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Dynamic Inversion for Nonaffine-in-Control Systems via Time-Scale Separation: Part I

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Abstract. This paper presents a new method for approximate dynamic inversion for nonaffine-in-control systems via time-scale separation. The control signal is sought as a solution of "fast" dynamics and is shown to asymptotically stabilize the original nonaffine system. Sufficient conditions are formulated, which are consistent with the assumptions of Tikhonov's theorem in singular perturbations theory. Several examples illustrate the theoretical results.

I. INTRODUCTION

Dynamic inversion, or equivalently feedback linearization, is one of the most popular control design methodologies for nonlinear systems that are *affine* in the control variables [1]–[4]. However, many practical applications give rise to *nonaffine* nonlinear systems, for which an explicit inversion is not possible. For example, the system $\dot{x} = u+\exp(u)$ is nonlinear, $u+\exp(u)$ is a monotonous function of u, yet an explicit inversion in terms of elementary functions is not possible: that is, given v, find u such that $u + \exp(u) = v$ is not possible. In this paper, we propose a control design methodology for a class of nonaffine nonlinear systems whose dynamic inversion solution exists but cannot be found explicitly. In other words, a transcendental equation arises when one attempts to invert the system dynamics.

In order to motivate our approach, consider a scalar nonlinear nonaffine in control system:

$$\dot{x} = f(x,u), \quad x(0) = x_0, \quad t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}$ is the system state at time t, $u(t) \in \mathbb{R}$ is the control input at time t, and f is a Lipschitz function of its arguments. Assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for $(x,u) \in \Omega_x \times \Omega_u \subset \mathbb{R} \times \mathbb{R}$, where Ω_x, Ω_u are compact sets that contain their respective origins; that is, there exists $b_0 > 0$, such that $\left|\frac{\partial f}{\partial u}\right| > b_0$. The control objective is to find a feedback law that will stabilize (1) from an arbitrary initial condition $x_0 \in \Omega_x$. Since $\text{sign}\left(\frac{\partial f}{\partial u}\right)$ is constant, it follows that $f(x,u)$ is monotonous in u, and, consequently, invertible with respect to its second argument. Dynamic inversion controller design leads to the (in general) transcendental equation $f(x,u) = -ax$, where given x, we seek a solution for u and $a > 0$. Note that it is assumed that the ideal dynamic inversion solution exists, but is not available explicitly. We attempt to find an approximate dynamic inversion controller using the so-called "fast" dynamics:

$$\epsilon\dot{u} = -\text{sign}\left(\frac{\partial f}{\partial u}\right)(f(x,u) + ax), \quad \epsilon \ll 1. \tag{2}$$

In fact, choosing the positive constant ϵ small enough, the dynamics in (2) become faster than the "slow" dynamics of the original system in (1). We propound that subject to a set of mild assumptions, the system in (1) can be stabilized via the solution of (2).

To briefly illustrate the heuristics behind our design approach, consider the problem of stabilization of the scalar nonlinear system given by

$$\dot{x} = \exp(x) + u + u^2\tanh(u). \tag{3}$$

A stabilizing dynamic inversion controller can be obtained by solving the following equation for u:

$$\exp(x) + u + u^2\tanh(u) = -x. \tag{4}$$

It can be checked that for $u \in \mathbb{R}$, $\frac{\partial f}{\partial u}$ has a constant sign. Hence the system is controllable. Notice however that the equation (4) cannot be solved explicitly for u, and hence the ideal dynamic inversion solution for u cannot be found. So we approximate the dynamic inversion solution via time-scale separation. Consider the following fast dynamics:

$$\epsilon\dot{u} = -\text{sign}\left(\frac{\partial f}{\partial u}\right)(\exp(x)+u+u^2\tanh(u)+x), \quad \epsilon \ll 1. \tag{5}$$

When $\epsilon = 0$, the relationship in (5) reduces to the algebraic relationship in (4), the solution of which renders the system (3) exponentially stable: $\dot{x} = -x$. It can be shown that for a suitably chosen small ϵ, the solution of differential equation (5), achieves asymptotic stabilization of the system (3), as shown in Figure 1.

The paper is organized as follows. In Section II, we recall Tikhonov's theorem from singular perturbation theory, which is the key result used in proving our main theorem. We give our main result on tracking a given reference signal for single input systems in Section III. A simulation example on tracking is given in Section IV. Finally, in Section V we give an extension to systems with multiple inputs.

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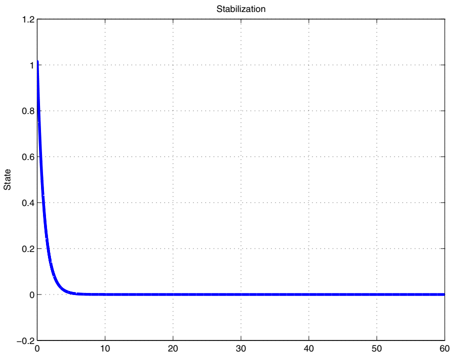

Fig. 1. Stabilization

II. PRELIMINARIES ON SINGULAR PERTURBATIONS

For proving our main result we will use Tikhonov's theorem on singular perturbations, which we recall below (see for instance Theorem 11.2 on page 439 of [1]).

