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Optimal Filtering for Multirate Systems Based on Lifted Models

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Abstract— This paper discusses filter design problems for multirate systems in H_2 and H_∞ settings by using the LMI machinery based on lifted models. The causality constraint on lifted filters is tackled, and the solution is provided in terms of a set of LMI conditions.

Keywords: H_2/H_∞ filtering, multirate systems, lifting technique, causality constraint, linear matrix inequality (LMI).

I. Introduction

Due to practical limitations, it is unrealistic, or sometimes impossible, to sample all variables in a complex system with a single rate. A system where several sampling rates co-exist is called a multirate system. Instead of treating general multirate systems, this paper considers the dual-rate case where the process input w is sampled with period mh, and the process output y is sampled with period nh. Without loss of generality, m and n are assumed to be coprime integers; h is called the base period, and mnh the frame period. This setup captures most of the essential features of multirate systems while maintains some clarity in exposition.

H_2 and H_∞ filter designs for the above mentioned multirate systems will be discussed in this paper. For simplicity, we only consider problems for single-input single-output systems. The extension to general multi-input multi-output multirate systems can be made following a similar line of research. The design objective of this work is to obtain the filtered information at the base period h, given the multirate input-output data. In this design, the linear matrix inequality (LMI) technique [1], as well as the lifting technique [3] will be employed. The former has been shown a numerically efficient tool to solve many convex optimization problems; and the latter is capable of converting a time-varying multirate system into an equivalent time-invariant single-rate system. However, using the lifting technique presents a challenge in the design of lifted controllers and lifted filters: they should satisfy the so-called causality constraint [2]. Causality constraint means that the feedthrough terms on lifted controllers/filters should satisfy a block lower triangular structure. In practical implementation, lifted controllers/filters should first be inverse lifted. To ensure that control signals or filtered signals depend only on the measurements available up to current time instant, the mapping between lifted signals should satisfy certain structure conditions. Causality constraint on lifted controllers has been discussed before in, e.g., [2], but this design condition on lifted filters has seldom been investigated based on LMI techniques. This work aims to show how to incorporate the causality constraint on lifted filters into H_2 and H_∞ filter designs by applying the LMI machinery.

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We note that similar work on multirate filtering problems has been presented in [7]. However, causality constraint was automatically satisfied in [7] because of the sampling scheme adopted there (a special case of dual-rate systems with $m = 1$). The multirate filter in [7] took a standard observer form $\hat{x}[k+1] = A\hat{x}[k] + K(y[k] - C\hat{x}[k])$ and the design was related to a series of static gains. In this work, however, the multirate filter will take a dynamic form.

The rest of the paper is organized as follows: in Section II, the problem is formulated by applying the lifting technique; in Section III, solutions are provided by a set of LMI conditions; in Section IV, a numerical example is presented to show the effectiveness of the proposed design approach; and finally, in Section V, concluding remarks are given.

II. Problem formulation

For the dual-rate system described in Section I, it is time-varying in general. The main purpose of this work is to design a time-varying dynamic filter Σ_{df}, which, based on the multirate input w and output y, will provide an accurate estimation $\hat{x}$ of true intersample states x every base period h, say, $\hat{x}[mnk+i]$ ($k = 0, 1\cdots,\infty, i = 1,\cdots, mn$), in H_2 and H_∞ formulations. The design problem in this work can be described as follows:

Problem: Given $\gamma > 0$, or $\beta > 0$, find a time-varying dynamic filter Σ_{df}, so that the H_2 norm, or the H_∞ norm of the transfer function from w to $e = x - \hat{x}$ is asymptotically stable and satisfies $\|G_{ew}\|_2^2 < \gamma$ (γ-suboptimal H_2 filtering), or $\|G_{ew}\|_\infty < \beta$ (β-suboptimal H_∞ filtering).

Similar to that in [7], we will convert this time-varying design problem into an equivalent time-invariant one, by applying the lifting technique. Assume the discrete-time model with underlying period h is known (otherwise it can be extracted from the multirate input-output data, see, e.g., [4]): the state x is with dimension n_a, and the model state space realization is $\Sigma_h : [A_h, B_h, C_h, D_h]$. The corresponding lifted model for the dual-rate system can then be obtained, say, $\Sigma_l : [A_l, B_l, C_l, D_l]$. Σ_l is with underlying period mnh; its state $x_l[k] = x[kT]$, its input $\underline{w}$, and output $\underline{y}$ are related with that of model Σ_h in the following way:

$$\begin{aligned} \underline{w}[k] &= \left[\begin{array}{cccc} w[kT] & w[kT+m] & \cdots & w[kT+T-m] \end{array}\right]', \\ \underline{y}[k] &= \left[\begin{array}{cccc} y[kT] & y[kT+n] & \cdots & y[kT+T-n] \end{array}\right]', \end{aligned}$$

where we have defined $T = mn$. For derivations of A_l, B_l, C_l, D_l, readers can refer to [4] for more details.

