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Order Reduction of n for Robust Adaptive Control Design of SISO Linear Systems

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Abstract— **In this paper, we propose an order-reduction design methodology to simplify the adaptive controller obtained in [1] by n integrators. We study the same class of linear systems as [1], make the same assumptions, and have the same formulation and approach to the problem. The main difference between our design methodology and that of [1] lies in the step 0 of the control design step. In this paper, we skip step 0 and immediately start the integrator backstepping procedure without stabilizing the filtered dynamics of the output. This relieves us from generating the reference trajectory for the filtered dynamics of the output and thus reducing the controller order by n. The trade-off for this order reduction is that the worst-case estimate for the expanded state vector has to be chosen as a suboptimal choice, rather than the optimal choice. Exactly the same robustness properties can be established for the reduced-order controllers as those of [1]. There is no definite performance comparison that can be made theoretically between the reduced-order controller and the full-order controller of [1]. Based on a few simulation examples, we observe that the reduced-order controller does not perform better than the full-order controller.**

Index Terms— **adaptive control, nonlinear H^∞ control, cost-to-come function, integrator backstepping.**

I. Introduction

Adaptive control has been an important research topic in control theory. Its early development since 1970s has been dominated by certainty equivalence principle [2], [3], which decouples the parameter estimation design from the control design, by making use of some standard parameter estimators and supplying the estimates to control law as if they were true parameters. The certainty equivalence based design simplifies the controller structure considerably and leads to many successful applications [4] for linear systems. On the other hand, early designs using this approach are shown to be nonrobust when the system has unmodeled dynamics and deterministic exogenous disturbance inputs [5]. Furthermore, it is unsuccessful to generalize this approach to the nonlinear systems with severe nonlinearity. All of these drawbacks motivate the study of nonlinear adaptive control design in 1990s and robust adaptive control design in 1980s and 1990s.

One of the major research focus for nonlinear adaptive control design is in (partially) feedback linearizable systems which are geometrically characterized in [6]. The introduction of integrator backstepping methodology [7] provides a systematic design tool to obtain adaptive control laws for the class of parametric strict- (or pure-) feedback nonlinear systems. This method admits great design flexibility evident in the selection of the value function and the virtual control laws. See the book [8] for a complete list of references. Adaptive control designs based on this method achieves better system performance for linear systems if the system has no disturbance input, in comparison with the certainty equivalence based adaptive controller design [8]. However, it is shown that such design may be nonrobust if the system is subject to exogenous disturbance inputs.

The main objectives of robust adaptive control are to improve transient performance, accommodate unmodeled dynamics, and tolerate exogenous disturbance inputs, which are consistent with the objectives of H^∞-optimal control. One key feature of H^∞-optimal control is that all the above objectives can be achieved by studying the disturbance attenuation property of the closed-loop system. Therefore, we formulate the robust adaptive control problem as a nonlinear H^∞-optimal control problem under imperfect state measurements. The game-theoretic approach to H^∞-optimal control problems offer the most promising tool to address nonlinear H^∞-optimal control problems [9] that has led to many successes [10], [11]. These motivate the worst-case analysis based approach to robust adaptive control design [12], [13], [1], where the measures of disturbance attenuation, asymptotic tracking, and transient performance are all incorporated into a single soft-constrained game theoretic cost function. In this approach, the unknown parameters are treated as part of the expanded state vector. An application of the cost-to-come function methodology [14] to the nonlinear H^∞-optimal control problem yields a finite dimensional estimator for the expanded system, and converts the H^∞-optimal control problem with imperfect state measurements into one with full-information measurements. Then, the integrator backstepping methodology is applied to solve this full information measurement problem. The above design paradigm has been successfully applied to identification problems [12] and robust adaptive control problems [13], [1], which indicates that the resulted identifiers and adaptive controllers have strong robustness properties. Encouraged by these successes, we continue to research further on this topic.

Motivated by the result of [1], we studied reduced-order adaptive controller design in [15]. The key to the order reduction is step 0 in the controller design step of [1]. In [15], instead of generating reference trajectory for the entire state vector of the filtered dynamics of the measured output as is done in [1], we generate only the reference trajectory for a particular linear combination of the state vector of the filtered dynamics. Thus, the controller order is reduced by $n-1$ or $n-2$ depending on the eigen structure of a feedback matrix, as compared with the full-order controller design proposed in [1]. It is proved that the closed-loop system, after order reduction, achieves the same strong robustness properties as [1]. Simulation results demonstrate a significant improvement in transient performance of the closed-loop system with the reduced-order controller.