Consider the problem of solving the system

$$\Sigma_0 : \left\{ \begin{array}{ll} \dot{x}(t) = f(t, x(t), u(t), \epsilon), & x(0) = \xi(\epsilon) \\ \epsilon\dot{u}(t) = g(t, x(t), u(t), \epsilon), & u(0) = \eta(\epsilon) \end{array} \right\}, \quad (6)$$

where $\xi : \epsilon \mapsto \xi(\epsilon)$ and $\eta : \epsilon \mapsto \eta(\epsilon)$ are smooth. Assume that f and g are continuously differentiable in their arguments for $(t, x, u, \epsilon) \in [0, \infty] \times D_x \times D_u \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_u \subset \mathbb{R}^m$ are domains, $\epsilon_0 > 0$. In addition, let Σ_0 be in *standard form*, that is,

$$0 = g(t, x, u, 0) \quad (7)$$

has $k \geq 1$ isolated real roots $u = h_i(t, x)$, $i \in \{1, \ldots, k\}$ for each $(t, x) \in [0, \infty] \times D_x$. We choose one particular i, which is fixed. We drop the subscript i henceforth. Let $v(t, x) = u - h(t, x)$. In singular perturbation theory, the system given by

$$\Sigma_{00} : \quad \dot{x}(t) = f(t, x(t), h(t, x(t)), 0), \quad x(0) = \xi(0), \quad (8)$$

is called the *reduced system*, and the system given by

$$\begin{array}{rcl} \Sigma_b : \dfrac{dv}{d\tau} & = & g(t, x, v+h(t, x), 0), \\ v(0) & = & \eta_0 - h(0, \xi_0) \end{array} \quad (9)$$

is called the *boundary layer system*, where $\eta_0 = \eta(0)$ and $\xi_0 = \xi(0)$, $(t, x) \in [0, \infty) \times D_x$ are treated as fixed parameters. The new time scale τ is related to the original time t via the relationship $\tau = \dfrac{t}{\epsilon}$. The following result is due to Tikhonov.

Theorem 2.1: Consider the singular perturbation system Σ_0 given in (6) and let $u = h(t, x)$ be an isolated root of (7). Assume that the following conditions are satisfied for all $[t, x, u - h(t, x), \epsilon] \in [0, \infty) \times D_x \times D_v \times [0, \epsilon_0]$ for some domains $D_x \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}^m$, which contain their respective origins:

A1. On any compact subset of $D_x \times D_v$, the functions f, g, their first partial derivatives with respect to (x, u, ϵ), and the first partial derivative of g with respect to t are continuous and bounded, $h(t, x)$ and $\left[\frac{\partial g}{\partial u}(t, x, u, 0)\right]$ have bounded first derivatives with respect to their arguments, $\left[\frac{\partial f}{\partial x}(t, x, h(t, x))\right]$ is Lipschitz in x, uniformly in t, and the initial data given by ξ and η are smooth functions of ϵ.

A2. The origin is an exponentially stable equilibrium point of the reduced system Σ_{00} given by equation (8). There exists a Lyapunov function $V : [0, \infty) \times D_x \to [0, \infty)$ that satisfies

$$\begin{array}{c} W_1(x) \leq V(t, x) \leq W_2(x) \\ \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x, h(t, x), 0) \leq -W_3(x) \end{array}$$

for all $(t, x) \in [0, \infty) \times D_x$, where W_1, W_2, W_3 are continuous positive definite functions on D_x, and let c be a nonnegative number such that $\{x \in D_x \mid W_1(x) \leq c\}$ is a compact subset of D_x.

A3. The origin is an equilibrium point of the boundary layer system Σ_b given by equation (9), which is exponentially stable uniformly in (t, x).

Let $R_v \subset D_v$ denote the region of attraction of the autonomous system $\frac{dv}{d\tau} = g(0, \xi_0, v + h(0, \xi_0), 0)$, and let Ω_v be a compact subset of R_v. Then for each compact set $\Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c,\ 0 < \rho < 1\}$, there exists a positive constant ϵ_* such that for all $t \geq 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(0, \xi_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, Σ_0 has a unique solution x_ϵ on $[0, \infty)$ and

$$x_\epsilon(t) - x_{00}(t) = O(\epsilon)$$

holds uniformly for $t \in [0, \infty)$, where $x_{00}(t)$ denotes the solution of the reduced system Σ_{00} in (8).

The following Remark will be useful in the sequel.

