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Generalized State Scaling-Based Robust Control of Nonlinear Systems and Applications to Triangular Systems

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Abstract— **In this paper, we propose a high-gain scaling based controller to achieve global state-feedback stabilization of a general class of nonlinear systems which are allowed to contain uncertain functions of all the states and the control input as long as polynomial bounds on ratios of some uncertain system terms are available. The design is based on a high-gain scaling involving appropriate powers of a high-gain scaling parameter which is a dynamic signal driven by the state. The designed controller has a very simple structure being essentially a dynamic extension and a linear feedback with state-dependent dynamic gains. The obtained results are applicable to both lower triangular (strict-feedback) and upper triangular (feedforward) structures and also to systems without any triangular structure as long as a set of inequalities involving powers of the polynomial bounds on the ratios of the uncertain system terms and scaling orders is solvable. The stability analysis is based on our recent results on uniform solvability of coupled state-dependent Lyapunov equations.**

I. INTRODUCTION

We consider a high-gain scaling based state-feedback controller design for the class of systems given by

$$\begin{aligned} \dot{x}_i &= \phi_{(i,i+1)}(t,x,u)x_{i+1} + \phi_i(t,x,u)\ ,\ i=1,\dots,n-1 \\ \dot{x}_n &= \mu(t,x,u)u + \phi_n(t,x,u) \end{aligned} \tag{1}$$

where $x=[x_1,\dots,x_n]^T$ is the system state and u the input. μ, $\phi_{(i,i+1)}$, and ϕ_i are uncertain time-varying functions.

High-gain techniques for controller and observer designs have been investigated in the literature. The basic adaptive high-gain controller given by $u=-ry, \dot{r}=y^2$ provides global stabilization under the assumption that the system is minimum-phase and of relative-degree one ([1–3] and references therein). A semiglobal observer design based on static high-gain scaling (using observer gains $r,\dots,r^n$ with a constant r) was considered in [4,5]. The observer analysis utilizes scaled observer errors $\frac{e_i}{r^i}$ (or $\frac{e_i}{r^{i-1}}$) with e_i being the estimation error of the i^{th} state. A global high-gain observer and controller with gains being powers of $\int_0^t y^2(\tau)d\tau$ were proposed in [6] for linear systems with appended stable nonlinear zero dynamics and input-matched nonlinearities. In [7], a high-gain observer and a backstepping based controller were designed for systems of form (1) under the assumptions that $\phi_{(i,i+1)} = 1, i = 1,\dots,n-1$, and with the terms $\phi_i, i = 1,\dots,n$, being known functions of $x_1,\dots,x_i$ incrementally linear in unmeasured states in the sense that $|\phi_i(x_1,\dots,x_i) - \phi_i(x_1,\hat{x}_2,\dots,\hat{x}_i)| \le \Gamma(x_1)\sum_{j=2}^n |\hat{x}_j - x_j|$ with $\Gamma(x_1)$ being a known function. The dynamics of the high-gain scaling parameter were a scalar differential Riccati equation driven by y.

In [8], it was shown that the high-gain scaling proposed in [7] essentially amplifies the *upper diagonal terms* ($\phi_{(i,i+1)}$) thus inducing the Cascading Upper Diagonal Dominance (CUDD) condition introduced in [9,10]. Motivated by duality considerations, a dynamic high-gain scaling based state-feedback controller and a dual high-gain observer/controller based output-feedback solution for strict-feedback (lower triangular, i.e., functions in dynamics of i^{th} state can be bounded as functions of $x_1,\dots,x_i$) systems were proposed in [11]. The dual high-gain design in [11] was based on the solution of a pair of coupled Lyapunov equations. The state-feedback controller in [11] requires the upper diagonal terms to satisfy the cascading dominance condition (which, in the controller context, requires the ratios $|\phi_{(i-1,i)}|/|\phi_{(i,i+1)}|$ to be upper bounded by a positive constant) and also requires the terms ϕ_i to be bounded as $|\phi_i| \le \Gamma(x_1)\sum_{j=1}^i \phi_{(i,j)}|x_j|$ with $\Gamma(x_1)$ and $\phi_{(i,j)}$ being known continuous nonnegative functions such that the ratios $|\phi_{(i,j)}|/|\phi_{(j-1,j)}|$, $\phi_{(i,2)}/|\phi_{(2,3)}|$, and $\phi_{(i,1)}/|\phi_{(1,2)}|$ are upper bounded by positive constants. These assumptions which essentially require that the upper diagonal terms should be dominant in the system and that there should be a cascading dominance relation between the upper diagonal terms such that the terms nearer to the control input are larger (in the sense that the ratios $|\phi_{(i-1,i)}|/|\phi_{(i,i+1)}|$ are bounded above) comprise the CUDD condition in the controller context. The output-feedback result in [11] provides a dual high-gain observer and controller for strict-feedback systems assuming that the upper diagonal terms also satisfy the observer-context cascading dominance condition (ratios $|\phi_{(i,i+1)}|/|\phi_{(i-1,i)}|$ bounded above). The functions $\phi_i, i = 1,\dots,n$, were allowed to contain functional and parametric uncertainties coupled with all the states.

The dual high-gain technique introduced in [11] was extended to the case of state-feedback and output-feedback control for feedforward (upper triangular, i.e, functions in dynamics of x_i can be bounded as functions of $x_{i+2},\dots,x_n,u$) systems in [12,13]. Previously available controller design techniques for feedforward systems include saturation-based designs [14–17] and forwarding [18]. Nested saturation designs rely on the use of small inputs (making the scheme sensitive to additive disturbances) and require ϕ_i to involve only quadratic or higher powers in their arguments. Forwarding is a recursive passivation scheme which proceeds by adding one integrator at a time through the design of cross terms (which, due to computational complexity, often need to be approximated numerically). However, due to a lack of robustness to additive disturbances in these designs [14–18], the extension to the output-feedback case was not feasible. In contrast, the dual high-gain approach in [12,13] provided a robustly stabilizing controller and enabled an output-feedback solution. The state-feedback controller in [12] required the upper diagonal terms $\phi_{(i,i+1)}$ to satisfy the controller-context cascading dominance condition and also required the ϕ_i terms to be bounded linearly (up to a factor of $\phi_{(1,2)}$ and a polynomial function of x_n) in the states and input.

In [19], a generalized high-gain scaling involving arbitrary powers of the high-gain scaling parameter was proposed for strict-feedback systems. The standard scaling x_i/r^{ai+b} with constants a and b can scale functions ϕ_i relative to the upper diagonal terms $\phi_{(i,i+1)}$ as shown in [8], but can not modify relative magnitudes of upper diagonal terms since all upper diagonal terms are scaled by r^a. Furthermore, this scaling requires CUDD-like assumptions as noted above and only attenuates arbitrary functions of x_1 (in the strict-feedback case) or x_n (in the feedforward case). In [19], it was seen that scaling using arbitrary powers of the high-gain scaling parameter can scale relative magnitudes of upper diagonal terms and also provides a technique to take into account the form of (polynomial) bounds on ratios of system terms and introduce a scaling specifically tailored to induce CUDD in the scaled system. In [19], a state-feedback controller was provided for strict-feedback systems with polynomial bounds on certain ratios of unknown system terms and an output-feedback controller was obtained without requiring cascading dominance.

