e_s(t), \tilde{W}(t)$, uniformly in z_0.

Proof: Consider the following Lyapunov function candidate

$$V(e_s, \tilde{W}) = e_s^\top P e_s + \tilde{W}^\top \Gamma^{-1} \tilde{W}. \tag{17}$$

Its derivative along the trajectories of (14), (16) can be upper bounded

$$\dot{V} \le -\lambda_{\min}(Q)\|e_s\|^2 + 2\|e_s\|PB\varepsilon^* \le 0,$$

where the well-known property of the Projection operator $\tilde{W}^\top(\text{Proj}(\hat{W}, y) - y) \le 0$ is used [8], which is true for all vectors y, while $\lambda_{\min}(Q)$ denotes the minimum eigenvalue of Q. Hence $\dot{V} \le 0$ outside the compact set

$$\left\{\|e_s\| \le 2\frac{\varepsilon^*\|PB\|}{\lambda_{\min}(Q)}\right\} \bigcap \left\{\|W\| \le W^*\right\}, \tag{18}$$

where W^* is the maximum allowable norm upper bound selected for the Projection operator, $\|\cdot\|$ denotes the 2-norm. Following standard invariant set arguments one can conclude that the prediction error dynamics (14), (15) is ultimately bounded with respect to $e_s(t), \tilde{W}(t)$, uniformly in z_0. ■

Remark 2: If $\varepsilon^* = 0$, then the adaptive law

$$\dot{\hat{W}}(t) = \Gamma\Phi(x(t), z(t), u(t))e_s^\top(t)PB, \quad \hat{W}(0) = W_0$$

yields asymptotic prediction, i.e. $e_s(t) \to 0$ as $t \to \infty$. Indeed, in that case, $\dot{V} = -e_s^\top Q e_s \le 0$, and application of Barbalat's lemma further implies that $\lim_{t\to\infty} e_s(t) = 0$ uniformly in z_0.

At this point, we are applying the methodology from [1], [2] to the one-step-ahead state predictor, defined via the series parallel model (13). Towards that end, we need the following assumption.

Assumption 3: Using Projection Operator and a proper choice of the regressor vector Φ (other than RBFs), the adaptive process in (16) is constructed such that the control effectiveness of the estimator is bounded away from zero for all $t > 0$:

$$\left|\hat{W}^\top(t)\frac{\partial\Phi(x, z, u)}{\partial u}\right| > a_0 > 0 \tag{19}$$

Remark 3: Assumption 3 is required to ensure exponential stability of the boundary layer system in application of Tikhonov's theorem as discussed below. One way to satisfy this assumption is to redefine the regressor vector, include the control signal as its first component, $bu + W^\top\Phi(x, z, u)$, and define the estimator to be $\hat{b}(t)u(t) + \hat{W}(t)\Phi(x, z, u)$. The requirement

$$b_0 > W^*\phi^*$$

where W^* is a norm bound imposed by the Projection operator $\|\hat{W}(t)\| \le W^*$, while $\phi^* \ge \|\frac{\partial\Phi(x,z,u)}{\partial u}\|$, will ensure that the control effectiveness of this redefined estimator $\hat{b}(t)u + \hat{W}(t)\Phi(x, z, u)$ is bounded away from zero:

$$\left|\hat{b}(t) + \hat{W}^\top(t)\frac{\partial\Phi(x, z, u)}{\partial u}\right| > a_0 > 0 \tag{20}$$

where $a_0 = b_0 - W^*\phi^*$.