Remark 1: Verification of Assumption A3 can be done via a Lyapunov argument: if there is a Lyapunov function $V(t, x, v)$ that satisfies

$$\begin{array}{l} c_1\|v\|^2 \leq V(t, x, v) \leq c_2\|v\|^2 \\ \dfrac{\partial V}{\partial v} g(t, x, v + h(t, x), 0) \leq -c_3\|v\|^2, \end{array}$$

for all $(t, x, v) \in [0, \infty) \times D_x \times D_v$, then Assumption A3 is satisfied. Alternately, Assumption A3 can be *locally* verified by linearization. Let φ denote the map $v \mapsto g(t, \xi, v + h(t, \xi), \epsilon)$. It can be shown that if there exists $\omega_0 > 0$ such that the Jacobian matrix $\left[\frac{\partial \varphi}{\partial v}\right]$ satisfies the eigenvalue condition

$$\text{Re}\left(\lambda\left[\frac{\partial \varphi}{\partial v}(t, x, h(t, x), 0)\right]\right) \leq -\omega_0 < 0,$$

for all $(t, x) \in [0, \infty) \times D_x$, then Assumption A3 is satisfied.

III. TRACKING DESIGN FOR SINGLE INPUT SYSTEMS

Consider the following nonlinear single-input system in normal form:

$$\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & x_3(t) \\ & \vdots & \\ \dot{x}_{r-1}(t) & = & x_r(t) \\ \dot{x}_r(t) & = & f(x(t), z(t), u(t)) \\ \dot{z}(t) & = & \zeta(x(t), z(t), u(t)), \end{array} \tag{10}$$

with $x(0) = x_0$, $z(0) = z_0$, for $(x, z, u) \in D_x \times D_z \times D_u$, where $D_x \subset \mathbb{R}^r$, $D_z \subset \mathbb{R}^{n-r}$ and $D_u \subset \mathbb{R}$ are domains containing their respective origins. Here $[x^\top(t)\ \ z^\top(t)]^\top$ denotes the state vector of the system, $x(t) = [x_1(t)\ \cdots\ x_r(t)]^\top \in \mathbb{R}^r$, $u(t)$ is the control input, r is the relative degree of the system, and $f : D_x \times D_z \times D_u \to \mathbb{R}$, $\zeta : D_x \times D_z \times D_u \to \mathbb{R}^{n-r}$ are continuously differentiable functions of their arguments. Furthermore, assume that $\frac{\partial f}{\partial u}$ is bounded away from zero for $(x, z, u) \in \Omega_{x,z,u} \subset D_x \times D_z \times D_u$, where $\Omega_{x,z,u}$ is a compact set of possible initial conditions; that is, there exists $b_0 > 0$, such that $\left|\frac{\partial f}{\partial u}\right| > b_0$. In addition, assume that the function f cannot be inverted explicitly with respect to u.

Let the reference model dynamics be given by:

$$\dot{x}_{\mathtt{r}}(t) = A_{\mathtt{r}} x_{\mathtt{r}}(t) + B_{\mathtt{r}}\mathtt{r}(t)\,, \quad x_{\mathtt{r}}(0) = x_{\mathtt{r},0}\,,$$

where $\mathtt{r}(t)$ is a continuously differentiable reference input signal, $x_{\mathtt{r}}(t) = [x_{\mathtt{r},1}(t)\ \cdots\ x_{\mathtt{r},r}(t)]^\top \in \mathbb{R}^r$ is the state of the reference model, and the Hurwitz matrix $A_{\mathtt{r}}$ and the column vector $B_{\mathtt{r}}$ have the following structure:

$$A_{\mathtt{r}} = \left[\begin{array}{cccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_1 & -a_2 & \dots & -a_r \end{array}\right], \quad B_{\mathtt{r}} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ b \end{array}\right].$$

Let $e(t) = x(t) - x_{\mathtt{r}}(t)$ be the tracking error signal. Then the open loop (time-varying) error dynamics are given by:

$$\dot{e}(t) = F(e(t) + x_{\mathtt{r}}(t), z(t), u(t)) - A_{\mathtt{r}} x_{\mathtt{r}}(t) - B_{\mathtt{r}}\mathtt{r}(t) \tag{11}$$

$$\dot{z}(t) = \zeta(e(t) + x_{\mathtt{r}}(t), z(t), u(t)) \tag{12}$$

where $F(x, z, u) = [x_2\ \cdots\ x_r\ f(x, z, u)]^\top$. Ideal dynamic inversion based control is found by solving the equation

$$f(x, z, u) = -a_r x_r - \cdots - a_2 x_2 - a_1 x_1 + b\mathtt{r} \tag{13}$$

resulting in the exponentially stable closed-loop tracking error dynamics $\dot{e}(t) = A_{\mathtt{r}} e(t)$. Since (13) cannot (in general) be solved explicitly for u, we construct an approximation of the dynamic inversion controller by introducing the following fast dynamics:

$$\epsilon \dot{u}(t) = -\text{sign}\left(\frac{\partial f}{\partial u}\right) \mathbf{f}(t, e, z, u), \quad u(0) = u_0, \tag{14}$$

where

$$\mathbf{f}(t, e, z, u) = f(e + x_{\mathtt{r}}(t), z, u) + a_r(e_r + x_{\mathtt{r},r}(t)) + \cdots + a_1(e_1 + x_{\mathtt{r},1}(t)) - b\mathtt{r}(t).$$

