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Nonlinear $\mathcal{L}_2$ gain and regional analysis for linear systems with anti-windup compensation

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***Abstract*— In this paper, we address regional stability and performance analysis on linear control systems with linear anti-windup augmentation. We use LMIs to compute 1) an upper bound for the nonlinear $\mathcal{L}_2$ gain, 2) an estimate of the reachable set under norm bounded exogenous input and 3) an estimate of the domain of attraction with zero input. The problems are studied in a general setting where the only requirement on the closed-loop system is well-posedness and internal stability of the linear closed-loop system.**

keywords: Anti-windup systems, nonlinear $\mathcal{L}_2$ gain, reachable set, domain of attraction, LMIs

I. INTRODUCTION

Input saturation has always existed in control systems even though it did not always receive sufficient attention. The reason for its being seemingly ignored during some periods of the development of control theory is largely due to the lack of efficient tools to take the theoretical challenges that it raises. Over the last decade tremendous attention has been given to control systems with input saturation and significant advances have been reported in the literature, mostly fueled by the developments in robustness and H_∞ theory, and the more recent LMI optimization techniques.

With fully-developed optimization tools available for linear systems, we are tempted to use these tools on systems with saturating actuators. At least the closed-loop system will be well behaved and predictable around its normal range of operation where saturation does not occur. The remaining issues are how to assess the performance when the system is driven off its normal range of operation and how to counteract the effect of saturation to minimize global performance degradation. The second issue has motivated the construction of anti-windup compensators.

Anti-windup compensators are intended to maintain the performances of a linear control system in the local operating range, while guaranteeing global stability or minimizing the degradation of the global performance. The main idea is to introduce correction terms (only when saturation occurs) in the controller equations to counteract the effect of saturation. The construction of the correction term from the difference between the input and the output of the actuator involves a lot of design freedom and has gone through decades of evolution.

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Earlier anti-windup compensators were constructed heuristically from experience and simulations. Over the last decade, systematic approaches have been proposed based on robustness and H_∞ optimal control [3], [4], [6], [13], [17], and extensive numerical design algorithms have been developed based on LMI optimization tools [1], [2], [5], [8], [10], [9], [16], [14], [15], [19]. Among these papers, [15] studied the general case where the controller is dynamic, the exogenous input directly enters the actuator and there is a correction term in the output equation of the controller. In [15], static anti-windup compensator was constructed for global stabilization and reduced $\mathcal{L}_2$ gain performance. These synthesis problems were first cast as convex optimization problems with LMI constraints for the general case. Further more, it was shown with an example that the correction term in the output equation of the controller may help to reduce the global $\mathcal{L}_2$ gain significantly from what could be achieved by earlier design methods where this correction term was absent.

The recent work [8] reached further by constructing dynamic anti-windup compensators for reduced global $\mathcal{L}_2$ gain. The synthesis problems were also formulated as convex optimization problems with LMI constraints (when the order of the compensator is no less than that of the plant). Moreover, numerical examples show that dynamic compensation may achieve much better performance recovery than static compensation.

Another significant contribution of [8] is the justification of the original intention of introducing anti-windup compensation through rigorous theoretical analysis rather than through numerical demonstration. It was concluded that, for a configuration with exponentially stable plants and stabilizing linear controller, the global $\mathcal{L}_2$ gain can always be made a finite value by designing the dynamic anti-windup compensator with the algorithm developed in the paper. This conclusion promises global stability before the anti-windup compensator is constructed, even before the linear controller is designed, thus giving us full confidence in designing a linear controller for the best local performances.

While the boundedness of the global $\mathcal{L}_2$ gain gives us a guaranteed global performance of the closed-loop system, it might be conservative for practical situations where the $\mathcal{L}_2$ norm of the exogenous input is bounded below a known constant. On the other hand, for plants which are not exponentially stable, the global $\mathcal{L}_2$ gain does not exist and we would also like to determine the $\mathcal{L}_2$ gain for a class of norm bounded inputs. It is certain that the $\mathcal{L}_2$ gain will diverge to infinity at certain bound on the input norm for exponentially unstable plants. The objective of designing anti-windup compensators is to enlarge this bound. Moreover, a global $\mathcal{L}_2$ gain may fail in characterizing the anti-windup compensation performance when operating in the small to medium signals range, because worst case signals that are unreasonably large may enforce bounds that are overly conservative (see the first of our examples). These situations motivate us to characterize the nonlinear $\mathcal{L}_2$ gain for general systems with anti-windup augmentation.