Let $e(t) = \hat{x}(t) - x_{\mathtt{r}}(t)$ be the tracking error signal between the series parallel model and the reference system. Then the open loop (time-varying) tracking error dynamics are given by:

$$\begin{aligned}\dot{e}(t) &= Ax(t) + B\hat{W}^\top(t)\Phi(x(t), z(t), u(t)) \\ &+ A_s e_s(t) - A_{\mathtt{r}} x_{\mathtt{r}}(t) - B_{\mathtt{r}}\mathtt{r}(t) \quad (21)\\ \dot{z}(t) &= \zeta(x_{\mathtt{r}}(t) - e_s(t) + e(t), z(t), u(t)) \quad (22)\end{aligned}$$

with $e(0) = \hat{x}_0 - x_{\mathtt{r}}(0), z(0) = z_0$. Dynamic inversion based controller is defined for the series parallel model as the solution of

$$\hat{W}^\top\Phi(x, z, u) - \sum_{i=1}^{i=r} a_{s,i}e_{s,i} = -\sum_{i=1}^{i=r} a_i\hat{x}_i + b\mathtt{r} \quad (23)$$

resulting in the asymptotically stable closed-loop tracking error dynamics $\dot{e}(t) = A_{\mathtt{r}}e(t)$. Since (23) cannot (in general) be solved explicitly for u, we construct an approximation of the dynamic inversion controller by introducing the following fast dynamics:

$$\epsilon\dot{u}(t) = -\mathrm{sign}\left(\frac{\partial \mathbf{f}}{\partial u}\right)\mathbf{f}(t, e, z, u), \quad u(0) = u_0, \quad (24)$$

where

$$\begin{aligned}\mathbf{f}(t, e, z, u) &= \hat{W}^\top(t)\Phi(e + x_{\mathtt{r}}(t) - e_s(t), z, u) \\ &- \sum_{i=1}^{i=r} a_{s,i}e_{s,i} + \sum_{i=1}^{i=r} a_i(e_i + x_{\mathtt{r},i}(t)) - b\mathtt{r}(t).\end{aligned}$$

Let $u = h(t, e, z)$ be an isolated root of $\mathbf{f}(t, e, z, u) = 0$. The reduced system for the dynamics in (21)-(22) is given by:

$$\dot{e}(t) = A_{\mathtt{r}}e(t), \quad (25)$$

$$\dot{z}(t) = \zeta(x_{\mathtt{r}}(t) + e(t) - e_s(t), z(t), h(t, e(t), z(t))) \quad (26)$$

with $e(0) = e_0, z(0) = z_0$. The boundary layer system is given by:

$$\frac{dv}{d\tau} = -\mathrm{sign}\left(\frac{\partial \mathbf{f}}{\partial u}\right)\mathbf{f}(t, e, z, v + h(t, e, z)). \quad (27)$$

Applying Theorem 1, we now get the following result for single input systems:

Theorem 4: Assume that the adaptive process is such that the following conditions are satisfied for all $[t, e, z, u - h(t, e, z), \epsilon] \in [0, \infty) \times D_{e,z} \times D_v \times [0, \epsilon_0]$ for some domains $D_{e,z} \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}$, which contain their respective origins:

B1. On any compact subset of $D_{e,z} \times D_v$, the functions $\mathbf{f}$, ζ, and their first partial derivatives with respect to (e, z, u), and the first partial derivative of $\mathbf{f}$ with respect to t are continuous and bounded, $h(t, e, z)$ and $\frac{\partial \mathbf{f}}{\partial u}(t, e, z, u)$ have bounded first derivatives with respect to their arguments, $\frac{\partial \mathbf{f}}{\partial e}$, $\frac{\partial \mathbf{f}}{\partial z}$ as functions of $(t, e, z, h(t, e, z))$ are Lipschitz in e, z, uniformly in t.

B2. The origin is an exponentially stable equilibrium point of the system

$$\dot{z}(t) = \zeta(x_{\mathtt{r}}(t) - e_s(t), z(t), h(t, 0, z(t))).$$

The map $(e, z) \mapsto \zeta(e + x_{\mathtt{r}}(t) - e_s(t), z, h(t, e, z))$ is continuously differentiable and Lipschitz in (e, z), uniformly in t.