Based on the model Σ_h with base period h, the real intersample states during the kth frame period: $x[mnk+1]$, $x[mnk+2]$, $\cdots$, $x[mnk+mn]$ can be calculated and lifted. This lifted signal is written as:

$$X[k] = \Phi x_l[k] + \Gamma \underline{w}[k], \tag{1}$$

where $X[k] = \left[\begin{array}{cccc} x[kT+1]' & x[kT+2]' & \cdots & x[kT+T]' \end{array}\right]'$,

$$\begin{aligned} \Phi &= \begin{bmatrix} \Psi \\ \Psi \cdot A_h^m \\ \vdots \\ \Psi \cdot A_h^{(n-1)m} \end{bmatrix}, \ \Psi = \begin{bmatrix} A_h \\ A_h^2 \\ \vdots \\ A_h^m \end{bmatrix}, \\ \Gamma &= \begin{bmatrix} \Omega & & & & \\ \Psi \cdot \theta & \Omega & & & \\ \Psi \cdot A_h^m \cdot \theta & \Psi \cdot \theta & \Omega & & \\ \vdots & & \ddots & \ddots & \\ \Psi \cdot A_h^{(n-2)m} \cdot \theta & \Psi \cdot A_h^{(n-3)m} \cdot \theta & \cdots & \cdots & \Omega \end{bmatrix}, \\ \Omega &= \begin{bmatrix} B_h \\ A_h B_h + B_h \\ \vdots \\ A_h^{m-1} B_h + A_h^{m-2} B_h + \cdots + B_h \end{bmatrix}, \\ \theta &= A_h^{m-1} B_h + A_h^{m-2} B_h + \cdots + B_h. \end{aligned}$$

Assume the dynamics of the lifted filter is $\Sigma_{\underline{df}}$: $[A_f, B_f, C_f, D_f]$, whose input is $\underline{y}$, output is $\hat{X}$ (lifted states estimation during the kth frame period) and state is ξ, then the lifted state estimation error $\underline{e} = X - \hat{X}$ during the kth frame period is: $\underline{e}[k] = \Phi x_l[k] + \Gamma \underline{w}[k] - C_f \xi[k] - D_f(C_l x_l[k] + D_l \underline{w}[k])$. The error dynamics, with input $\underline{w}$, output $\underline{e}$, and state ζ, is $\Sigma_{\underline{ed}} : \left[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\right]$. Here $\zeta = \begin{pmatrix} x_l[k] \\ \xi[k] \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} A_l & 0 \\ B_f C_l & A_f \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} B_l \\ B_f D_l \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} \Phi - D_f C_l & -C_f \end{pmatrix}$, $\tilde{D} = \Gamma - D_f D_l$.

The proposed time-varying filter design problem is now equivalent to a time-invariant filter design problem, see Lemma 1.

Lemma 1: Given $\gamma > 0$, or $\beta > 0$, find a lifted filter $\Sigma_{\underline{df}}$ so that the corresponding lifted error dynamics $\Sigma_{\underline{ed}}$ is asymptotically stable and the H_2 norm, or the H_∞ norm of the lifted error dynamics transfer function $G_{\underline{ew}}$ satisfies $\|G_{\underline{ew}}\|_2^2 < mn\gamma$ or $\|G_{\underline{ew}}\|_\infty < \beta$; moreover, the matrix D_f of $\Sigma_{\underline{df}}$ should satisfy the causality constraint.

Proof: Results can be obtained by using the lifting technique, and analyzing the relationship between H_2/H_∞ norms of the original time-varying but periodic system G_{ew} and the lifted linear time-invariant (LTI) system $G_{\underline{ew}}$. For a reference, see, e.g., [7], [8]. ■

III. Main Results

Lemma 2: 1) Let $\gamma > 0$, $\|G_{\underline{ew}}\|_2^2 < mn\gamma$ if and only if there exists $P = P' > 0$ such that

$$\begin{aligned} \begin{bmatrix} P & \tilde{A}P & \tilde{B} \\ P\tilde{A}' & P & 0 \\ \tilde{B}' & 0 & I \end{bmatrix} &> 0, \\ \begin{bmatrix} mn\gamma I & \tilde{C}P & \tilde{D} \\ P\tilde{C}' & P & 0 \\ \tilde{D}' & 0 & I \end{bmatrix} &> 0. \end{aligned} \tag{2}$$

2) Let $\beta > 0$, $\|G_{\underline{ew}}\|_\infty < \beta$ if and only if there exists $P = P' > 0$ such that

$$\begin{bmatrix} P^{-1} & \tilde{A} & \tilde{B} & 0 \\ \tilde{A}' & P & 0 & \tilde{C}' \\ \tilde{B}' & 0 & \beta I & \tilde{D}' \\ 0 & \tilde{C} & \tilde{D} & \beta I \end{bmatrix} > 0, \tag{3}$$