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In this paper, we extend the results of [19] to state-feedback controller design for the general class of systems (1) without requiring a triangular structure. Furthermore, we allow ϕ_i, $\phi_{(i,i+1)}$, and μ to be uncertain functions of time, state, and input while all the earlier designs required upper diagonal terms $\phi_{(i,i+1)}$ to be known functions of the state. The results are applicable to both lower triangular and upper triangular systems and also to non-triangular systems as long as a set of linear inequalities involving powers of the polynomial bounds and scaling orders is solvable. We show that this set of inequalities is always solvable in the case of strict-feedback systems (recovering the result of [19]) and solvable in the case of feedforward systems under certain assumptions. Moreover, the solvability of the set of inequalities can be easily checked numerically even for non-triangular systems and the proposed design yields a globally asymptotically stabilizing state-feedback controller whenever a solution exists.

The required assumptions are listed in Section II. The state scaling and dynamic extension are explained in Section III. The controller design is provided in Section IV. The stability analysis is contained in Section V. The application of the results to strict-feedback and feedforward systems are illustrated in Sections VI and VII, respectively. An illustrative example is provided in Section VIII.

II. Assumptions and Problem Statement

Definition 1: A function $f : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \mapsto \mathcal{R}$ is said to be a multinomial if it is of the form

$$f(z_1, \ldots, z_l) \quad = \quad \sum_{k=1}^{N} \chi_k \prod_{i=1}^{l} z_i^{\beta_{(i,k)}} \tag{2}$$

where N is a positive integer and χ_k and $\beta_{(i,k)}, i = 1, \ldots, l, k = 1, \ldots, N$, are nonnegative real numbers. A multinomial $f(z_1, \ldots, z_l)$ is said to be proper if a continuous nonnegative function $\overline{f}$ (called its bounding function) exists such that $|f(z_1, \ldots, z_l)| \leq \overline{f}(z_1, \ldots, z_l)\sqrt{\sum_{i=1}^{l} z_i^2}$. A real number ζ is said to dominate f relative to real numbers $\zeta_1, \ldots, \zeta_l$ if

$$\zeta \quad > \quad \zeta_1\beta_{(1,k)} + \ldots + \zeta_l\beta_{(l,k)} \ , \ \ k = 1, \ldots, N. \tag{3}$$

To denote that ζ dominates f relative to $\zeta_1, \ldots, \zeta_l$, we use the notation $\zeta \succ f|_{(\zeta_1,\ldots,\zeta_l)}$. It can be shown that the multinomial in (2) is proper if and only if $\sum_{i=1}^{l} \beta_{(i,k)} \geq 1$, $k = 1, \ldots, N$.

Lemma 1: If $f(z_1, \ldots, z_l)$ is a multinomial and $\zeta \succ f|_{(\zeta_1,\ldots,\zeta_l)}$, then for all $\eta \geq 1$, and all real numbers $z_1, \ldots, z_l$,

$$\left| \frac{f(\eta^{\zeta_1} z_1, \ldots, \eta^{\zeta_l} z_l)}{\eta^{\zeta}} \right| \quad \leq \quad f(|z_1|, \ldots, |z_l|). \tag{4}$$

Proof of Lemma 1: Using (2) and (3),

$$\begin{aligned} \left| \frac{f(\eta^{\zeta_1} z_1, \ldots, \eta^{\zeta_l} z_l)}{\eta^{\zeta}} \right| \quad &= \quad \left| \sum_{k=1}^{N} \chi_k \frac{\eta^{\sum_{i=1}^{l} \zeta_i \beta_{(i,k)}}}{\eta^{\zeta}} \prod_{i=1}^{l} z_i^{\beta_{(i,k)}} \right| \\ &\leq \quad \left| \sum_{k=1}^{N} \chi_k \prod_{i=1}^{l} |z_i|^{\beta_{(i,k)}} \right| \end{aligned} \tag{5}$$

yielding (4). $\diamond$

The control objective in this paper is to regulate the state x of system (1) to the origin using dynamic state feedback in the presence of the uncertain terms ϕ_i, $\phi_{(i,i+1)}$, and μ. The design will be carried out under the following assumptions.

Assumption A1: (*Controllability of system (1)*) A positive constant σ is known such that for all $t \geq 0$, $x \in \mathcal{R}^n$, and $u \in \mathcal{R}$, $|\phi_{(i,i+1)}(t, x, u)| \geq \sigma > 0$, $i = 1, \ldots, n-1$ and $|\mu(t, x, u)| \geq \sigma > 0$. The sign of each $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, is independent of its argument and known. Furthermore, continuous nonnegative functions $\overline{\phi}_{(i,i+1)}, i = 1, \ldots, n-1$, and $\overline{\mu}$ are known such that $|\phi_{(i,i+1)}(t, x, u)| \leq \overline{\phi}_{(i,i+1)}(x, u)$ and $|\mu(t, x, u)| \leq \overline{\mu}(x)$.

Assumption A2: The functions $\phi_i, i = 1, \ldots, n$ can be bounded as $|\phi_i(t, x, u)| \leq \sum_{j=1}^{i} \phi_{(i,j)}(t, x, u)|x_j| + \tilde{\phi}_i(t, x, u)$ with $\phi_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, and $\tilde{\phi}_i, i = 1, \ldots, n$, being continuous nonnegative functions. Furthermore,

- proper multinomials $f_i, i = 1, \ldots, n$,
- (not necessarily proper) multinomials $f_{(i,j)}, i = 1, \ldots, n-1, j = 1, \ldots, i$, and $\tilde{f}_i, i = 2, \ldots, n-1$, and
- continuous nonnegative functions γ_u and $\overline{\phi}_{(n,j)}, j = 1, \ldots, n$,

are known such that[1]

$$\frac{\tilde{\phi}_i}{|\phi_{(1,2)}|} \leq f_i\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right), i = 1, \ldots, n \tag{6}$$

$$\frac{\phi_{(i,j)}}{\sqrt{|\phi_{(i,i+1)}||\phi_{(j-1,j)}|}} \leq f_{(i,j)}\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right), \quad i = 1, \ldots, n-1, j = 1, \ldots, i \tag{7}$$

$$\phi_{(n,j)} \leq \overline{\phi}_{(n,j)}(x, u) \ , \ \ j = 1, \ldots, n \tag{8}$$

$$\frac{|\phi_{(i-1,i)}|}{|\phi_{(i,i+1)}|} \leq \tilde{f}_i\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right), i = 2, \ldots, n-1 \tag{9}$$

with $\phi_{(0,1)} \stackrel{\triangle}{=} \phi_{(1,2)}$.