B3. The adaptive process is such that $(t, e, z, v) \mapsto \frac{\partial \mathbf{f}}{\partial u}(t, e, z, v + h(t, e, z))$ is bounded away from zero for all $(t, e, z) \in [0, \infty) \times D_{e,z}$.

Then the origin of (27) is exponentially stable. Moreover, let Ω_v be a compact subset of R_v, where $R_v \subset D_v$ denotes the region of attraction of the autonomous system

$$\frac{dv}{d\tau} = -\mathrm{sign}\left(\frac{\partial \mathbf{f}}{\partial u}\right)\mathbf{f}(0, e_0, z_0, v + h(0, e_0, z_0)).$$

Then for each compact subset $\Omega_{z,e} \subset D_{z,e}$ there exists a positive constant ϵ_* and a $T > 0$ such that for all $t \geq 0$, $(e_0, z_0) \in \Omega_{e,z}$, $u_0 - h(0, e_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations (13), (24) has a unique solution $\hat{x}_\epsilon(t)$ on $[0, \infty)$ and

$$\hat{x}_\epsilon(t) = x_{\mathtt{r}}(t) + O(\epsilon) \quad (28)$$

holds uniformly for $t \in [T, \infty)$.

Proof: We need to verify that Assumptions A1, A2, A3 in Theorem 1 are satisfied. Assumption B1 clearly implies that A1 holds.

We now show that Assumption A2 holds. Assumption B2 implies (see Lemma 4.6, page 176 of [4]), that the system

$$\dot{z} = \zeta(x_{\mathtt{r}}(t) - e_s(t) + e, z, h(t, x_{\mathtt{r}}(t) - e_s(t) + e, z)$$

(with e viewed as the input) is input to state stable. Thus there exists class $\mathcal{K}$ and class $\mathcal{KL}$ functions γ and β, respectively, such that

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t}\|e(\tau)\|\right)$$

for all $t \geq t_0$, $t_0 \in [0, \infty)$. Furthermore from the proof of Lemma 4.6 of [4], it follows that $\gamma(\rho) = c\rho$, for some constant $c > 0$. Using the fact that the unforced system $\dot{z} = \zeta(x_{\mathtt{r}} - e_s, z, h(t, 0, z))$ has 0 as an exponentially stable equilibrium point, it can be seen from the proof of Lemma 4.6 of [4] that $\beta(\rho, t) = k\rho\exp(-\omega t)$ for some positive constants k and ω. Thus the solution to the reduced system (25)-(26) satisfies $\|e(t)\| \leq \|e_0\|c_1\exp(-\omega_0 t)$ and $\|z(t)\| \leq (\|x_0\| + \|z_0\|)c_2\exp(-\omega_0 t)$ for all $t \geq 0$ and for some $\omega_0 > 0$. Hence, the origin $(0, 0)$ is an exponentially stable equilibrium point of (25)-(26). From a converse Lyapunov theorem (Theorem 4.14 on pages 162-163 of [4]), it follows that there exists a Lyapunov function $V : [0, \infty) \times D_{e,z} \to \mathbb{R}$ such that $w_1\|(e, z)\|^2 \leq V(t, e, z) \leq w_2\|(e, z)\|^2$ and $\frac{\partial V}{\partial t}(t, e, z) + \nabla_{e,z}V \cdot \mathbf{F}(t, e, z) \leq -w_3\|(e, z)\|^2$, where

$$\mathbf{F}(t, e, z) = \begin{bmatrix} A_{\mathtt{r}}e \\ \zeta(e + x_{\mathtt{r}}(t) - e_s(t), z, h(t, e, z)) \end{bmatrix}.$$

We note that any positive c can be chosen in A2 of Theorem 1, and so a compact $\Omega_{e,z} \subset \{(e,z) \in D_{e,z} \mid W_2(e,z) \leq \rho c,\ 0 < \rho < 1\}$ can be chosen to be any subset of $D_{e,z}$.