Let $u = h(t, e, z)$ be an isolated root of $\mathbf{f}(t, e, z, u) = 0$. The reduced system for the dynamics in (11)-(12) is given by:

$$\dot{e}(t) = A_{\mathtt{r}} e(t) \tag{15}$$

$$\dot{z}(t) = \zeta(x_{\mathtt{r}}(t) + e(t), z(t), h(t, e(t), z(t)) \tag{16}$$

with $e(0) = e_0, z(0) = z_0$. The boundary layer system is given by:

$$\frac{dv}{d\tau} = -\text{sign}\left(\frac{\partial f}{\partial u}\right) \mathbf{f}(t, e, z, v + h(t, e, z)). \tag{17}$$

Applying Theorem 2.1, we now get our main result for single input systems:

Theorem 3.1: Assume that the following conditions are satisfied for all $[t, e, z, u - h(t, e, z), \epsilon] \in [0, \infty) \times D_{e,z} \times D_v \times [0, \epsilon_0]$ for some domains $D_{e,z} \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}$, which contain their respective origins:

B1. On any compact subset of $D_{e,z} \times D_v$, the functions f, ζ, and their first partial derivatives with respect to (e, z, u), and the first partial derivative of f with respect to t are continuous and bounded, $h(t, e, z)$ and $\frac{\partial f}{\partial u}(t, e, z, u)$ have bounded first derivatives with respect to their arguments, $\frac{\partial f}{\partial e}$, $\frac{\partial f}{\partial z}$ as functions of $(t, e, z, h(t, e, z))$ are Lipschitz in e, z, uniformly in t.

B2. The origin is an exponentially stable equilibrium point of the system

$$\dot{z}(t) = \zeta(x_{\mathtt{r}}(t), z(t), h(t, 0, z(t)).$$

The map $(e, z) \mapsto \zeta(e + x_{\mathtt{r}}(t), z, h(t, e, z))$ is continuously differentiable and Lipschitz in (e, z), uniformly in t.

B3. $(t, e, z, v) \mapsto \frac{\partial f}{\partial u}(t, e, z, v + h(t, e, z))$ is bounded below by some positive number for all $(t, e, z) \in [0, \infty) \times D_{e,z}$.

Then the origin of (17) is exponentially stable. Moreover, let Ω_v be a compact subset of R_v, where $R_v \subset D_v$ denotes the region of attraction of the autonomous system

$$\frac{dv}{d\tau} = -\text{sign}\left(\frac{\partial f}{\partial u}\right) \mathbf{f}(0, e_0, z_0, v + h(0, e_0, z_0)).$$

Then for each compact subset $\Omega_{z,e} \subset D_{z,e}$ there exists a positive constant ϵ_* and a $T > 0$ such that for all $t \geq 0$, $(e_0, z_0) \in \Omega_{e,z}$, $u_0 - h(0, e_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations (10), (14) has a unique solution $x_\epsilon(t)$ on $[0, \infty)$ and

$$x_\epsilon(t) = x_{\mathtt{r}}(t) + O(\epsilon) \tag{18}$$

holds uniformly for $t \in [T, \infty)$.

Proof. We need to verify that Assumptions A1, A2, A3 in Theorem 2.1 are satisfied. Assumption B1 clearly implies that A1 holds.

We now show that Assumption A2 holds. Assumption B2 implies (see Lemma 4.6, page 176 of [1]), that the system

$$\dot{z}(t) = \zeta(x_{\mathtt{r}}(t) + e(t), z(t), h(t, x_{\mathtt{r}}(t) + e(t), z(t))$$

(with e viewed as the input) is input to state stable. Thus there exists class $\mathcal{K}$ and class $\mathcal{KL}$ functions γ and β, respectively, such that

$$\|z(t)\| \leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|e(\tau)\|\right)$$

for all $t \geq t_0$, $t_0 \in [0, \infty)$. Furthermore from the proof of Lemma 4.6 of [1], it follows that $\gamma(\rho) = c\rho$, for some constant $c > 0$. Using the fact that the unforced system $\dot{z} = \zeta(x_{\mathtt{r}}, z, h(t, 0, z))$ has 0 as an exponentially stable equilibrium point, it can be seen from the proof of Lemma 4.6 of [1] that $\beta(\rho, t) = k\rho\exp(-\omega t)$ for some positive constants k and ω. Thus the solution to the reduced system (15)-(16) satisfies $\|e(t)\| \leq \|e_0\| c_1 \exp(-\omega_0 t)$ and $\|z(t)\| \leq (\|x_0\| + \|z_0\|)c_2 \exp(-\omega_0 t)$ for all $t \geq 0$ and for some $\omega_0 > 0$. Hence, the origin $(0, 0)$ is an exponentially stable equilibrium point of (15)-(16). From a converse Lyapunov theorem (Theorem 4.14 on pages 162-163 of [1]), it follows that there exists a Lyapunov function $V : [0, \infty) \times D_{e,z} \to \mathbb{R}$ such that $w_1\|(e, z)\|^2 \leq V(t, e, z) \leq w_2\|(e, z)\|^2$ and $\frac{\partial V}{\partial t}(t, e, z) + \nabla_{e,z} V \cdot \mathbf{F}(t, e, z) \leq -w_3\|(e, z)\|^2$, where