Proof: (2) and (3) are standard LMI conditions for LTI H_2 and H_∞ design problems stated in Lemma 1, see, e.g., [5], [6] and the references therein. ■

Theorem 1: 1) Let $\gamma > 0$ be given, $\Sigma_{\underline{df}}$ is an admissible filter assuring $\|G_{\underline{ew}}\|_2^2 < mn\gamma$ if and only if there exist $R = R' > 0$, $X = X' > 0$, M, Z, N, and D_f satisfying

$$\begin{bmatrix} R & X & E_1 & E_2 & E_3 \\ X & X & XA_l & XA_l & XB_l \\ E_1' & A_l'X & R & X & 0 \\ E_2' & A_l'X & X & X & 0 \\ E_3' & B_l'X & 0 & 0 & I \end{bmatrix} > 0, \tag{4}$$

$$\begin{bmatrix} mn\gamma I & \Xi_1 & \Xi_2 & \Xi_3 \\ \Xi_1' & R & X & 0 \\ \Xi_2' & X & X & 0 \\ \Xi_3' & 0 & 0 & I \end{bmatrix} > 0. \tag{5}$$

where $E_1 = RA_l + ZC_l$, $E_2 = RA_l + ZC_l + M$, $E_3 = RB_l + ZD_l$, and $\Xi_1 = \Phi - D_f C_l$, $\Xi_2 = \Phi - D_f C_l - N$, $\Xi_3 = \Gamma - D_f D_l$.

2) Let $\beta > 0$ be given, $\Sigma_{\underline{df}}$ is an admissible filter assuring $\|G_{\underline{ew}}\|_\infty < \beta$ if and only if there exist $R = R' > 0$, $X = X' > 0$, M, Z, N, and D_f satisfying

$$\begin{bmatrix} R & R & RA_l & RA_l & RB_l & 0 \\ R & X & K_1 & K_2 & K_3 & 0 \\ A_l'R & K_1' & R & R & 0 & \Theta_1' \\ A_l'R & K_2' & R & X & 0 & \Theta_2' \\ B_l'R & K_3' & 0 & 0 & \beta I & \Theta_3' \\ 0 & 0 & \Theta_1 & \Theta_2 & \Theta_3 & \beta I \end{bmatrix} > 0, \tag{6}$$

where $K_1 = XA_l + ZC_l + M$, $K_2 = XA_l + ZC_l$, $K_3 = XB_l + ZD_l$, and $\Theta_1 = \Phi - D_f C_l - N$, $\Theta_2 = \Phi - D_f C_l$, $\Theta_3 = \Gamma - D_f D_l$.

Proof: Conditions (4), (5) (for H_2 filtering problem) and (6) (for H_∞ filtering problem) are derived by using similar methods shown in, e.g., [5], [6], and considering the causality constraint on D_f. ■

Remark 1: Causality constraint on D_f in this work means that D_f should have a block lower triangular structure as follows:

$$D_f = \begin{bmatrix} D_f^{11} & 0 & \cdots & & 0 \\ D_f^{21} & D_f^{22} & 0 & \cdots & 0 \\ & & \ddots & & \\ D_f^{m1} & D_f^{m2} & & & D_f^{mm} \end{bmatrix}, \qquad (7)$$

where D_f^{ij} $(i, j = 1, 2, \cdots, m)$ is a full matrix with $n * n_a$ rows and one column since we consider single-input single-output dual-rate systems only.

Remark 2: Similar to [5], [6], the lifted H_2 and H_∞ filters can be written out explicitly as follows:

1) Any matrices R, X, M, N, and Z satisfying (4) and (5) yield admissible H_2 optimal lifted filter $\Sigma_{\underline{df}}$ with $A_f = (X-R)^{-1}M$, $B_f = (X-R)^{-1}Z$, $C_f = N$; D_f can be solved directly from LMI conditions (4) and (5).
2) Any matrices R, X, M, N, and Z satisfying (6) yield admissible H_∞ optimal lifted filter $\Sigma_{\underline{df}}$ with $A_f = (R-X)^{-1}M$, $B_f = (R-X)^{-1}Z$, $C_f = N$; D_f can be solved directly from LMI condition (6).

Remark 3: Once the lifted filter $\Sigma_{\underline{df}}$ is found, i.e., matrices A_f, B_f, C_f, and D_f are determined with D_f satisfying the causality constraint, the time-varying filter to be implemented is $\Sigma_{df} = L_{mn}^{-1}\Sigma_{\underline{df}}L_m$.