An attempt to characterize the $\mathcal{L}_2$ gain for norm bounded inputs has been made in [18] for systems that may include exponentially unstable plants. This paper contains no explicit algorithm to compute the $\mathcal{L}_2$ gain and the main result is based on the assumption that the deadzone function lies within a sector $[0, K]$ with $K < I$ during the operation of the system as long as the input norm is below a certain bound (Such an assumption is also made in some other papers to explain how the global results can be adapted for regional synthesis). The relation between K and the bound on inputs is left unresolved except for a special case. However, for the general case where the exogenous input directly enters the actuator, the input of the actuator can be arbitrarily large at certain instant and there exists no $K < I$ such that the deadzone function can be bounded by $[0, K]$, even if the $\mathcal{L}_2$norm of the exogenous input is arbitrarily small. It is therefore clear that, for a general anti-windup configuration, the idea of using a narrowed sector to replace the global sector $[0, I]$ for regional performance analysis will not go through.

In this paper, we will generalize and enhance the tool developed in [12], [11] (also used in [1], [2], [7], [5], [16]) for dealing with saturation and deadzone nonlinearities. This tool has been proved more effective and flexible than the conventional idea of bounding the saturation (or deadzone) with a conic sector smaller than $[0, I]$. As a matter of fact, the condition for regional quadratic stability developed using this tool is less conservative than that obtained with the conventional method. Moreover, the condition can be stated with LMIs. In [12], [11], [1], [2], [7], [5], the input to the saturating actuator is a linear function Fx. In this paper, we will consider more complicated situations where the input to the actuator could be a nonlinear function (resulting from an algebraic loop) of the augmented state and of the exogenous input.

The main objective of this paper is to characterize the nonlinear $\mathcal{L}_2$ gain and the reachable set for a general anti-windup system where the only assumptions are well-posedness and local stability. While determining the reachable set is a meaningful problem by itself, it also facilitates the characterization of the nonlinear $\mathcal{L}_2$ gain. For completeness, we will also include results for the estimation of the domain of attraction for closed-loop systems that are not globally stable. This problem has been studied in [1], [2], [5] for cases where the input of the saturation depends linearly on the state. In this paper, we will deal with the general case where there may exist algebraic loops in the anti-windup configuration. Under such a situation, the relation between the state and the input to the actuator is nonlinear and may not be explicitly described.

This paper is organized as follows. Section II describes the anti-windup system and present three problems to be studied in the paper. Section III presents three main results including the characterization of the nonlinear $\mathcal{L}_2$ gain, the reachable sets and the estimation of the domain of attraction. Section IV uses an example to demonstrate the main results.

Notation For compact presentation of matrices, given a square matrices X and $P = P^T > 0$, we denote $\mathrm{He}X := X + X^T$ and $\mathcal{E}(P) := \{x : \ x^T P x \le 1\}$.

II. Problem statement

Consider a linear plant,

$$\mathcal{P} \quad \begin{cases} \dot{x}_p = A_p x_p + B_{p,u} u + B_{p,w} w \\ y = C_{p,y} x_p + D_{p,yu} u + D_{p,yw} w \\ z = C_{p,z} x_p + D_{p,zu} u + D_{p,zw} w \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the plant state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the exogenous input (possibly containing disturbance, reference and measurement noise), $y \in \mathbb{R}^{n_y}$ is the measurement output and $z \in \mathbb{R}^{n_z}$ is the performance output. Assume that an unconstrained controller has been designed,

$$\mathcal{C} \begin{cases} \dot{x}_c &= A_c x_c + B_{c,y} y + B_{c,w} w + v_1, \\ y_c &= C_c x_c + D_{c,y} y + D_{c,w} w + v_2, \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state and $y_c \in \mathbb{R}^{n_u}$ is the controller output, v_1 and v_2 will be used for anti-windup augmentation. In the case without plant input saturation (therefore, without any anti-windup compensation), the so-called *unconstrained closed-loop* is formed by setting

$$u = y_c, \quad v_1 = 0, \ v_2 = 0. \tag{3}$$

For our study we will assume that the unconstrained closed-loop system satisfies the following property.