$$\mathbf{F}(t, e, z) = \left[\begin{array}{c} A_{\mathtt{r}} e \\ \zeta(e + x_{\mathtt{r}}, z, h(t, e, z)) \end{array}\right].$$

We note that any positive c can be chosen in A2 of Theorem 2.1, and so a compact $\Omega_{e,z} \subset \{(e, z) \in D_{e,z} \mid W_2(e, z) \leq \rho c,\ 0 < \rho < 1\}$ can be chosen to be any subset of $D_{e,z}$.

In light of the Remark 2.1, it is easy to see that with the definition of the boundary layer system given by (17), its exponential stability can be verified locally by linearization with respect to v.

Hence Theorem 2.1 applies and so it follows that for each compact set $\Omega_{e,z} \subset D_{e,z}$ there exists a positive constant ϵ_* and such that for all $(e_0, z_0) \in \Omega_{e,z}$, $u_0 - h(0, e_0, z_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations given by (10), (14) has a unique solution x_ϵ, z_ϵ on $[0, \infty)$ and

$$\begin{aligned} x_\epsilon(t) &= x_{\mathtt{r}}(t) + O(\epsilon), \\ z_\epsilon(t) &= z_{\mathtt{r}}(t) + O(\epsilon) \end{aligned}$$

hold uniformly for $t \in [T, \infty)$, where $z_{\mathtt{r}}$ denotes the solution of

$$\begin{aligned} \dot{e}(t) &= A_{\mathtt{r}} e(t), \quad e(0) = e_0, \\ \dot{z}(t) &= \zeta(x_{\mathtt{r}}(t) + e(t), z(t), h(t, e(t), z(t))), \quad z(0) = z_0, \end{aligned}$$

and $T \geq 0$ is such that $\|\exp(TA_{\mathtt{r}})x_0 - \exp(TA_{\mathtt{r}})x_{\mathtt{r},0}\| \leq \epsilon$.

Remark 2: The reference system in Theorem 3.1 is *linear*. However an application of Theorem 2.1 to the scalar system

$$\dot{x}(t) = f(x(t), u(t))$$

$$\epsilon\dot{u}(t) = -\mathrm{sign}\left(\frac{\partial f}{\partial u}\right)(f(x(t), u(t)) - \mathtt{g}(x(t), \mathtt{r}(t)))$$

yields a similar result for tracking the state of the scalar *nonlinear* reference system $\dot{x}_{\mathtt{r}}(t) = \mathtt{g}(x_{\mathtt{r}}(t), \mathtt{r}(t))$.

IV. SIMULATIONS

Consider the nonlinear system given by:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t)\exp(x_2(t)) + u(t) + (u(t))^2\tanh(u(t)) \\ \dot{z}(t) &= -z(t) - ((x_1(t))^2 + (x_2(t))^2)(z(t))^3. \end{aligned}$$

The control objective is to design u such that $x = \left[\begin{array}{cc} x_1 & x_2 \end{array}\right]^\top$ tracks the state of the linear system

$$\dot{x}_{\mathtt{r}}(t) = \left[\begin{array}{cc} 0 & 1 \\ -4 & -4 \end{array}\right] x_{\mathtt{r}}(t) + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \sum_{k=0}^{\infty} \mathbf{1}_{[2k, 2k+1]}\left(\frac{2}{25}t\right)$$

with $x_{\mathtt{r}}(0) = [0.5 \quad -0.5]^\top$, where $\mathbf{1}_{[a,b]}$ denotes the indicator function of the interval $[a, b]$:

$$\mathbf{1}_{[a,b]}(t) = \left\{\begin{array}{ll} 1 & \text{if } t \in [a, b], \\ 0 & \text{if } t \notin [a, b]. \end{array}\right.$$