Assumption A3: A continuous positive function $\tilde{\mu}(x)$, a positive constant μ^*, and a multinomial $\tilde{f}_n$ are known such that for all $t \geq 0, x \in \mathcal{R}^n$, and $u \in \mathcal{R}$,

$$\tilde{\mu}(x)\gamma_u(x) \quad \leq \quad \mu^* \tag{10}$$

$$\frac{|\phi_{(n-1,n)}(t, x, u)|}{\tilde{\mu}(x)|\mu(t, x, u)|} \quad \leq \quad \tilde{f}_n\left(|x_1|, \ldots, |x_n|, \gamma_u(x)|u|\right). \tag{11}$$

Assumption A4: Positive constants $q_i, i = 1, \ldots, n$, and a constant q_{n+1} exist such that

$$q_i + q_2 - q_1 \succ f_i|_{(q_1,\ldots,q_{n+1})} \ , \ i = 1, \ldots, n \tag{12}$$

$$\frac{q_{i+1} + q_i - q_j - q_{j-1}}{2} \succ f_{(i,j)}|_{(q_1,\ldots,q_{n+1})} \ , \ i = 2, \ldots, n-1, \quad j = 2, \ldots, i \tag{13}$$

$$\frac{q_{i+1}+q_i+q_2-3q_1}{2} \succ f_{(i,1)}|_{(q_1,\ldots,q_{n+1})}, i=1, \ldots, n-1 \tag{14}$$

$$q_{i+1} + q_{i-1} - 2q_i \succ \tilde{f}_i|_{(q_1,\ldots,q_{n+1})} \ , \ i = 2, \ldots, n. \tag{15}$$

If any of the multinomials among $f_i, i = 1, \ldots, n$, are zero, the corresponding inequalities in (12) can be dropped. Note that none of the f_i can be a non-zero constant since $f_i, i = 1, \ldots, n$, are required to be proper multinomials. If any of $f_{(i,j)}, i = 1, \ldots, n-1, j = 1, \ldots, i$, or $\tilde{f}_i, i = 2, \ldots, n$, are non-zero constants, the right hand sides of the corresponding inequalities in (13), (14), and (15) reduce to zero. If any of $f_{(i,j)}, i = 1, \ldots, n-1, j = 1, \ldots, i$, are zero, the corresponding inequalities in (13) and (14) can be dropped. None of the $\tilde{f}_i$ can be zero since $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, are lower bounded in magnitude by σ.

Remark 1: The conditions (12) - (15) form a finite set of strict linear inequalities. Hence, if Assumption A4 is satisfied, a positive constant c exists such that the inequalities (12) - (15) also hold with c subtracted from the left hand side of the inequalities. It will be seen in Section III that Assumption A4 implies the existence of a high-gain state scaling that can attenuate system uncertainties.

Remark 2: The decomposition of ϕ_i in Assumption A2 allows considerable freedom in incorporating a term in a known bound on ϕ_i into one of the $\phi_{(i,j)}$ or $\tilde{\phi}_i$ terms. For instance, given $\phi_3 = x_1x_2 + x_3^2$, ϕ_3 can be decomposed in a variety of ways including (a) $\phi_{(3,1)} = |x_2|$, $\phi_{(3,2)} = \phi_{(3,3)} = 0$, $\tilde{\phi}_3 = x_3^2$; (b) $\phi_{(3,1)} = \phi_{(3,3)} = 0$, $\phi_{(3,2)} = |x_1|$, $\tilde{\phi}_3 = x_3^2$; and (c) $\phi_{(3,1)} = \phi_{(3,2)} = 0$,

[1]For notational convenience, we drop the arguments of functions when no confusion will result.

$\phi_{(3,3)} = |x_3|$, $\tilde{\phi}_3 = |x_1||x_2|$. This freedom can be exploited to aid the feasibility of (12)-(15).

III. A Dynamic Extension and a State Scaling

The control input u is designed as

$$u = r^{q_{n+1}} \tilde{\mu}(x) \xi_{n+1} \quad (16)$$

where ξ_{n+1} is a new state variable with the dynamics

$$\dot{\xi}_{n+1} = v - b_v \frac{\dot{r}}{r} \xi_{n+1} \quad (17)$$

with v being the new control input, b_v a positive constant, and r a dynamic scaling parameter. The design of the dynamics of r will ensure that $r(t) \geq 1$ for all $t \geq 0$. The control input transformation (16) and (17) corresponds to a dynamic extension of the state so that, in the extended system, the bounds on ratios of uncertain system terms in Assumption A2 are functions of the states x and ξ_{n+1} and do not involve the new input v.

A state scaling is introduced as

$$\xi_i = \frac{x_i}{r^{q_i}} \ , \ i = 1, \ldots, n \ ; \ \xi = [\xi_1, \ldots, \xi_{n+1}]^T. \quad (18)$$

The scaled dynamics can be written in matrix form as

$$\dot{\xi} = A(t,x,u,r)\xi + Bv + \Phi(t,x,u,r) - \frac{\dot{r}}{r} D\xi \quad (19)$$

where

$$B = [0, \ldots, 0, 1]^T \quad (20)$$

$$\Phi(t,x,u,r) = [\frac{1}{r^{q_1}}\phi_1(t,x,u), \ldots, \frac{1}{r^{q_n}}\phi_n(t,x,u), 0]^T \quad (21)$$

$$D = \mathrm{diag}(q_1, \ldots, q_n, b_v) \quad (22)$$

and $A(t,x,u,r)$ is the $(n+1) \times (n+1)$ matrix with $(i,j)^{th}$ element

$$A_{(i,i+1)}(t,x,u,r) = r^{q_{i+1}-q_i} \phi_{(i,i+1)}(t,x,u) \ , \ i = 1, \ldots, n$$
$$A_{(i,j)} \equiv 0, \ j \neq i+1 \ , \ i = 1, \ldots, n+1 \quad (23)$$

with $\phi_{(n,n+1)}(t,x,u) \stackrel{\triangle}{=} \tilde{\mu}(x)\mu(t,x,u)$.

Crucial properties of the matrices appearing in the scaled dynamics are provided in the following results.

Theorem 1: A continuous positive function $R(\xi)$, a $1 \times (n+1)$ vector function $K(x, \xi_{n+1}, r)$, a constant symmetric positive-definite $(n+1) \times (n+1)$ matrix P, and positive constants ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ can be found such that[2] for all $r \geq R(\xi)$, and all $t \geq 0$, $x \in \mathcal{R}^n$, and $\xi_{n+1} \in \mathcal{R}$,

$$\begin{aligned} & P[A(t,x,u,r) + Q_1\overline{\Phi}(t,x,u,r)Q_2 + BK(x,\xi_{n+1},r)] \\ & \quad + [A(t,x,u,r) + Q_1\overline{\Phi}(t,x,u,r)Q_2 + BK(x,\xi_{n+1},r)]^T P \\ & \leq -\nu_1 \frac{|\phi_{(1,2)}(t,x,u)|}{r^{q_1-q_2}} I \end{aligned} \quad (24)$$

$$\underline{\nu}_2 I \leq PD + DP \leq \overline{\nu}_2 I \quad (25)$$

where $\overline{\Phi}(t,x,u,r)$ is the lower triangular $(n+1) \times (n+1)$ matrix with $(i,j)^{th}$ element equal to $r^{q_j - q_i}\phi_{(i,j)}(t,x,u)$ if $1 \leq i \leq n, 1 \leq j \leq i$, and zero otherwise. Q_1 and Q_2 are arbitrary $(n+1) \times (n+1)$ diagonal matrices with each diagonal entry $+1$ or -1.