Assumption 1: The unconstrained closed-loop system (1), (2), (3) is well posed and internally stable.

In the presence of actuator saturation, the relation between u and y_c is described as $u = \mathrm{sat}(y_c)$, where $\mathrm{sat}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a symmetric decentralized saturation function with its ith component depending on the ith input component y_{ci} as follows:

$$u_i = \mathrm{sat}_{\bar{u}_i}(y_{ci}) = \begin{cases} \bar{u}_i, & \text{if } y_{ci} \ge \bar{u}_i, \\ y_{ci}, & \text{if } -\bar{u}_i \le y_{ci} \le \bar{u}_i, \\ -\bar{u}_i, & \text{if } y_{ci} \le -\bar{u}_i. \end{cases} \tag{4}$$

To avoid or to minimize performance degradation caused by saturation, the closed-loop system can be augmented with the following anti-windup compensator

$$\mathcal{AW} \begin{cases} \dot{x}_{aw} &= A_{aw} x_{aw} + B_{aw}(\mathrm{sat}(y_c) - y_c) \\ v &= C_{aw} x_{aw} + D_{aw}(\mathrm{sat}(y_c) - y_c), \end{cases} \tag{5}$$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and the unconstrained interconnection (3) is replaced by the following *anti-windup interconnection*

$$u = \mathrm{sat}(y_c). \tag{6}$$

The resulting nonlinear closed-loop (1), (2), (5), (6) is depicted in Figure 1 and will be denoted *anti-windup closed-loop* henceforth and its state will be denoted by $x := [x_p^T \ x_c^T \ x_{aw}^T]^T$.

Stability and performance of the anti-windup closed-loop are usually studied by using the conic sector $[0, I]$

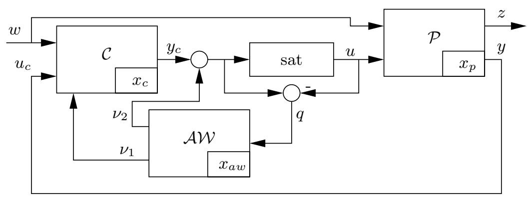

Fig. 1. The linear anti-windup closed-loop system.

to describe the saturating actuator (e.g.,[8], [15]). This description may be conservative when the system operates in a bounded region of the state space, for instance, when the $\mathcal{L}_2$ norm of w is bounded by a known value. Attempt has been made in [18] to characterize the $\mathcal{L}_2$ gain for norm bounded w by replacing the sector $[0, I]$ with a smaller one $[0, K]$ where $0 < K < I$. It was expected that this K could be obtained or estimated for a class of w such that $\|w\|_2 \leq s$. However, this approach will not work for the general situation where the disturbance directly enters the actuator, i.e., when $D_{cw} + D_{cy}D_{pyw} \neq 0$. Under this situation, y_c contains the disturbance and it can take arbitrarily large value for any number s. As a result, there exists no sector $[0, K]$ with $K < I$ to contain the saturation nonlinearity.

This paper will use a tool originally developed in [12], [11] to deal with saturation and deadzone functions. The main idea of the tool is as follows. For a scalar saturation function $\mathrm{sat}_{\bar{u}}(\cdot)$, if $|v| \leq \bar{u}$ then $\mathrm{sat}_{\bar{u}}(u)$ is between u and v for all $u \in \mathbb{R}$. Applying this tool to deal with the saturating actuator in Figure 1, we will have $\mathrm{sat}_{\bar{u}_i}(y_{ci})$ between y_{ci} and $H_i x$ as long as $|H_i x| \leq \bar{u}_i$. Here x is the combined state in Figure 1 and H_i can be any row vector of appropriate dimension. It turns out that the choice of H_i can be incorporated into LMI optimization problems.

To use this new tool of dealing with saturation, the crucial point is to guarantee that $|H_i x| \leq \bar{u}_i$ is satisfied during the operation of the closed-loop system in Figure 1. This means that we need to find a subset of the combined state space to confine all the possible trajectories (as tight as possible) for a class of norm bounded w. In other words, for the purpose of evaluating the $\mathcal{L}_2$ gain for norm bounded w, it is important to characterize the reachable set. Here we should remark that characterizing the reachable set is itself a meaningful problem for a system which has a desirable operating region of the state.