IV. Example

In this example, we assume input and output sampling periods of the considered system are mh and nh, respectively, where $m = 2$, $n = 3$, and $h = 0.25$ seconds. Σ_h is given by $A_h = [1.0168 \;\; 0.2059; -1.8117 \;\; 0.3991]$, $B_h = [0.0317; 0.0111]$, $C_h = [-0.8 \;\; 0.6]$, and $D_h = 1.5$.

Two states of the system are required to be estimated every base period h, and the proposed LMI approach to both H_2 and H_∞ designs is applied. It is found that the smallest H_2 performance level is $mn\gamma = 0.016$, i.e., $\gamma = 0.016/6$. And the optimal lifted H_2 filter is described by: $A_f = [-0.5378 \;\; -0.1579; 0.5619 \;\; -0.1832]$, $B_f = [-0.0300 \;\; 0.0359; -0.0343 \;\; -0.1301]$,

$$C_f = \begin{bmatrix} 0.5822 & 0.1485 \\ -0.7395 & 0.4285 \\ 0.4432 & 0.2267 \\ -1.3487 & -0.1024 \\ 0.1731 & 0.2095 \\ -1.3413 & -0.4516 \\ -0.0514 & 0.1427 \\ -0.8622 & -0.5660 \\ -0.2298 & 0.0286 \\ -0.2510 & -0.4845 \\ -0.2853 & -0.0707 \\ 0.3161 & -0.2451 \end{bmatrix}, D_f = \begin{bmatrix} 0.0185 & 0 \\ 0.0153 & 0 \\ 0.0430 & 0 \\ -0.0200 & 0 \\ 0.0396 & 0 \\ -0.0858 & 0 \\ 0.0269 & 0.0518 \\ -0.1071 & -0.0143 \\ 0.0053 & 0.0497 \\ -0.0915 & -0.0995 \\ -0.0135 & 0.0301 \\ -0.0461 & -0.1298 \end{bmatrix}.$$

For the H_∞ design, the smallest H_∞ level calculated by LMI is $\beta = 0.447$. And the optimal lifted H_∞ filter is described by: $A_f = [-0.4307 - 0.1302; 0.3951 - 0.0683]$, $B_f = [-0.03050.0407; -0.0524 - 0.1735]$,

$$C_f = \begin{bmatrix} 0.6278 & 0.1347 \\ -0.9365 & 0.1826 \\ 0.4523 & 0.1678 \\ -1.5088 & -0.1736 \\ 0.1259 & 0.1419 \\ -1.4299 & -0.3707 \\ -0.0455 & 0.0752 \\ -0.8833 & -0.4030 \\ -0.2966 & -0.0363 \\ -0.2942 & -0.3074 \\ -0.4307 & -0.1299 \\ 0.3959 & -0.0674 \end{bmatrix}, D_f = \begin{bmatrix} 0.0207 & 0 \\ 0.0549 & 0 \\ 0.0525 & 0 \\ -0.0086 & 0 \\ 0.0549 & 0 \\ -0.0974 & 0 \\ 0.0353 & 0.0693 \\ -0.1348 & -0.0356 \\ 0.0011 & 0.0657 \\ -0.1203 & -0.1389 \\ -0.0307 & 0.0407 \\ -0.0525 & -0.1735 \end{bmatrix}.$$

Notice that both D_f satisfy the block lower triangular structure, i.e., they are causal to be implemented. Optimal lifted H_2 and H_∞ filters are implemented after being inverse lifted. Simulation results showed that both H_2 and H_∞ filters can provide good estimations for the real states. Figure 1 is a comparison between the real state x_1 (solid line) and its estimations (dotted line by H_2 filter and dash line by H_∞ filter). For state x_2, similar results have been obtained. The simulation time is 25 seconds, the initial state variables are $x_1(0) = x_2(0) = 0.1$, and the exogenous input ω is a random signal with zero mean and variance 0.04.

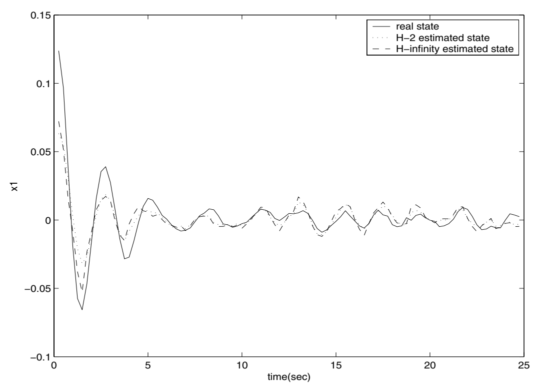

Fig. 1. State estimation (x_1)

V. Conclusions

H_2 and H_∞ filter design for multirate systems based on lifted models has been studied in this paper. Causality constraint on the lifted filters was tackled; and the causal solution was provided in terms of a set of LMI conditions. The effectiveness of the proposed method has been verified by a numerical example.

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