Proof of Theorem 1: Let ρ be any positive constant. Pick

$$R(\xi) = \max\left\{1, \max_{i=1,\ldots,n}\left\{\left[\frac{\tilde{f}_i(|\xi_1|, \ldots, |\xi_n|, \mu^*|\xi_{n+1}|)}{\rho}\right]^{\frac{1}{c}}\right\}, \max_{i=1,\ldots,n-1, j=1,\ldots,i}\left\{\left[\frac{f_{(i,j)}(|\xi_1|, \ldots, |\xi_n|, \mu^*|\xi_{n+1}|)}{\rho}\right]^{\frac{1}{c}}\right\}\right\}. \quad (26)$$

[2] I denotes an identity matrix of appropriate dimensions.

Define $\overline{A} = A + Q_1\overline{\Phi}Q_2$ and denote by $\overline{A}_{(i,j)}$ its $(i,j)^{th}$ element. Using (16), (23), and Assumptions A2 and A3,

$$\frac{|\overline{A}_{(i,j)}|}{\sqrt{|\overline{A}_{(i,i+1)}||\overline{A}_{(j-1,j)}|}} \leq \frac{\hat{f}_{(i,j)}}{r^{\frac{q_{i+1}+q_i-q_j-q_{j-1}}{2}}} \ , \ i = 2, \ldots, n-1, \quad j = 2, \ldots, i$$

$$\frac{|\overline{A}_{(i,1)}|}{\sqrt{|\overline{A}_{(i,i+1)}||\overline{A}_{(1,2)}|}} \leq \frac{\hat{f}_{(i,1)}}{r^{\frac{q_{i+1}+q_i+q_2-3q_1}{2}}} \ , \ i = 1, \ldots, n-1$$

$$\frac{|\overline{A}_{(i-1,i)}|}{|\overline{A}_{(i,i+1)}|} \leq \frac{\hat{\tilde{f}}_i}{r^{q_{i+1}+q_{i-1}-2q_i}} \ , \ i = 2, \ldots, n-1. \quad (27)$$

where the symbols $\hat{f}_{(i,j)}$ and $\hat{\tilde{f}}_i$ have been used to denote $f_{(i,j)}(r^{q_1}|\xi_1|, \ldots, r^{q_n}|\xi_n|, r^{q_{n+1}}\mu^*|\xi_{n+1}|)$ and $\tilde{f}_i(r^{q_1}|\xi_1|, \ldots, r^{q_n}|\xi_n|, r^{q_{n+1}}\mu^*|\xi_{n+1}|)$, respectively. Using (12)-(15), Remark 1, and Lemma 1, it follows, using Definition A1 in the Appendix, that $\overline{A}$ is dual w-CUDD (ρ) for all $r \geq R(\xi)$ with $R(\xi)$ defined in (26). Noting that D is a diagonal matrix with positive diagonal entries, and applying Theorem 1 in [19] (also see Theorem 2 in [20]), the result of Theorem 1 follows. Note that, by Theorem 1 in [19], the choice of K depends only on the known upper and lower bounds on $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, and the known upper bounds on $\phi_{(n,j)}, j = 1, \ldots, n$, and does not require knowledge of the uncertain functions $\phi_{(i,j)}, i = 1, \ldots, n, j = 1, \ldots, i$, and $\phi_{(i,i+1)}, i = 1, \ldots, n-1$, themselves. Hence, K is a known function of (x, ξ_{n+1}, u, r), and hence of (x, ξ_{n+1}, r) by (16). The choice of P, ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ depends only on the choice of ρ which is free to be arbitrarily picked and the signs of $\phi_{(i,i+1)}$ which are known and constant by Assumption A1. Furthermore, K, P, ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ do not depend on the diagonal matrices Q_1 and Q_2. $\diamond$

Theorem 2: A continuous positive function $\overline{R}(\xi)$, a $1 \times (n+1)$ vector function $K(x, \xi_{n+1}, r)$, a constant symmetric positive-definite $(n+1) \times (n+1)$ matrix P, and positive constants $\underline{\nu}_1$, $\underline{\nu}_2$, and $\overline{\nu}_2$ can be found such that for all $r \geq \overline{R}(\xi)$, and all $t \geq 0, x \in \mathcal{R}^n$ and $\xi_{n+1} \in \mathcal{R}$,

$$\begin{aligned} & \xi^T \Big\{ P[A(t,x,u,r) + BK(x,\xi_{n+1},r)] \\ & \quad + [A(t,x,u,r) + BK(x,\xi_{n+1},r)]^T P \Big\} \xi \\ & + 2\xi^T P \Phi(t,x,u,r) \qquad \leq -\underline{\nu}_1 r^{q_2-q_1}|\phi_{(1,2)}||\xi|^2 \end{aligned} \quad (28)$$

$$\underline{\nu}_2 I \leq PD + DP \leq \overline{\nu}_2 I. \quad (29)$$

Proof of Theorem 2: Let ρ be any positive constant. Obtain $R(\xi)$, $K(x, \xi_{n+1}, r)$, P, ν_1, $\underline{\nu}_2$, and $\overline{\nu}_2$ as in Theorem 1. Let

$$\overline{R}(\xi) = \max\left(R(\xi), \left[\frac{4\lambda_{max}(P)f^*(\xi)}{\nu_1}\right]^{\frac{1}{c}}\right) \quad (30)$$

where $\lambda_{max}(P)$ denotes the maximum eigenvalue of P and

$$f^*(\xi) = (1+\mu^*)\sqrt{\sum_{i=1}^{n} \overline{f}_i^2(|\xi_1|, \ldots, |\xi_n|, \mu^*|\xi_{n+1}|)} \quad (31)$$

with $\overline{f}_i$ being the bounding functions of $f_i, i = 1, \ldots, n$. Using Assumption A2 and (21),[3]

$$\begin{aligned} 2\xi^T P\Phi & \leq \xi^T PS(P\xi)\overline{\Phi}S(\xi)\xi + \xi^T S(\xi)\overline{\Phi}^T S(P\xi)P\xi \\ & \quad + 2\lambda_{max}(P)|\xi||\tilde{\Phi}| \end{aligned} \quad (32)$$

where $\tilde{\Phi} = [\tilde{\phi}_1/r^{q_1}, \ldots, \tilde{\phi}_n/r^{q_n}, 0]^T$. Using (6),

$$|\tilde{\Phi}(t,x,u,r)| \leq |\phi_{(1,2)}(t,x,u)|\frac{r^{q_2-q_1}}{r^c} f^*(\xi)|\xi| \quad (33)$$

with f^* given in (31). Invoking Theorem 1 with $Q_1 = S(P\xi)$ and $Q_2 = S(\xi)$, and using (24), (30), (32), and (33), we obtain (28) with $\underline{\nu}_1 = \nu_1/2$. (29) is obtained directly from (25). As in

[3] If $\eta \in \mathcal{R}^{n_\eta}$, $S(\eta)$ denotes the $n_\eta \times n_\eta$ diagonal matrix with i^{th} diagonal entry being the sign (± 1) of η_i. Hence, $S(\eta)\eta$ is a vector of the same dimension as η obtained by replacing each element of η by its magnitude.