For a configuration in Figure 1 which is not globally asymptotically stable, another important problem is to estimate the domain of attraction. This problem has been addressed in [1], [2], [5] for relatively simpler situations where there exist no algebraic loop.

Based on the aforementioned motivations, we propose to address the following problems in this paper:

Problem 1: Given the anti-windup closed-loop (1), (2), (5), (6). For $s > 0$, determine an estimate of the reachable set from bounded inputs, namely a set $\mathcal{R}$, such that with $x(0) = 0$, we have

$$x(t) \in \mathcal{R} \qquad \forall t \geq 0,$$

for all $\|w\|_2 \leq s$, where $\|\cdot\|_2$ represents the $\mathcal{L}_2$ norm of its argument.

Problem 2: Given the anti-windup closed-loop (1), (2), (5), (6), determine a nonlinear $\mathcal{L}_2$ gain from w to z, namely a nondecreasing function $\gamma(\cdot) : \mathbb{R} \to \mathbb{R}$, such that the following holds:

$$\|z\|_2 \leq \gamma(\|w\|_2) \cdot \|w\|_2. \tag{7}$$

Problem 3: Given the anti-windup closed-loop (1), (2), (5), (6), determine a set $\mathcal{S}$, as large as possible, such that with $w = 0$, $\lim_{t\to\infty} x(t) = 0$ for all $x(0) \in \mathcal{S}$.

III. MAIN RESULTS

For the statement of our main results, it is useful to define the deadzone function $\mathrm{dz}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ as

$$\mathrm{dz}(y_c) := y_c - \mathrm{sat}(y_c).$$

The ith component of dz, denoted as $\mathrm{dz}_{\bar{u}_i}$, depends only on y_{ci},

$$\mathrm{dz}_{\bar{u}_i}(y_{ci}) = y_{ci} - \mathrm{sat}_{\bar{u}_i}(y_{ci}).$$

Based on this deadzone function, the anti-windup closed-loop system of Figure 1 can be represented in the following compact form, graphically represented in Figure 2:

$$\mathcal{H} \left\{ \begin{array}{rcl} \dot{x} &=& Ax + B_q q + B_w w \\ y_c &=& C_y x + D_{yq} q + D_{yw} w \\ z &=& C_z x + D_{zq} q + D_{zw} w \\ q &=& \mathrm{dz}(y_c) \ . \end{array} \right. \tag{8}$$

where $x = [x_p^T \ x_c^T \ x_{aw}^T]^T \in \mathbb{R}^n$, $n = n_p + n_c + n_{aw}$ and, by Assumption 1, the matrices appearing in (8) are uniquely defined based on the plant, controller and anti-windup matrices of (1), (2), (5), by equations (9), shown at the top of next page, where $\Delta_u := (I - D_{c,y}D_{p,yu})^{-1}$ and $\Delta_y := (I - D_{p,yu}D_{c,y})^{-1}$, and D_{aw1}, D_{aw2} correspond to v_1 and v_2, respectively.

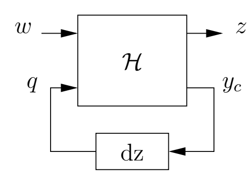

Fig. 2. Compact representation of the anti-windup closed-loop system.

Based on the compact representation (8) we are now ready to state our main results, whose proofs are omitted due to space constraints.

Theorem 1: (Reachable set by bounded inputs) Given $Q = Q^T > 0$ and $s > 0$. For the anti-windup closed-loop (1), (2), (5), (6), with $x(0) = 0$, we have

$$x(t) \in \mathcal{E}(Q^{-1}/s^2), \tag{10}$$

for all w such that $\|w\|_2 \leq s$ if there exist a diagonal $U \in \mathbb{R}^{n_u}, U > 0$ and $Y \in \mathbb{R}^{n_u \times n}$ such that

$$\mathrm{He} \begin{bmatrix} AQ & B_q U & B_w \\ C_y Q - Y & -U + D_{yq} U & D_{yw} \\ 0 & 0 & -\frac{I}{2} \end{bmatrix} \leq 0 \tag{11}$$

$$\begin{bmatrix} \frac{\bar{u}_i^2}{s^2} & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u, \tag{12}$$