It can be checked that the Assumptions B1 and B2 of Theorem 3.1 are satisfied with the domains $D_e = \mathbb{R}^2$ and $D_v = \mathbb{R}$ (which contain their respective origins), $\epsilon = 1$, h as the map

$$\begin{aligned} (t, e) \mapsto \psi^{-1}\Big(&-(e_1 + x_{\mathtt{r},1}(t))\exp(e_2 + x_{\mathtt{r},2}(t)) \\ &-4x_1(t) - 4x_2(t) + \sum_{k=0}^{\infty} \mathbf{1}_{[2k,2k+1]}\left(\frac{2}{25}t\right)\Big), \end{aligned}$$

where ψ denotes the diffeomorphism $u \mapsto u + u^2\tanh(u)$ from $\mathbb{R}$ onto $\mathbb{R}$, and $\mathbf{f}(t, e, u) = (e_1 + x_{\mathtt{r},1}(t))\exp(e_2 + x_{\mathtt{r},2}(t)) + u + u^2\tanh(u) + 4x_1(t) + 4x_2(t) - \sum_{k=0}^{\infty} \mathbf{1}_{[2k,2k+1]}\left(\frac{2}{25}t\right)$. Figure 2 shows the tracking performance of the components of the state vector versus the states of the reference model. Figure 3 shows the stabilization of the internal state and the input history. The values of the parameters used in simulations are: $\epsilon = 0.04$, $x_0 = [1\ 1]^\top$, $u_0 = 0$.

V. EXTENSION TO SYSTEMS WITH MULTIPLE INPUTS

Consider the following nonlinear system in nonaffine normal form:

$$\left[\begin{array}{c} \dot{x}_{k,1}(t) \\ \vdots \\ \dot{x}_{k,r_k-1}(t) \\ \dot{x}_{k,r_k}(t) \end{array}\right] = \left[\begin{array}{cccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & 0 & \dots & 0 \end{array}\right] \left[\begin{array}{c} x_{k,1}(t) \\ \vdots \\ x_{k,r_k-1}(t) \\ x_{k,r_k}(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ f_k(x(t), z(t), u(t)) \end{array}\right],\ k \in \{1, \dots, m\},$$

$$\dot{z}(t) = \zeta(x(t), z(t), u(t)), \tag{19}$$

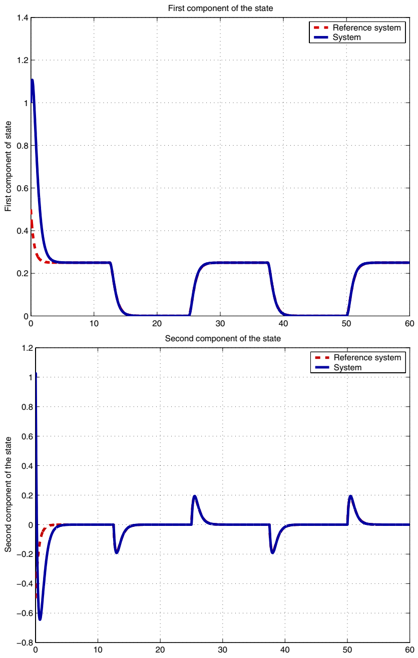

Fig. 2. Tracking

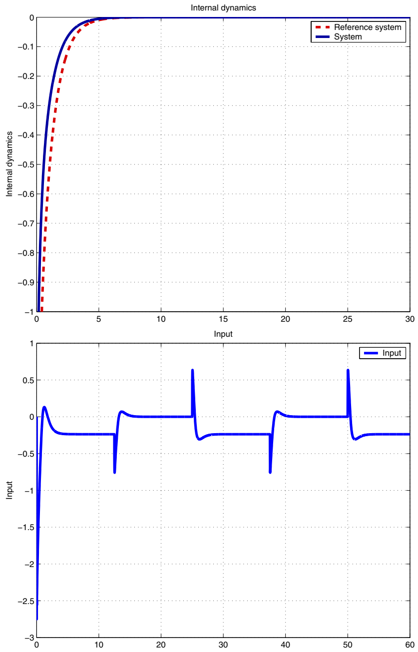

Fig. 3. Internal state and the input

with $x(0) = x_0$, $z(0) = z_0$, where $x = \begin{bmatrix} x_{1,1} & \dots & x_{1,r_1} & \dots & x_{m,1} & \dots & x_{m,r_m} \end{bmatrix}^\top$, $u = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix}^\top$ and $[x^\top \ z^\top]^\top \in \mathbb{R}^n$. The control objective is to design u such that the state x tracks the state $x_\mathtt{r}$ of the reference system

$$\dot{x}_\mathtt{r}(t) = A_\mathtt{r} x_\mathtt{r}(t) + B_\mathtt{r} \mathtt{r}(t), \quad t \geq 0, \quad x_\mathtt{r}(0) = x_{\mathtt{r},0} \tag{20}$$