In light of the Remark 2.1, it is easy to see that with the definition of the boundary layer system given by (27), its exponential stability can be verified locally by linearization with respect to v.

Hence Theorem 1 applies and so it follows that for each compact set $\Omega_{e,z} \subset D_{e,z}$ there exists a positive constant ϵ_* and such that for all $(e_0, z_0) \in \Omega_{e,z}$, $u_0 - h(0, e_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations given by (13), (24) has a unique solution $\hat{x}_\epsilon$, z_ϵ on $[0, \infty)$ and

$$\hat{x}_\epsilon(t) = x_{\mathtt{r}}(t) + O(\epsilon), \tag{29}$$

$$z_\epsilon(t) = z_{\mathtt{r}}(t) + O(\epsilon) \tag{30}$$

hold uniformly for $t \in [T, \infty)$, where $z_{\mathtt{r}}$ denotes the solution of

$$\begin{aligned} \dot{e}(t) &= A_{\mathtt{r}} e(t), \\ \dot{z}(t) &= \zeta(x_{\mathtt{r}}(t) - e_s(t) + e(t), z(t), h(t, e(t), z(t))) \end{aligned}$$

with $e(0) = e_0, z(0) = z_0$, and $T \geq 0$ is such that $\|\exp(TA_{\mathtt{r}})\hat{x}_0 - \exp(TA_{\mathtt{r}})x_{\mathtt{r},0}\| \leq \epsilon$. ■

Corollary 5: From Theorems 2 and 4, it follows that $x(t)$ tracks $x_{\mathtt{r}}(t)$ with bounded errors. Moreover, if $\varepsilon^* = 0$, then $x(t) - x_{\mathtt{r}}(t) = O(\epsilon)$.

Proof. Indeed, application of Tikhonov's Theorem (Theorem 1) implies that:

$$\hat{x}_\epsilon(t) = x_{\mathtt{r}}(t) + O(\epsilon).$$

Recalling that $x_{\mathtt{r}}(t) = \hat{x}(t) - e(t) = x(t) + e_s(t) - e(t)$, one gets:

$$\hat{x}_\epsilon(t) = x(t) + e_s(t) + O(\epsilon)$$

Comparing to (29), implies that $x(t) - x_{\mathtt{r}}(t) = e_s(t) + O(\epsilon)$. Finally, using Theorem 2 one concludes that $x(t)$ tracks $x_{\mathtt{r}}(t)$ with bounded errors. On the other hand, if $\varepsilon^* = 0$, then it follows from Remark 2 that $x(t) - x_{\mathtt{r}}(t) = O(\epsilon)$.

IV. Simulations

Consider the tracking problem for the scalar nonlinear system given by

$$\dot{x}(t) = 0.5x(t) + \tanh(u(t)+3) + \tanh(u(t)-3) + 0.01u(t) \tag{31}$$

with $x(0) = -0.5$. It is easy to see that the system dynamics is invertible, but not in terms of elementary functions. This system is motivated by aircraft applications, in which control effectiveness $\dfrac{\partial f}{\partial u}(\cdot, u)$ is small for both small and large control inputs u, as it is shown in Figure 1. The series parallel model is designed with the use of 25 RBFs, distributed over the grid $x \in [-2, 2]$, $u \in [-2, 2]$ with the step size equal to 1 in both dimensions:

$$\Phi_i(x, u) = \exp(-3((x - x_i)^2 + (u - u_i)^2)/2)\,,$$

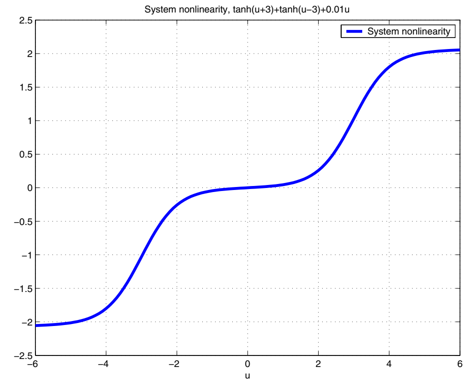

Fig. 1. System nonlinearity: $\tanh(u + 3) + \tanh(u - 3) + 0.01u$

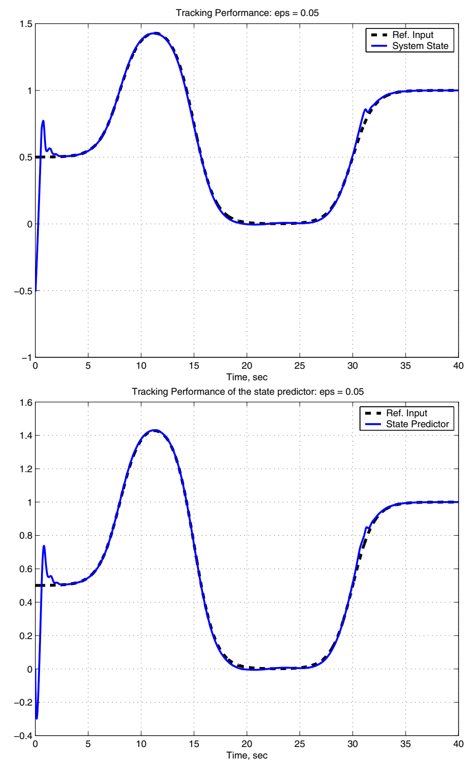

Fig. 2. Tracking performance

where the point (x_i, u_i) represents the center of the i^{th} RBF. The eigenvalue of the one step ahead predictor is set to be $a = 10$:

$$\dot{\hat{x}}(t) = \hat{W}^\top(t)\Phi(x(t), u(t)) - 10(\hat{x}(t) - x(t))\,, \hat{x}(0) = 0\,. \tag{32}$$

The norm upper bound for the projection operator is set to $W^* = 10$, adaptation gain is set to $\Gamma = 100$. Simulation is performed using the following reference input:

$$\begin{aligned} \mathtt{r}(t) &= \frac{1}{1 + \exp(t - 8)} - \frac{1.5}{1 + \exp(t - 15)} \\ &+ \frac{1}{1 + \exp(t - 30)} + 0.5\,. \end{aligned}$$