$$\left[\begin{array}{c} A \\ \hline C_y \\ \hline C_z \end{array}\right] = \left[\begin{array}{ccc} A_p + B_{pu}\Delta_u D_{cy} C_{py} & B_{pu}\Delta_u C_c & B_{pu}\Delta_u C_{aw2} \\ B_{cy}\Delta_y C_{py} & A_c + B_{cy}\Delta_y D_{pyu} C_c & C_{aw1} + B_{cy}\Delta_y D_{pyu} C_{aw2} \\ 0 & 0 & A_{aw} \\ \hline \Delta_u D_{cy} C_{py} & \Delta_u C_c & \Delta_u C_{aw2} \\ \hline D_{pzu}\Delta_u D_{cy} C_{py} + Cpz & D_{pzu}\Delta_u C_c & D_{pzu}\Delta_u C_{aw2} \end{array}\right] \tag{9a}$$

$$\left[\begin{array}{c|c} B_q & B_w \\ \hline D_{yq} & D_{yw} \\ \hline D_{zq} & D_{zw} \end{array}\right] = \left[\begin{array}{c|c} -B_{pu}\Delta_u + B_{pu}\Delta_u D_{aw2} & B_{pw} + B_{pu}\Delta_u (D_{cy} D_{pyw} + D_{cw}) \\ -B_{cy}\Delta_y D_{pyu} + D_{aw1} + B_{cy}\Delta_y D_{pyu} D_{aw2} & B_{cw} + B_{cy}\Delta_y (D_{pyu} D_{cw} + D_{pyw}) \\ B_{aw} & 0 \\ \hline I - \Delta_u + \Delta_u D_{aw2} & \Delta_u (D_{cw} + D_{cy} D_{pyw}) \\ \hline -D_{pzu}\Delta_u + D_{pzu}\Delta_u D_{aw2} & D_{pzw} + D_{pzu}\Delta_u (D_{cy} D_{pyw} + D_{cw}) \end{array}\right] \tag{9b}$$

where Y_i is the ith row of Y.

Under conditions (11) and (12), an estimate of the reachable set is given by $\mathcal{E}(Q^{-1}/s^2)$. Since smaller estimates are desirable, we may formulate an optimization problem to minimize the size of $\mathcal{E}(Q^{-1}/s^2)$ under the constraints (11) and (12), which are LMIs in Q, Y and U. There are different measures of size for ellipsoids, such as the trace of Q and the determinant of Q, among which the trace of Q is a convex measure and is much easier to handle.

In practical application, we may be interested to know the range of a certain state or an output during the operation of the system. For instance, given a row vector $C \in \mathbb{R}^{1\times n}$, we would like to determine the maximal value of $|Cx(t)|$ for all $t \geq 0$. Since $x(t) \in \mathcal{E}(Q^{-1}/s^2)$, the maximal value of $|Cx(t)|$ is less than

$$\begin{aligned} \bar{\alpha} &:= (\max\{x^T C^T C x : x^T (s^2 Q)^{-1} x \leq 1\})^{1/2} \\ &= \min\{\alpha : C^T C \leq \alpha^2 (s^2 Q)^{-1}\} \\ &= \min\{\alpha : CQC^T \leq \alpha^2/s^2\}. \end{aligned}$$

To minimize $\bar{\alpha}$, we can minimize α such that $CQC^T \leq \alpha^2/s^2$ with Q satisfying (11) and (12). With α determined this way, we have $|Cx(t)| \leq \alpha$ for all t. We may choose different C's, such as $C_i, i = 1, 2, \cdots, N$, and obtain a bound α_i on $|C_i x(t)|$ for each i. The polytope formed as $\{x \in \mathbb{R}^n : |C_i x| \leq \alpha_i, i = 1, \cdots, N\}$ will also be an estimate of the reachable set.

Theorem 2: (Nonlinear $\mathcal{L}_2$ gain) Given the anti-windup closed-loop (1), (2), (5), (6). For $s > 0$, define

$$\gamma(s) := \min_{Q,U,Y,\gamma} \gamma, \text{ subject to}$$

$$\text{He}\begin{bmatrix} AQ & B_q U & B_w & 0 \\ C_y Q - Y & -U + D_{yq} U & D_{yw} & 0 \\ 0 & 0 & -\frac{I}{2} & 0 \\ C_z Q & D_{zq} U & D_{zw} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0 \tag{13}$$

$$\begin{bmatrix} \frac{\bar{u}_i^2}{s^2} & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u, \tag{14}$$

where $U > 0$ is diagonal and $Q = Q^T > 0$. Then $\gamma : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is nondecreasing and satisfies the nonlinear $\mathcal{L}_2$ bound (7).

