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Efficient Computation of Fair Communication Equilibria in Generalized Coordination Games

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Abstract— **We consider the problem of computing a fair communication equilibrium in a repeated game while making efficient use of communicative and computational resources. At each stage of a game, several controllers (game players) observe a random process and evaluate the utilities that they associate with all possible joint actions given their observations. These utilities are reported to a central communication device, which computes and recommends a fair joint action, subject to game-theoretic equilibrium constraints. It is shown how, under a class of generalized coordination games, such a communication device fulfill its role, while trading off performance and efficiency.**

I. INTRODUCTION

Decentralized control problems can often be formulated in a game-theoretic setting, with each controller behaving strategically in light of how others are believed to behave. Under the assumption that each controller is autonomous, solution concepts such as Nash, correlated, and communication equilibrium can be used to characterize how the controllers may act to optimize their own local performance. In this paper, we consider the problem of implementing a communication equilibrium for a decentralized control system, while paying particular attention to fairness and communicative and computational overhead.

Consider a system of L controllers, observing some random process $\{X_t, t = 0, 1, 2, \ldots\}$. At time t, each controller l observes X_t and executes an action $u^l \in U^l$. Each controller has a private valuation $h^l(u, X_t)$ over joint controls $u = (u^1, \ldots, u^L)$ ($u \in U^{(L)}$). Controllers are considered rational, strategic entities (players), and hence the situation is a repeated game with random payoffs. We assume that players use a communication device (see [1], [2]) to impose a degree of fairness and coordination in the game. Such a device is defined as in the following definition.

Definition 1.1: A (canonical) optimal fair communication equilibrium consists of a two-way transaction between players and a communication device as follows. At each stage t, players observe X_t and communicate their valuations $h^l(u, X_t)$ to the communication device. The communication device calculates a probability distribution π over joint actions according to the following linear program:

$$\begin{aligned} &\text{maximize} && z \\ &\text{subject to} && (\forall u, y \in U^l, l = 1, \ldots, L), \\ & && \sum_{u \in U^{(L)}} \pi(u) h^l(u, X_t) \geq z \\ & && \sum_{u^{-l} \in U^{-l}} \pi(u, u^{-l}) h^l((u, u^{-l}), X_t) \\ & && \geq \sum_{u^{-l} \in U^{-l}} \pi(u, u^{-l}) h^l((y, u^{-l}), X_t) \\ & && \sum_{u \in U^{(L)}} \pi(u) = 1, \ \pi \geq 0. \end{aligned} \tag{1}$$

(As is usual in game theory, $u^{-l} \in U^{-l}$ represents the actions of all players but l.) A joint action is randomly selected according to π, and each player is sent a private recommendation to play their part.

Any distribution π over joint controls satisfying the last two constraints in (1) is a correlated equilibrium (see [3], [4]). A rational player who knows π and observes a recommendation drawn from it will always find it most profitable to follow the recommendation. Optimal fairness follows from the objective and first constraint, and maximizes the minimum expected reward over all players (see [5]).

Optimal fairness appears to be the most natural notion of fairness to consider in a game theoretic setting; A minimum level of safety can be imposed while still respecting the decision makers' autonomy. This is in contrast with max-min and proportional fairness ([6] and [7]), for which players are expected to be considerate of others, a concept inappropriate for games. Indeed, these latter types of fairness are a generalization and a restriction, respectively, of Pareto optimality, which is generally at odds with game-theoretic equilibrium.

We assume the communication device is designed for maximal flexibility, so that players must communicate their valuations by transmitting all values of $h^l(u, X_t)$ at each stage. Program (1) can quickly become complex, with approximately $|U^{(L)}|$ variables and $L + \sum_{l=1}^{L} |U^l|$ constraints. Moreover, players must submit a total of $L \times |U^{(L)}|$ values to the communication device at each play of the game! Since both the computational complexity and communication requirements grow exponentially with L, we search for more tractable solutions by restricting our attention to an appropriate class of games, called generalized coordination games.