Theorem 1, the choice of K, P, $\underline{\nu}_1$, $\underline{\nu}_2$, and $\overline{\nu}_2$ depends only on the known bounds on uncertain functions guaranteed in Assumptions A1-A3 and not the uncertain functions themselves and can be obtained by a constructive procedure. $\diamond$

Lemma 2: Given any $1 \times (n+1)$ vector function $K(x, \xi_{n+1}, r)$ and a constant symmetric positive-definite $(n+1)\times(n+1)$ matrix P, a function $\Delta(x, \xi_{n+1}, r)$ can be found such that for all $t \geq 0, r \geq 1, x \in \mathcal{R}^n$, and $\xi_{n+1} \in \mathcal{R}$

$$\xi^T \Big\{ P[A(t,x,u,r) + BK(x,\xi_{n+1},r)] + [A(t,x,u,r) + BK(x,\xi_{n+1},r)]^T P \Big\} \xi + 2\xi^T P\Phi(t,x,u,r) \leq \Delta(x,\xi_{n+1},r)|\xi|^2. \quad (34)$$

Proof of Lemma 2: Using Assumption A1,

$$\frac{\xi^T[PA + A^T P]\xi}{\lambda_{max}(P)} \leq 2|\xi|^2 \sqrt{\sum_{i=1}^{n} \left[\frac{\overline{\phi}_{(i,i+1)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})}{r^{q_i - q_{i+1}}} \right]^2} \quad (35)$$

where $\overline{\phi}_{(n,n+1)}(x,u) = |\tilde{\mu}(x)|\overline{\mu}(x)$. Using Assumption A2, $|\overline{\Phi}(t,x,u,r)| \leq \hat{\overline{\Phi}}(x,\xi_{n+1},r)$ where

$$\begin{aligned}
\hat{\overline{\Phi}}(x,\xi_{n+1},r) \triangleq \Bigg[& \sum_{i=2}^{n}\sum_{j=2}^{i} f^2_{(i,j)}(|x_1|,\ldots,|x_n|, r^{q_{n+1}}\mu^*|\xi_{n+1}|) \\
& \times \overline{\phi}_{(i,i+1)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})\overline{\phi}_{(j-1,j)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1}) \\
& + \sum_{i=1}^{n} f^2_{(i,1)}(|x_1|,\ldots,|x_n|, r^{q_{n+1}}\mu^*|\xi_{n+1}|) \\
& \times \overline{\phi}_{(i,i+1)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})\overline{\phi}_{(1,2)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1}) \\
& + \sum_{j=1}^{n} \overline{\phi}^2_{(n,j)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1}) \Bigg]^{\frac{1}{2}}. \quad (36)
\end{aligned}$$

Using (32), (33), and (36),

$$\begin{aligned}
2\xi^T P\Phi \leq \Bigg[& 2\lambda_{max}(P)\hat{\overline{\Phi}}(x,\xi_{n+1},r) \\
& + 2\lambda_{max}(P)\overline{\phi}_{(1,2)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})\frac{r^{q_2-q_1}}{r^c}f^*(\xi) \Bigg] |\xi|^2. \quad (37)
\end{aligned}$$

Using the inequalities (35) and (37), (34) follows with

$$\begin{aligned}
\Delta(x,\xi_{n+1},r) = {} & 2\lambda_{max}(P)\sqrt{\sum_{i=1}^{n}\left[\frac{\overline{\phi}_{(i,i+1)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})}{r^{q_i-q_{i+1}}}\right]^2} \\
& + 2\lambda_{max}(P)|K(x,\xi_{n+1},r)| + 2\lambda_{max}(P)\hat{\overline{\Phi}}(x,\xi_{n+1},r) \\
& + 2\lambda_{max}(P)\overline{\phi}_{(1,2)}(x, r^{q_{n+1}}\tilde{\mu}(x)\xi_{n+1})\frac{r^{q_2-q_1}}{r^c}f^*(\xi). \quad (38)
\end{aligned}$$

IV. Controller Design

The control input v is designed as

$$v = K(x,\xi_{n+1},r)\xi \quad (39)$$

and the dynamics of r are designed as

$$\dot{r} = \frac{r}{\underline{\nu}_2} q\left(\overline{R}(\xi) - r\right)[\Delta(x,\xi_{n+1},r) + \Delta_0] \; ; \; r(0) \geq 1 \quad (40)$$

with K and $\overline{R}$ obtained as in Theorem 2 and Δ as in Lemma 2. Δ_0 is any positive constant and q is any continuous nonnegative function such that

$$q(b) = \begin{cases} 1 & , \; b > 0 \\ 0 & , \; b \leq -\epsilon_r \end{cases} \quad (41)$$

with ϵ_r being any positive constant. From the dynamics (40), $r(t)$ is a monotonically non-decreasing function of time. Hence, $r(t) \geq 1$ for all time t.

The overall dynamic controller designed for the control input u is given by (16), (17), (39), and (40).

V. Stability Analysis

The stability of the closed-loop system can be demonstrated using the Lyapunov function $V = \xi^T P \xi$ where P is the symmetric positive-definite $(n+1)\times(n+1)$ matrix obtained in Theorem 2. Using (19), (29), and (40),

$$\begin{aligned}
\dot{V} \leq {} & \xi^T\Big\{P[A+BK] + [A+BK]^T P\Big\}\xi + 2\xi^T P\Phi \\
& - q\left(\overline{R}(\xi) - r\right)\left[\Delta(x,\xi_{n+1},r) + \Delta_0\right]|\xi|^2. \quad (42)
\end{aligned}$$

We consider the two cases $r < \overline{R}(\xi)$ and $r \geq \overline{R}(\xi)$. In the case that $r < \overline{R}(\xi)$, the definition (41) of q implies that $q(\overline{R}(\xi) - r) = 1$. Using Lemma 2,

$$\dot{V} \leq -\Delta_0|\xi|^2. \quad (43)$$

If $r \geq \overline{R}(\xi)$, using Theorem 2, (42) reduces to

$$\dot{V} \leq -\underline{\nu}_1 r^{q_2-q_1}|\phi_{(1,2)}(t,x,u)||\xi|^2. \quad (44)$$

From (43) and (44), $V(t)$ is a non-increasing function of time over the maximal interval of existence of solutions $[0,t_f)$ implying that $\xi(t)$ is uniformly bounded on $[0,t_f)$. Noting from (40) and (41) that $\dot{r} = 0$ if $r \geq \overline{R}(\xi) + \epsilon_r$, the boundedness of ξ implies the boundedness of r. Also, from (40), $r(t) \geq 1$ for all $t \in [0,t_f)$. Therefore, $x_i = r^{q_i}\xi_i, i = 1,\ldots,n$, and hence the control input u are uniformly bounded over $[0,t_f)$. This implies that $t_f = \infty$ and solutions exist for all time with all closed-loop signals being uniformly bounded. By the above arguments, $r(t) \in [1,\overline{r}] \;\; \forall t \geq 0$ with $\overline{r}$ being some positive constant. From (43) and (44), we obtain using Assumption A1 that

$$\dot{V} \leq -\frac{1}{\lambda_{max}(P)}\min(\Delta_0, \underline{\nu}_1\sigma, \underline{\nu}_1\sigma\overline{r}^{q_2-q_1})V. \quad (45)$$

The inequality (45) implies that $V(t)$ and hence $|\xi(t)|$ go to zero exponentially as $t \to \infty$. Since $|x_i| \leq \overline{r}^{q_i}|\xi_i|$, the states $x(t)$ and the control input $u(t)$ also go to zero exponentially.