Remark 1: If we set $Y = 0$, then the constraint (14) vanishes and (13) is equivalent to (10a) of [8], which characterizes the global finite $\mathcal{L}_2$ gain γ_G of the anti-windup closed-loop system. Hence $\gamma(\cdot)$ is always smaller than the global finite $\mathcal{L}_2$ gain. Moreover, $\gamma(s) \to \gamma_G$ as $s \to +\infty$. As a matter of fact, as $s \to +\infty$, (14) enforces $Y \to 0$. ◦

Remark 2: If $D_{yw} = 0$, and if there exists a diagonal U_0 such that $-2U_0 + D_{yq} U_0 + U_0 D_{yq}^T < 0$, then for sufficiently small s, $\gamma(s)$ coincides with the $\mathcal{L}_2$ gain of the unconstrained closed-loop system (1), (2), (3) (namely, the linear closed-loop system when the anti-windup correction is not in operation). This can be seen as follows. Let γ_0 be any number strictly larger than the $\mathcal{L}_2$ gain of the closed-loop linear system, which is simply obtained by setting $q = 0$ in (8). Then there exists $Q = Q^T > 0$ such that

$$\text{He}\begin{bmatrix} AQ & B_w & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_z Q & D_{zw} & -\frac{\gamma_0^2}{2} I \end{bmatrix} < 0$$

If we set $Y = C_y Q$, then (14) is satisfied with s sufficiently small. Moreover, the right-hand side of (13) with γ replaced with γ_0 is equivalent to (with permutation)

$$\text{He}\begin{bmatrix} AQ & B_w & 0 & B_q U \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z Q & D_{zw} & -\frac{\gamma_0^2}{2} I & D_{z,q} U \\ 0 & 0 & 0 & -U + D_{yq} U \end{bmatrix} < 0,$$

which, by Schur complement can be satisfied by letting $U = \varepsilon U_0$ for sufficiently small ε. (Note: the existence of such a U_0 guarantees the well-posedness of the unconstrained closed-loop – see [8] for details). ◦

Remark 3: The property in Remark 2 is generally not true if $D_{yw} \neq 0$. One may expect that for sufficiently small s, the state of the system will stay in a small neighborhood of the origin and the anti-windup correction should not be in effect. This misconception needs to be clarified. Although it is certain that the state will stay in a small neighborhood of the origin, the input of the actuator, y_c, can be arbitrarily large even if the $\mathcal{L}_2$ norm of w is sufficiently small, so that saturation will be activated and the $\mathcal{L}_2$ gain will be larger than that of the unconstrained closed-loop system. ◦

Theorem 3: (Estimation of the domain of attraction) Given $Q = Q^T > 0$, define $V(x) = x^T Q^{-1} x$. For the anti-windup closed-loop (1), (2), (5), (6), with $w \equiv 0$, we have $\dot{V} < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$ if there exist a diagonal $U \in \mathbb{R}^{n_u}, U > 0$ and $Y \in \mathbb{R}^{n_u \times n}$ such that

$$\mathrm{He}\begin{bmatrix} AQ & B_q U \\ C_y Q - Y & -U + D_{yq} U \end{bmatrix} < 0 \qquad (15)$$

$$\begin{bmatrix} \bar{u}_i^2 & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u \qquad (16)$$

In contrast to the reachable set, larger estimate of the domain of attraction is desirable. We may formulate different LMI optimization problems to maximize $\mathcal{E}(Q^{-1})$ with respect to different measures of set size, such as the trace of Q and the determinant of Q. We may also choose a group of vectors $x_i \in \mathbb{R}^n, i = 1, 2, \cdots, N$ and formulate a problem to maximize α such that $\alpha x_i \in \mathcal{E}(Q^{-1})$ for $i = 1, 2, \cdots, N$.