II. GENERALIZED COORDINATION GAMES

Generalized coordination games are amenable to a significant reduction in communicative and computational overhead in the scenario above. Essentially, they are strategic games in which players can restrict their attention to a

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subset of actions, provided that this subset is common among players. The following definitions make this clear.

Definition 2.1: Given an L-player strategic game with action sets U^l, a complete restricted subgame is defined by joint action set $V^{(L)} \subset U^{(L)}$ such that, for any player l, any pure action $a^l \in (V^l)^c$ is strictly dominated, given that all other players restrict themselves to V^{-l}.

Definition 2.2: A generalized coordination game is a game that can be partitioned into complete restricted subgames with disjoint action sets.

The following lemma allows us to simplify equilibrium computations significantly by considering only one subgame at a time.

Lemma 2.1: An equilibrium (Nash or correlated) for a complete restricted subgame is an equilibrium for the original game.

Proof: This follows directly from the definitions above. In such an equilibrium, no player will deviate unilaterally, hence no player needs to plan for deviations by others, hence the equilibrium holds. ■

Note that the set of complete restricted subgames is not unique, in the sense that complete restricted subgames can often be further subdivided. In general, more subgames lead to greater efficiency at the cost of reduced performance, as will be made clear below.

III. COMPUTING COMMUNICATION EQUILIBRIA IN GENERALIZED COORDINATION GAMES

In this section, we present an algorithm for implementing an optimal fair communication device under the generalized coordination game structure, and discuss the associated reduction in communicative and computational overhead.

Algorithm 3.1: Fair Mixture of Optimal Fair Correlated Equilibria.

1) Players observe X_t, identify complete restricted subgames, and transmit their payoffs for each one.
2) The communication device computes an optimal fair correlated equilibrium for each subgame as in (1).
3) The communication device computes a probability distribution over subgames that maximizes the minimum expected reward over all players, and generates private recommendations for each player.

Some rule for identifying subgames must be implemented for consistency. For example, each player might restrict themselves to the subdivision with the smallest cardinality.

Lemma 3.1: Any mixture of equilibria of the complete restricted subgames is a correlated equilibrium for the original game.

Proof: The proof follows directly from Lemma 2.1, and the fact that the set of correlated equilibria is convex, [4]. ■

To quantify the savings associated with a decomposition into subgames, consider the case where the original game is decomposed into x subgames. Define $C(x) = \sum |V^{(L)}|$ over all subgames. Assuming, for simplicity, that $|U^l| = N$ for all l, we have

$$(N-x)^L + x^L \leq C(x) \leq N^L x^{1-L}, \tag{2}$$

where the actual value depends on the sizes of the subgames. We define communicative savings to be the reduction in the number of values transmitted to the correlation device, and computational savings to be the reduction in computational complexity at the device. These savings are related to $\Delta(x) = C(1) - C(x)$, given by

$$\sum_{k=1}^{L-1} \binom{L}{k} (N-x)^{L-k} x^k \leq \Delta(x) \leq N^L - N^L x^{1-L}. \tag{3}$$

To be precise, the communicative savings is $L\Delta(x)$, while the computational savings is at least $\Delta(x)^{3.5}$ (under the best known linear programming algorithms). Note also that the computational savings is further enhanced since computations for separate subgames may be computed in parallel.

There is a degradation in performance associated with these savings. This degradation is difficult to quantify, but experimental results show that for a 2×4 generalized coordination games with random payoffs, Algorithm (3.1) gives an equilibrium that is, on average, only 1.2% worse than the equilibrium given by (1). Compared with a 50% cost reduction in communication, and a 98% savings in computational complexity, this degradation is worthwhile.

The degradation can be larger, for example, the game of chicken exhibits a 14% degradation due to this method. In a 2×2 symmetric game, it can be shown that the degradation can approach 50%, thus exactly cancelling the communications savings. However, this represents an extreme case, in which the game is highly polarized, and a small concession by one player will reduce his utility by ε, while increasing his opponent's utility to his maximum payoff minus ε.

IV. CONCLUSION

We have characterized a fair communication equilibrium for a decentralized control problem cast as a repeated game. We give an algorithm that implements such an equilibrium with significant communicative and computational savings, under certain structural properties.

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