The stability properties of the closed-loop system are summarized in the following theorem.

Theorem 3: Under Assumptions A1-A4, the proposed dynamic compensator given by (16), (17), (39), and (40) guarantees global boundedness of all closed-loop states. Furthermore, $\lim_{t\to\infty} x_i(t) = 0, i = 1,\ldots,n, \lim_{t\to\infty}\xi_{n+1}(t) = 0$, and $\lim_{t\to\infty} u(t) = 0$.

VI. Application to Strict-Feedback Systems

Systems for which f_i, $f_{(i,j)}$, and $\tilde{f}_i$ have a lower triangular structure given by

$$\begin{aligned}
f_i &= f_i(|x_1|,\ldots,|x_{i-1}|) \;,\; i = 1,\ldots,n \\
f_{(i,j)} &= f_{(i,j)}(|x_1|,\ldots,|x_i|) \;,\; i = 1,\ldots,n-1, j = 1,\ldots,i \\
\tilde{f}_i &= \tilde{f}_i(|x_1|,\ldots,|x_i|) \;,\; i = 2,\ldots,n \quad (46)
\end{aligned}$$

and $\overline{\phi}_{(i,i+1)}, i = 1,\ldots,n-1$, and $\overline{\phi}_{(n,j)}, j = 1,\ldots,n$, do not depend on the control input u form a class of strict-feedback systems. In this case, $\gamma_u(x) \equiv 0$ and (10) is trivially satisfied. Also, (11) can be satisfied with

$$\tilde{\mu}(x) = \frac{\overline{\phi}_{(n-1,n)}(x)}{\sigma} \;,\; \tilde{f}_n = 1. \quad (47)$$

Hence, Assumption A3 is automatically satisfied for strict-feedback systems. In fact, since f_i and $\tilde{f}_i$ do not involve the control input u, the dynamic extension (16) and (17) is not needed and Assumption A3 (which is needed specifically to enforce w-CUDD structure with the dynamic extension and to handle the dependence of f_i and $\tilde{f}_i$ on u) can be eliminated altogether. If the dynamic extension (16) and (17) is not introduced, the scaled state vector is defined as $\xi = [\xi_1,\ldots,\xi_n]^T$ with $\xi_i, i = 1,\ldots,n$, given by (18) and using results analogous to Theorem 2 and Lemma 2, the controller is designed as $u = K(x,r)\xi$ with the dynamics of r given by (40).

The following theorem guarantees that constants $q_i, i = 1,\ldots,n+1$, can be obtained to satisfy the inequalities (12) - (15) given any multinomials $f_i, i = 1,\ldots,n$, $f_{(i,j)}, i = 1,\ldots,n-$

$1, j = 1, \ldots, i$, and $\tilde{f}_i, i = 2, \ldots, n$, of the lower triangular structure (46).

Theorem 4: Given any multinomials $f_i(z_1, \ldots, z_{i-1}), i = 1, \ldots, n$, $f_{(i,j)}(z_1, \ldots, z_i), i = 1, \ldots, n-1, j = 1, \ldots, i$, and $\tilde{f}_i(z_1, \ldots, z_i)$, $i = 2, \ldots, n$, positive constants $q_i, i = 1, \ldots, n+1$, can be found such that

$$q_i + q_2 - q_1 \succ f_i|_{(q_1,\ldots,q_{i-1})}, \ i = 1, \ldots, n \quad (48)$$

$$\frac{q_{i+1} + q_i - q_j - q_{j-1}}{2} \succ f_{(i,j)}|_{(q_1,\ldots,q_i)}, \ i = 2, \ldots, n-1, \ j = 2, \ldots, i \quad (49)$$

$$\frac{q_{i+1} + q_i + q_2 - 3q_1}{2} \succ f_{(i,1)}|_{(q_1,\ldots,q_i)}, \ i = 1, \ldots, n-1 \quad (50)$$

$$q_{i+1} + q_{i-1} - 2q_i \succ \tilde{f}_i|_{(q_1,\ldots,q_i)}, \ i = 2, \ldots, n. \quad (51)$$

Proof of Theorem 4: Pick any $q_1 > 0$. The constants $q_2, \ldots, q_{n+1}$ can be picked recursively. Assuming that constants $q_1, \ldots, q_i$ have been picked, pick q_{i+1} such that

$$q_{i+1} > \max\Big\{0, q_1 - q_2 + \hat{f}_{i+1}, \max_{j=2,\ldots,i}\{q_j + q_{j-1} - q_i + 2\hat{f}_{(i,j)}\}, 3q_1 - q_2 - q_i + 2\hat{f}_{(i,1)}, 2q_i - q_{i-1} + \hat{\tilde{f}}_i\Big\} \quad (52)$$

where $\hat{f}_{i+1}$, $\hat{f}_{(i,j)}$, and $\hat{\tilde{f}}_i$ are given by

$$\begin{aligned} \hat{f}_{i+1} &= \inf\{\theta \in \mathcal{R} | \theta \succ f_{i+1}|_{(q_1,\ldots,q_i)}\} \\ \hat{f}_{(i,j)} &= \inf\{\theta \in \mathcal{R} | \theta \succ f_{(i,j)}|_{(q_1,\ldots,q_i)}\} \\ \hat{\tilde{f}}_i &= \inf\{\theta \in \mathcal{R} | \theta \succ \tilde{f}_i|_{(q_1,\ldots,q_i)}\} \end{aligned} \quad (53)$$

with $f_{n+1} \equiv 1$. The sequence $q_1, \ldots, q_{n+1}$ generated by this recursive algorithm satisfies inequalities (48)-(51). $\diamond$

Remark 3: Theorem 4 shows that Assumption A4 is always satisfied for strict-feedback systems. Hence, the design for strict-feedback systems requires only Assumptions A1 and A2. Assumption A2 can be further weakened using the technique proposed in [19] to include arbitrary (not necessarily polynomially bounded) continuous nonnegative functions of x_1 as multiplicative factors in the bounds (6)-(9). In [19], the state x_1 is not scaled and appears as a separate x_1^2 term in the Lyapunov function along with a quadratic $\xi^T P \xi$ term with $\xi = [(x_2 + \zeta(x_1))/r^{q_2}, x_3/r^{q_3}, \ldots, x_n/r^{q_n}]$ where the design freedom ζ which generates a cros term in the Lyapunov function derivative is introduced to handle the additional x_1 dependent terms in the bounds on uncertain functions. The approach in this paper, however, has the advantage that it provides a unified design for general nonlinear systems including both lower triangular and upper triangular systems.