Here $x_\mathtt{r} = [x^\mathtt{r}_{1,1} \ \dots \ x^\mathtt{r}_{1,r_1} \ \dots \ x^\mathtt{r}_{m,1} \ \dots \ x^\mathtt{r}_{m,r_m}]^\top$ is the state of the reference model, $\mathtt{r} = [\mathtt{r}_1 \ \cdots \ \mathtt{r}_m]$ is a vector of continuously differentiable reference input signals. The pair $(A_\mathtt{r}, B_\mathtt{r})$ is assumed to be in block-diagonal Brunovsky canonical form, that is,

$$A_\mathtt{r} = \begin{bmatrix} A_{\mathtt{r},1} & & \\ & \ddots & \\ & & A_{\mathtt{r},m} \end{bmatrix}, B_\mathtt{r} = \begin{bmatrix} B_{\mathtt{r},1} & & \\ & \ddots & \\ & & B_{\mathtt{r},m} \end{bmatrix}$$

where $A_{\mathtt{r},k}, B_{\mathtt{r},k}$ are of the form:

$$A_{\mathtt{r},k} = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a^\mathtt{r}_{k,1} & -a^\mathtt{r}_{k,2} & \dots & -a^\mathtt{r}_{k,r_k} \end{bmatrix},$$

$$B_{\mathtt{r},k} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_k \end{bmatrix}, \ j \in \{1, \dots, \mathtt{r}_k\}, \ k \in \{1, \dots, m\}.$$

We also assume that $a^\mathtt{r}_{k,j} > 0$ for all $j \in \{1, \dots, \mathtt{r}_k\}$, $k \in \{1, \dots, m\}$. Let $e(t) = x(t) - x_\mathtt{r}(t)$ be the tracking error. The open-loop time-varying error dynamics are given by:

$$\dot{e}(t) = F(e(t) + x_\mathtt{r}(t), z(t), u(t)) - A_\mathtt{r} x_\mathtt{r}(t) - B_\mathtt{r} \mathtt{r}(t) \tag{21}$$

$$\dot{z}(t) = \zeta(x(t), z(t), u(t)), \quad k \in \{1, \dots, m\} \tag{22}$$

where $F(x, z, u) = [x_{1,2} \ \dots \ x_{1,r_1} \ \ f_1(x, z, u) \ \dots \ x_{m,2} \ \dots \ x_{m,r_m} \ \ f_m(x, z, u)]^\top$. For dynamic inversion based control, we seek a m-dimensional solution u of the following system of m equations

$$\begin{bmatrix} f_1(x, z, u) \\ \vdots \\ f_m(x, z, u) \end{bmatrix} = \begin{bmatrix} -a^\mathtt{r}_{1,1} x_{1,1} - \dots - a^\mathtt{r}_{1,r_1} x_{1,r_1} + b_1 \mathtt{r}_1 \\ \vdots \\ -a^\mathtt{r}_{m,1} x_{m,1} - \dots - a^\mathtt{r}_{m,r_m} x_{m,r_m} + b_m \mathtt{r}_m \end{bmatrix} \tag{23}$$

resulting in asymptotically stable closed loop tracking error dynamics $\dot{e}(t) = A_\mathtt{r} e(t)$. Since the exact solution of (23) cannot be found explicitly, we consider its approximation via the fast dynamics:

$$\epsilon \dot{u}(t) = P \mathbf{f}(t, e(t), z(t), u(t)), \quad u(0) = u_0 \,, \tag{24}$$

where $P \in \mathbb{R}^{m \times m}$ and

$$\mathbf{f}(t, e, z, u) = \begin{bmatrix} f_1(e + x_\mathtt{r}(t), z, u) + a^\mathtt{r}_{1,1}(x^\mathtt{r}_{1,1}(t) + e_{1,1}) \\ + \dots + a^\mathtt{r}_{1,r_1}(x^\mathtt{r}_{1,r_1}(t) + e_{1,r_1}) - b_1 \mathtt{r}_1(t) \\ \vdots \\ f_m(e + x_\mathtt{r}(t), z, u) + a^\mathtt{r}_{m,1}(x^\mathtt{r}_{m,1}(t) + e_{m,1}) \\ + \dots + a^\mathtt{r}_{m,r_m}(x^\mathtt{r}_{m,r_m}(t) + e_{m,r_m}) - b_m \mathtt{r}_m(t) \end{bmatrix}.$$

Let $u = h(t, e, z)$ be an isolated root of $\mathbf{f}(t, e, z, u) = 0$. The reduced system for (21)-(22) is given by:

$$\begin{array}{rcl}\dot{e}(t) & = & A_{\mathtt{r}}e(t), \quad e(0) = e_0 \\ \dot{z}(t) & = & \zeta(e(t) + x_{\mathtt{r}}(t), z(t), u(t)), \quad z(0) = z_0.\end{array}$$

The boundary layer system is given by:

$$\frac{dv}{d\tau} = P\mathbf{f}(t, e, z, v + h(t, e, z)). \tag{25}$$

A straightforward extension of Theorem 3.1 yields the following result:

Theorem 5.1: Let the following conditions be satisfied for all $[t, e, u - h(t, z, e), \epsilon] \in [0, \infty) \times D_{z,e} \times D_v \times [0, \epsilon_0]$ for some domains $D_{e,z} \subset \mathbb{R}^n$ and $D_v \subset \mathbb{R}^m$, which contain their respective origins:

C1. On any compact subset of $D_{e,z} \times D_v$, the function $\mathbf{f}$, ζ, and their first partial derivatives with respect to (e, z, u), and the first partial derivative of $\mathbf{f}$ with respect to t are continuous and bounded, $h(t, e, z)$ and $\frac{\partial f}{\partial u}(t, e, z, u)$ have bounded first derivatives with respect to their arguments, $\frac{\partial f}{\partial z}(t, e, z, h(t, e, z)$, $\frac{\partial f}{\partial e}(t, e, z, h(t, e, z)$ is Lipschitz in e, z, uniformly in t.