IV. Examples

Example 1: Consider the simple damped mass-spring system used in [19]. The plant matrices for this example are given by

$$\begin{bmatrix} A_p & B_{p,u} & B_{p,w} \\ C_{p,y} & D_{p,yu} & D_{p,yw} \\ C_{p,z} & D_{p,zu} & D_{p,zw} \end{bmatrix} = \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ -6.667 & -0.007 & 8.333 & 0 \\ \hline -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 \end{array}\right],$$

which arise from choosing the following parameters in the mass-spring system commented in [19]:

$$m = 0.12\, kg, \quad k = 0.8\, \frac{kg}{s^2}, \quad f_0 = 0.0008\, \frac{kg}{s}.$$

The unconstrained controller is selected according to the two degree of freedom design strategy given in [19]. The resulting matrices using the notation in (2)'s is

$$\begin{bmatrix} A_c & B_{c,y} & B_{c,w} \\ C_c & D_{c,y} & D_{c,w} \end{bmatrix} = \left[\begin{array}{ccc|c|c} -2.5 & 0 & 0 & 0 & 1.5811 \\ 193.65 & -80 & 19.365 & 122.47 & 0 \\ 50 & 0 & 0 & 31.623 & 0 \\ \hline 316.23 & -122.47 & 31.623 & 200 & 0 \end{array}\right],$$

so that the input w corresponds to a reference signal r for the mass position, and the performance output z corresponds to the tracking error $z = r - y$ (with y being the mass position).

The interesting peculiarity of this example is that both static and dynamic anti-windup designs leading to finite global $\mathcal{L}_2$ gain of the closed-loop can be determined using the algorithms in [8], [15]. In particular, the static anti-windup compensation scheme leads to the following gain

$$D_{aw} = \begin{bmatrix} -0.2658 & 0.32596 & 3.775 & 0.99 \end{bmatrix}^T,$$

which induces a global $\mathcal{L}_2$ gain of 64.43 on the compensated closed-loop. The dynamic anti-windup compensation construction leads to the following plant-order compensator

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \left[\begin{array}{cc|c} 0.011673 & 3.9564 & -0.9775 \\ -1.5433 & -305.42 & -0.11299 \\ \hline 0.00056336 & 0.0025947 & -0.0036511 \\ 4.1806 & 0.039359 & -0.011064 \\ 1.0731 & -0.019946 & 0.034351 \\ 6.8492 & -1.1084 & 0.99 \end{array}\right],$$

which, by way of the extra degrees of freedom available in dynamic anti-windup design is able to improve the finite $\mathcal{L}_2$ gain of the closed-loop to 34.5.

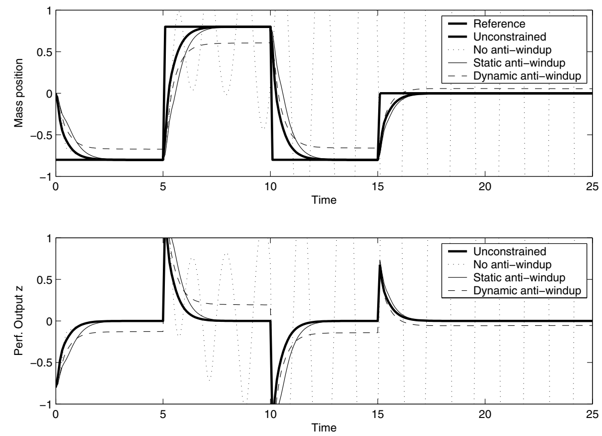

Fig. 3. Response of the system in Example 1.

Surprisingly, the simulation results reported in Figure 3 show that the compensation arising from the static anti-windup compensation (solid) is extremely more desirable than the compensation arising from the dynamic anti-windup compensation scheme (dashed). Indeed, the dashed curve converges to the desired set point at an extremely slow rate, whereas static anti-windup reaches the set-point quite rapidly. Global finite $\mathcal{L}_2$ gain analysis, however, predicted that the former should have performed better than the latter, due to its reduced closed-loop $\mathcal{L}_2$ gain.

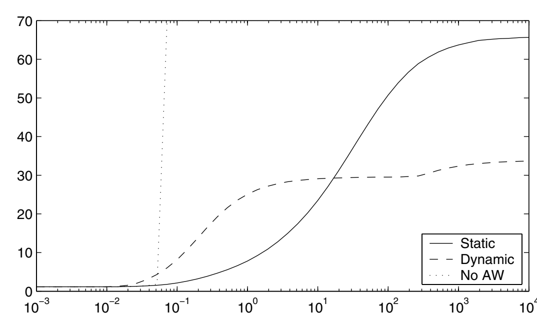

Fig. 4. Nonlinear $\mathcal{L}_2$ gains for the system in Example 1.