VII. Application to Feedforward Systems

Systems for which f_i and $\tilde{f}_i$, $i = 1, \ldots, n$, have an upper triangular structure

$$\begin{aligned} f_i &= f_i(|x_{i+2}|, \ldots, |x_n|, \gamma_u(x)|u|), \ i = 1, \ldots, n-2 \\ \tilde{f}_i &= \tilde{f}_i(|x_{i+2}|, \ldots, |x_n|, \gamma_u(x)|u|), \ i = 2, \ldots, n-2 \\ f_{n-1} &= f_{n-1}(|x_n|, \gamma_u(x)|u|), \ f_n \equiv 0 \\ \tilde{f}_i &= \tilde{f}_i(|x_n|, \gamma_u(x)|u|), \ i = n-1, n \end{aligned} \quad (54)$$

and $f_{(i,j)} \equiv 0, i = 1, \ldots, n, j = 1, \ldots, i$, form a class of feedforward systems. Unlike the case of strict-feedback systems considered in Section VI, the triangularity in (54) is not sufficient to guarantee solvability of the inequalities in Assumption A4. A particular case in which solvability can be guaranteed and an explicit solution for $q_1, \ldots, q_{n+1}$ obtained is provided in Theorem 5. The conditions imposed in Theorem 5 essentially require that $|x_{i+2}|, \ldots, |x_{n-1}|$ appear linearly in the multinomials f_i, but allow $|x_n|$ and $\gamma_u(x)|u|$ to appear with arbitrary powers in f_i and $\tilde{f}_i$. In other cases, Assumption A4 can be numerically tested for feasibility. Since the inequalities in Assumption A4 are all linear in $q_1, \ldots, q_{n+1}$, it is numerically straightforward to check feasibility and obtain a solution when the inequalities are feasible. In the example in Section VIII, a system which is neither strict-feedback nor feedforward is considered and a numerical solution for the inequalities in Assumption A4 is provided.

Theorem 5: Given multinomials $f_i, i = 1, \ldots, n$, and $\tilde{f}_i, i = 2, \ldots, n$, of the form

$$f_i(z_{i+2}, \ldots, z_{n+1}) = f_{(i,n)}(z_n) f_{(i,n+1)}(z_{n+1}) \sum_{j=i+2}^{n+1} |z_j|, \quad i = 1, \ldots, n-2 \quad (55)$$

$$f_{n-1}(z_n, z_{n+1}) = f_{(i,n)}(z_n) f_{(i,n+1)}(z_{n+1}) |z_{n+1}| \quad (56)$$

$$\tilde{f}_i(z_n, z_{n+1}) = \tilde{f}_{(i,n)}(z_n) \tilde{f}_{(i,n+1)}(z_{n+1}), \ i = 2, \ldots, n \quad (57)$$

with $f_{(i,n)}$, $f_{(i,n+1)}$, $\tilde{f}_{(i,n)}$, and $\tilde{f}_{(i,n+1)}$ being multinomials (not necessarily proper), positive constants $q_1, \ldots, q_n$, and a constant q_{n+1} can be found such that

$$q_i + q_2 - q_1 \succ f_i|_{(q_{i+2},\ldots,q_{n+1})}, \ i = 1, \ldots, n-2 \quad (58)$$

$$q_{n-1} + q_2 - q_1 \succ f_{n-1}|_{(q_n, q_{n+1})} \quad (59)$$

$$q_{i+1} + q_{i-1} - 2q_i \succ \tilde{f}_i|_{(q_n, q_{n+1})}, \ i = 2, \ldots, n. \quad (60)$$

Proof of Theorem 5: Defining

$$\begin{aligned} q_i &= (n-i) + a(n-i)^2, \ i = 1, \ldots, n-1 \\ q_{n+1} &= -1 + a \ ; \ a \le \frac{1}{2n+1}, \end{aligned} \quad (61)$$

we have the relations

$$\begin{aligned} q_i + q_2 - q_1 &= q_{i+2} + 1 + a(2n - 4i - 1) \\ &\ge q_{i+2} + 4a, \ i = 1, \ldots, n-3 \\ q_{n-2} + q_2 - q_1 &\ge 8a \\ q_{n-1} + q_2 - q_1 &\ge q_{n+1} + 4a \\ q_{i+1} + q_{i-1} - 2q_i &= 2a, \ i = 2, \ldots, n-2 \\ q_{n-2} - 2q_{n-1} &= 2a \\ q_{n+1} + q_{n-1} &= 2a \end{aligned} \quad (62)$$

so that

$$\begin{aligned} q_i + q_2 - q_1 - \epsilon &\succ f_i|_{(q_{i+2},\ldots,q_{n-1},0,q_{n+1})}, i = 1, \ldots, n-3 \\ q_i + q_2 - q_1 - \epsilon &\succ f_i|_{(0,q_{n+1})}, i = n-2, n-1 \\ q_{i+1} + q_{i-1} - 2q_i - \epsilon &\succ \tilde{f}_i|_{(0,q_{n+1})}, \ i = 2, \ldots, n-1 \\ q_{n+1} + q_{n-1} - \epsilon &\succ \tilde{f}_n|_{(0,q_{n+1})} \end{aligned} \quad (63)$$

for any positive constant $\epsilon < 2a$ and any positive q_n. Choose q_n to be a small enough positive constant such that

$$\begin{aligned} \epsilon - q_n &\succ f_{(i,n)}|_{q_n}, \ i = 1, \ldots, n-1 \\ \epsilon - q_n &\succ \tilde{f}_{(i,n)}|_{q_n}, \ i = 2, \ldots, n-1 \\ \epsilon - 3q_n &\succ \tilde{f}_{(n,n)}|_{q_n}. \end{aligned} \quad (64)$$

Such a choice of q_n is always possible for any multinomials $f_{(i,n)}$ and $\tilde{f}_{(i,n)}$ since $\epsilon > 0$. This choice of q_n along with the choice of $q_1, \ldots, q_{n-1}$, q_{n+1} in (61) satisfies inequalities (58)-(60). $\diamond$

Remark 4: The conditions imposed in Theorem 5 essentially require that $|x_{i+2}|, \ldots, |x_{n-1}|$ appear linearly in the multinomials f_i, but allow $|x_n|$ and $\gamma_u(x)|u|$ to appear with arbitrary powers. Also, the multinomials $\tilde{f}_i$ are allowed to involve arbitrary powers but are required to depend only on $|x_n|$ and $\gamma_u(x)|u|$. Note that since $f_n \equiv 0$, the inequalities in (58) and (59) for $i = n$ is not required.