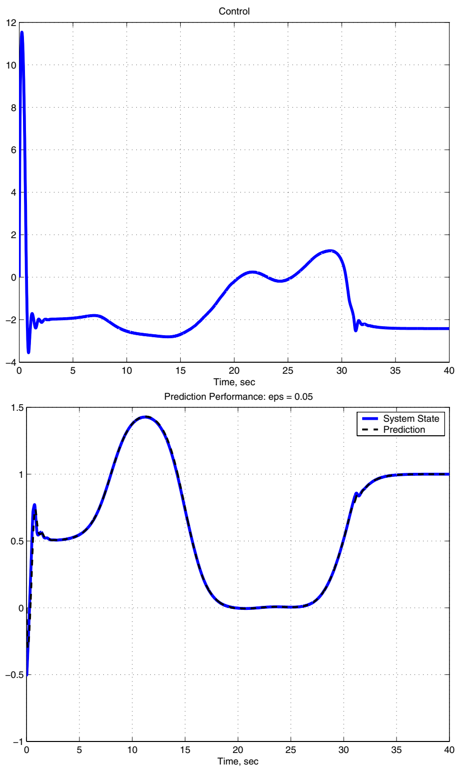

Fig. 3. Control history and prediction performance

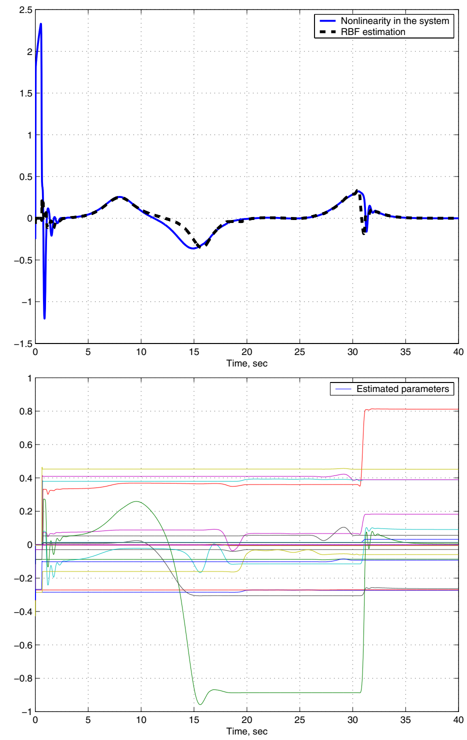

Fig. 4. Adaptive estimation of system nonlinearity

The fast dynamics is designed as:

$$\begin{aligned} 0.05\dot{u} &= -(\hat{W}^\top(t)\Phi(x(t),u(t)) \\ &- 10(\hat{x}(t)-x(t)) + 2(\hat{x}(t)-\mathtt{r}(t)) - \dot{\mathtt{r}}(t)). \end{aligned}$$

Figure 2 shows the tracking performance of the reference input $\mathtt{r}(t)$ by both the estimated state $\hat{x}(t)$ and the actual system state $x(t)$. At the same time Figure 3 demonstrates the convergence ability of the one-step-ahead predictor and the required control effort. Finally, Figure 4 demonstrates the function approximation performance of the RBF NN via a set of bounded weights.

Remark 4: We note that the exponential stability of the boundary layer system, required by Tikhonov's Theorem, is not satisfied in this simulation example. This implies that such a requirement is overly conservative. Current research is directed towards relaxing or replacing that assumption with less restrictive requirement.

V. Conclusions

In this paper, we have presented a new design technique for adaptive dynamic inversion of nonaffine-in-control uncertain systems. First, a one-step-ahead adaptive state predictor was developed for the uncertain system using a series parallel model. Using parameter adaptation, the state of this model was shown to track the original system with bounded errors. Next, approximate dynamic inversion for the series parallel model was performed via appropriately defined fast dynamics. The latter was shown to satisfy assumptions of Tikhonov's theorem from singular perturbation theory, implying that the state of the series parallel model tracks the reference input. Finally, the two results together achieved the desired tracking objective. Open problem in this context is the choice of a regressor vector used in forming the adaptive process such that the corresponding boundary layer system becomes exponentially stable. One method for achieving this was discussed in Remark 3. Research in alternative directions is currently underway.

References

[1] N. Hovakimyan, E. Lavretsky and A. Sasane. Dynamic Inversion for Nonaffine-in-Control Systems via Time-Scale Separation: Part I. *American Control Conference*, 2005.

[2] N. Hovakimyan, E. Lavretsky and A. Sasane. Approximate Dynamic Inversion for Nonaffine-in-Control Systems via Time-Scale Separation. *Submitted to Systems and Control Letters*, 2004.

[3] R.A. Brualdi and S. Mellendorf. Regions in the complex plane containing the eigenvalues of a matrix. *The American Mathematical Monthly*, 101:975-985, 1994.

[4] H.K. Khalil. *Nonlinear Systems.* Prentice Hall, 2002.

[5] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer, 1990.

[6] A.Isidori. *Nonlinear Control Systems.* Springer, 1995.

[7] M.Krstic, I.Kanellakopoulos and P.Kokotovic. *Nonlinear and Adaptive Control Design.* John Wiley, New York, 1995.

[8] J.B. Pomet and L. Praly. Adaptive Nonlinear Regulation: Estimation from the Lyapunov Equation. *IEEE Trans. Autom. Contr.*, 37(6):729-740, 1992.