A possible explanation for this unexpected behavior can be found when looking at the nonlinear $\mathcal{L}_2$ gain characterizing the two compensation schemes. In particular, Figure 4 compares the nonlinear $\mathcal{L}_2$ gain for the scheme with static anti-windup (solid) to the nonlinear $\mathcal{L}_2$ gain for the scheme with dynamic anti-windup (dashed). While for large signals the dynamic scheme improves upon the static one, in the intermediate signal range, that we are interested in, the static scheme is characterized by an improved regional $\mathcal{L}_2$ gain, which predicts the simulation results shown in Figure 3.

In light of the nonlinear curves of Figure 4, it is now clear why the simulations of Figure 3 lead to such a surprising result. (Note that for those simulations, the input w corresponds to $\|w\|_2 = \sqrt{15 \cdot 0.8^2} \approx 3$.) This is one example where nonlinear $\mathcal{L}_2$ gains are necessary to appropriately characterize the $\mathcal{L}_2$ performance of anti-windup closed-loop systems operating within a reasonable signal range.

Example 2: We take an example that has been used in [15]. The plant (1) has the following matrices

$$\begin{bmatrix} A_p & B_{p,u} & B_{p,w} \\ C_{p,y} & D_{p,yu} & D_{p,yw} \\ C_{p,z} & D_{p,zu} & D_{p,zw} \end{bmatrix} = \left[\begin{array}{rr|rr|rr} -0.01 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.01 & 0 & 1 & 0 & 0 \\ \hline 0.4 & -0.5 & 0 & 0 & 0 & 0 \\ -0.3 & 0.4 & 0 & 0 & 0 & 0 \\ \hline 0.4 & -0.5 & 0 & 0 & -1 & 0 \\ -0.3 & 0.4 & 0 & 0 & 0 & -1 \end{array}\right]$$

The controller (2)'s matrices are

$$\begin{bmatrix} A_c & B_{c,y} & B_{c,w} \\ C_c & D_{c,y} & D_{c,w} \end{bmatrix} = \left[\begin{array}{rr|rr|rr} 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ \hline 0.020 & 0.025 & -2 & -2.5 & 2.0 & 2.5 \\ 0.015 & 0.020 & -1.5 & -2 & 1.5 & 2.0 \end{array}\right]$$

The $\mathcal{L}_2$ gain for the linear unconstrained closed-loop system is 1. An upper bound of the $\mathcal{L}_2$ gain resulting from the static anti-windup design in [15] is 1.55. A particular anti-windup gain achieving this is

$$D_{aw} = \begin{bmatrix} -1.4123 & -1.0593 \\ -1.7654 & -1.4124 \\ -6.2381 & -5.6498 \\ -5.6492 & -3.4139 \end{bmatrix}$$

An upper bound for the nonlinear $\mathcal{L}_2$ gain under this particular static anti-windup is plotted in Fig. 5 as a function of $\|w\|_2$ (solid curve). An upper bound for the $\mathcal{L}_2$ gain of the closed-loop system without anti-windup compensation is plotted in dashed curve as comparison. Actually, the upper bound for the $\mathcal{L}_2$ gain is bounded for bounded $\|w\|_2$ and it grows at a rate about 2.84 as $\|w\|_2$ goes to infinity.

V. CONCLUSIONS

In this paper we have addressed tools for regional stability and nonlinear performance analysis of linear closed-loop systems with linear anti-windup augmentation. Constructive techniques for numerical estimation of several closed-loop properties are given and illustrated on two simulation examples. The examples illustrate cases where nonlinear gains are mandatory for a correct interpretation of the performance achieved by the closed-loop with anti-windup compensation.

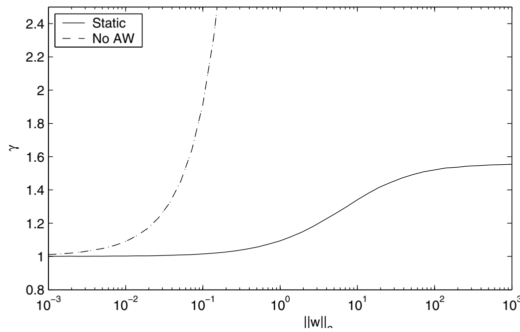

Fig. 5. Nonlinear $\mathcal{L}_2$ gains for Example 2.

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