VIII. An Illustrative Example

Consider the fifth order system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2 \\ \dot{x}_2 &= (1 + x_1^4 x_2^2) x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (1 + x_1^2 x_2^4) x_5 + x_3^3 |u|^{\frac{1}{5}} + x_2^5 \\ \dot{x}_5 &= u + x_2^4 x_3^3. \end{aligned} \quad (65)$$

System (65) is not in either a lower triangular (strict-feedback) form or an upper triangular (feedforward) form. The results in the literature on controller designs for triangular systems can not be applied to systems such as (65). However, the design proposed in this paper can be applied to system (65). System (65) is of the form (1) with $\phi_{(1,2)} = 1$, $\phi_{(2,3)} = 1+x_1^4x_2^2$, $\phi_{(3,4)} = 1$, $\phi_{(4,5)} = 1 + x_1^2x_2^4$, $\mu = 1$, $\phi_1 = x_3^2$, $\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = x_3^3|u|^{\frac{1}{5}} + x_2^5$, and $\phi_5 = x_2^4x_3^3$. Decomposing $\phi_1,\ldots,\phi_5$ with $\phi_{(i,j)} = 0, i = 1\ldots,5, j = 1,\ldots,i$, and $\tilde{\phi}_i \equiv |\phi_i|, i = 1,\ldots,5$, Assumptions A1, A2, and A3 are satisfied with $\sigma = 1$, $\overline{\phi}_{(i,i+1)} \equiv \phi_{(i,i+1)}, i = 1,\ldots,4$, $\overline{\mu} \equiv \mu$, $\gamma_u = \tilde{\mu} = \mu^* = 1$, $f_1 = x_3^2$, $f_2 = f_3 = 0$, $f_4 = |x_3|^3|u|^{\frac{1}{5}} + |x_2|^5$, $f_5 = x_2^4|x_3|^3$, $\tilde{f}_2 = 1$, $\tilde{f}_3 = 1 + x_1^4x_2^2$, $\tilde{f}_4 = 1$, $\tilde{f}_5 = 1+x_1^2x_2^4$, $f_{(i,j)} = 0, i = 1,\ldots,4, j = 1,\ldots,i$, and $\overline{\phi}_{(5,j)} = 0, j = 1,\ldots,5$. For system (65), the set of inequalities in Assumption A4 reduces to

$$\begin{aligned}
q_1 + q_2 - q_1 &\geq 2q_3 \\
q_4 + q_2 - q_1 &\geq 3q_3 + 0.2q_6 \\
q_4 + q_2 - q_1 &\geq 5q_2 \\
q_5 + q_2 - q_1 &\geq 4q_2 + 3q_3 \\
q_3 + q_1 - 2q_2 &\geq 0 \\
q_4 + q_2 - 2q_3 &\geq 4q_1 + 2q_2 \\
q_5 + q_3 - 2q_4 &\geq 0 \\
q_6 + q_4 - 2q_5 &\geq 2q_1 + 4q_2.
\end{aligned} \tag{66}$$

Positive constants $q_1,\ldots,q_5$, and a constant q_6 to satisfy (66) are obtained numerically as

$$\begin{array}{lllllllll} q_1 & = & 6 & q_2 & = & 3 & q_3 & = & 1 \\ q_4 & = & 30 & q_5 & = & 60 & q_6 & = & 115. \end{array} \tag{67}$$

Hence, the system (65) satisfies Assumptions A1-A4 and the proposed control design technique is applicable to system (65).

As noted in Remark 2, the decomposition $|\phi_i| \leq \sum_{j=1}^{i} \phi_{(i,j)}|x_j| + \tilde{\phi}_i$ in Assumption A2 allows flexibility in assigning the terms in the bound on ϕ_i into one of the $\phi_{(i,j)}$ or $\tilde{\phi}_i$. This freedom can be utilized to aid the feasibility of the system of inequalities in Assumption A4. To illustrate this, consider system (65) with an additional term $x_1^2x_2^2x_4^2$ added to the x_4 dynamics in (65), i.e., with ϕ_4 changed to $\phi_4 = x_3^3|u|^{\frac{1}{5}} + x_2^5 + x_1^2x_2^2x_4^2$. If ϕ_4 is decomposed with $\phi_{(4,j)} \equiv 0, j = 1,\ldots,4$, and $\tilde{\phi}_4 = |\phi_4|$, Assumption A2 is satisfied with $f_4 = |x_3|^3|u|^{\frac{1}{5}} + |x_2|^5 + x_1^2x_2^2x_4^2$. The term $x_1^2x_2^2x_4^2$ in f_4 results in the introduction of the additional inequality

$$q_4 + q_2 - q_1 > 2q_1 + 2q_2 + 2q_4 \tag{68}$$

into the system of inequalities (66). The system of inequalities given by (66) and (68) does not admit a solution with positive $q_1,\ldots,q_5$. However, note that, in the solution (67) for (66), q_1 and q_2 are much smaller than q_4. This suggests that ϕ_4 should be decomposed as $|\phi_4| \leq \phi_{(4,4)}|x_4| + \tilde{\phi}_4$ with $\phi_{(4,4)} = x_1^2x_2^2|x_4|$ and $\tilde{\phi}_4 = |x_3|^3|u|^{\frac{1}{5}} + |x_2|^5$. Assumption A2 can be satisfied with $f_{(4,4)} = x_1^2x_2^2|x_4|$ which from (13) yields the inequality

$$\frac{q_5 + q_4 - q_4 - q_3}{2} \geq 2q_1 + 2q_2 + q_4. \tag{69}$$

The system of inequalities given by (66) and (69) is feasible and a solution is given by

$$\begin{array}{lllllllll} q_1 & = & 6 & q_2 & = & 3 & q_3 & = & 1 \\ q_4 & = & 69 & q_5 & = & 177 & q_6 & = & 310. \end{array} \tag{70}$$

Furthermore, noting that $\phi_{(4,5)} = 1 + x_1^2x_2^4$ and $\phi_{(2,3)} = 1 + x_1^4x_2^2$ and that $f_{(4,4)}$ is required to be an upper bound on the ratio of $\phi_{(4,4)}$ and $\sqrt{\phi_{(4,5)}\phi_{(2,3)}}$, the bound f_4 can be sharpened to $f_{(4,4)} = |x_1||x_4|$ so that the right hand side of (69) is replaced by $q_1 + q_4$. This, as could be expected, has the effect of reducing $q_4,\ldots,q_6$ so that Assumption A4 is satisfied with q_1, q_2, and q_3 as in (70), $q_4 = 45$, $q_5 = 105$, and $q_6 = 190$.

IX. Appendix

Definition A1 [20]: Let ρ be a positive constant. An $n \times n$ matrix A is said to be dual w-CUDD(ρ) if the following hold:
1) A is in lower Hessenberg form, i.e., $A_{(i,j)} \equiv 0$ for $j \geq i+2$.
2) The upper diagonal elements of A are non-zero, i.e., $A_{(i,i+1)} \neq 0$, $i = 1,\ldots,n-1$.
3) The inequalities

$$\begin{aligned}
\frac{|A_{(i,j)}|}{\sqrt{|A_{(i,i+1)}||A_{(j-1,j)}|}} &\leq \rho\,,\ i = 2,\ldots,n-1, j = 2,\ldots,i \\
\frac{|A_{(i,1)}|}{\sqrt{|A_{(i,i+1)}||A_{(1,2)}|}} &\leq \rho\,,\ i = 1,\ldots,n-1 \\
\frac{|A_{(i-1,i)}|}{|A_{(i,i+1)}|} &\leq \rho\,,\ i = 2,\ldots,n-1
\end{aligned} \tag{71}$$

are satisfied.

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