C2. The origin is an exponentially stable equilibrium point of the system $\dot{z}(t) = \zeta(x_{\mathtt{r}}(t), z(t), h(t, 0, z(t)))$. The map $(z, e) \mapsto \zeta(e + x_{\mathtt{r}}(t), z, h(t, z, e))$ is continuously differentiable and Lipschitz in (z, e), uniformly in t.

C3. $(t, e, z, v) \mapsto \text{dist}(\text{co spec}(P[\frac{\partial \mathbf{f}}{\partial u}(t, e, z, v + h(t, e, z))]), i\mathbb{R})$ is bounded below by a positive number for all $(t, e, z) \in [0, \infty) \times D_{e,z}$, where $\text{co spec}(M)$ denotes the convex hull of the eigenvalues of the square matrix M and $\text{dist}(\cdot, i\mathbb{R})$ denotes the distance from the imaginary axis.

Then the origin of (25) is exponentially stable. Moreover, let Ω_v be a compact subset of R_v, where $R_v \subset D_v$ denotes the region of attraction of the autonomous system $\frac{dv}{d\tau} = P\mathbf{f}(0, z_0, e_0, v + h(0, z_0, e_0))$. Then for each compact $\Omega_{z,e} \subset D_{z,e}$ there exists a positive constant ϵ_* and a $T > 0$ such that for all $t \geq 0$, $(z_0, e_0) \in \Omega_{z,e}$, $u_0 - h(0, z_0, e_0) \in \Omega_v$ and $0 < \epsilon < \epsilon_*$, the system of equations (19),(24) has a unique solution $x_\epsilon(t)$ on $[0, \infty)$ and

$$x_\epsilon(t) = x_{\mathtt{r}}(t) + O(\epsilon)$$

holds uniformly for $t \in [T, \infty)$.

The proof is a straightforward extension of the proof of Theorem 3.1, and is therefore omitted. We make several remarks regarding verification of Assumption C3.

Remark 3:

1) a) If $\frac{\partial f_i}{\partial u_j} = 0$ for all i and j such that $i \neq j$ (and so $\left[\frac{\partial \mathbf{f}}{\partial u}\right]$ is diagonal) and $(t, z, e, v) \mapsto \text{sign}\left(\frac{\partial f_k}{\partial u_k}(t, z, e, v + h(t, z, e))\right)$ is bounded away from zero, then Assumption C3 is satisfied with

$$P = \begin{bmatrix} -\text{sign}\left(\frac{\partial f_1}{\partial u_1}\right) & & \\ & \ddots & \\ & & -\text{sign}\left(\frac{\partial f_m}{\partial u_m}\right) \end{bmatrix}.$$

   b) If $(t, z, e, v) \mapsto \text{dist}\Big(\text{co spec}\Big(\Big[\frac{\partial \mathbf{f}}{\partial u}(t, z, e, v + h(t, z, e))\Big]\Big), i\mathbb{R}\Big)$ is bounded below by a positive number for all $(t, z, e) \in [0, \infty) \times D_{z,e}$, then Assumption C3 is satisfied with $P = \text{sign}\left(\text{tr}\left(\left[\frac{\partial \mathbf{f}}{\partial u}(t, z, e, u)\right]\right)\right)$.

2) We notice that since Theorem 2.1 is true for time-varying systems, the approximate dynamic inversion control methodology can also be applied to solve stabilization and tracking problems for time-varying systems as well.

Remark 4: We notice that Tikhonov's theorem allows for the equation $\mathbf{f}(t, e, z, u) = 0$ to have multiple isolated roots, and *not just single isolated root*. In that case, one needs the knowledge of the $\text{sign}\left(\frac{\partial f}{\partial u}\right)$ in the neighborhood of every isolated root to construct the boundary layer system with exponentially stable origin. Then the tracking problem can be solved via a set of controllers, provided that the initialization of the fast dynamics is such that the corresponding boundary layer system has an exponentially stable equilibrium.

VI. CONCLUSIONS

In this paper, we have considered systems that are non-affine in control and for which ideal dynamic inversion is well-defined, but the solution is not explicitly available. For these systems, we have developed an approximate dynamic inversion control method using time-scale separation. We have given sufficient conditions for tracking in single-input systems and extended it to systems with multiple inputs. In Part II we extend these results to uncertain systems via adaptive dynamic inversion [5].

VII. ACKNOWLEDGMENT

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