In this paper, we continue to study the reduced-order adaptive controller design methodology, in comparison with the full-order controller achieved in [1]. We study the same class of linear

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systems as [1], make the same assumptions, and have the same problem formulation and solution approach to the problem. We assume the system under consideration has known upper bound for its dynamic order, is observable, admits a transfer function that is strictly minimum phase with a known relative degree. The true system may be uncontrollable, as long as the uncontrollable part is stable in the sense of Lyapunov, and the uncontrollable modes on the imaginary axis are uncontrollable from the disturbance input. Based on these assumptions, the unknown system is transformed into the design model, where it is linear in all of the unknown quantities. We assume that the measurement channel is noisy to avoid singularity in the estimation design. We also assume that the unknown parameter vector belongs to a convex compact set characterized by a known smooth nonnegative radially unbounded and strictly convex function $P(\bar{\theta})$. Furthermore, for any parameter vector which belongs to the convex compact set, the corresponding high frequency gain is never zero. The robust adaptive control problem is then formulated as a nonlinear H^{∞}-optimal control problem under imperfect state measurements. We adopt a game theoretic solution to this problem by separating it into estimation design and controller design steps. In estimation design, we apply the cost-to-come function methodology to obtain the finite dimensional estimator. We apply a soft projection algorithm to relieve the persistency of excitation assumption. The result of this estimation step is exactly the same as [1]. Therefore, we summarize the result of the estimation design in [1] for ease of reference. The main difference between this paper and [1] starts in the controller design step. As [1], we still apply the integrator backstepping methodology to derive the control law. But, the controller design begins from step 1, without first stabilizing the filtered dynamics of the output as step 0 does in [1]. This relieves us from generating the reference trajectory for state of the filtered dynamics of the output to track. Therefore, the controller structure can be simplified by n integrators, where n is the upper bound of the order of the unknown system. The rest of the backstepping design procedure is similar to that of [1]. The lack of step 0 results in the lack of one nonpositive drift term in the derivative of the closed-loop value function, which may degrade the performance of the reduced-order controller. In addition, in order to guarantee the boundness of the closed-loop signals, the worst-case estimate for the expanded state has to be chosen suboptimally, rather than optimally. Exactly the same robustness results are established for the reduced-order controller as the full-order counterpart of [1]. It is shown that, whenever the disturbance input is bounded and the reference trajectory and its derivatives up to rth order, where r being the relative degree of the transfer function of the true system, are bounded, then, all signals in the closed-loop system are bounded. It is also shown that the reduced-order controller achieves the desired disturbance attenuation level, whose ultimate lower bound is the noise intensity in the measurement channel. When the disturbance input is bounded and of finite energy, and the reference trajectory and its derivatives up to rth order are bounded, then we have asymptotic tracking. This completes the preview of the results of this paper.

The organization of this paper is as follows. In Section II, we list the notations to be used in the paper. We present the problem formulation of the robust adaptive control problem in Section III. Then, we present the summary of estimation design in Section IV. In Section V, the controller design is presented. In Section VI, we present the main robustness results in terms of a theorem. An example is presented in Section VII. The paper ends with some concluding remarks in Section VIII. Due to page limitation, some details, for example, the detailed proof of the theorem, detailed derivations, and simulation details of the example, are omitted in this shortened version. For interested readers, please contact us for a copy of the full version of the paper.

II. NOTATIONS

We denote the real line by $\mathbb{R}$, the set of natural numbers by $\mathbb{N}$. We say that a function f belongs to $\mathcal{C}$ if it is continuous; we say that it belongs to $\mathcal{C}_k$ if it is continuously (partial) differentiable up to kth order; we say that it is smooth if it belongs to $\mathcal{C}_\infty$. For a vector or matrix A, A' denotes its transpose. For any $z \in \mathbb{R}^n$ and any $n \times n$-dimensional symmetric matrix M, $n \in \mathbb{N}$, $|z|_M^2$ denotes $z'Mz$ and $|z|^2$ denotes $z'z$. For any $b \in \mathbb{R}$, $\operatorname{sgn}(b) = \begin{cases} -1 & b<0 \\ 0 & b=0 \\ 1 & b>0 \end{cases}$. For any matrix M, the vector $\overrightarrow{M}$ is formed by stacking up its column vectors. For any symmetric matrix M, the vector $\overleftarrow{M}$ is formed by stacking up the column vectors of the lower triangular part of M. For $n \times n$ -dimensional symmetric matrices M_1 and M_2, where $n \in \mathbb{N}$, we write $M_1 > M_2$ if $M_1 - M_2$ is positive definite; we write $M_1 \geq M_2$ if $M_1 - M_2$ is positive semi-definite. For $n \in \mathbb{N}$, the set of $n \times n$-dimensional positive definite matrices is denoted by $\mathcal{S}_{+n}$. Denote $e_{n,i}$ to be $[\mathbf{0}_{1\times(i-1)}\ 1\ \mathbf{0}_{1\times(n-i)}]'$, for $i = 1, \ldots, n$ and $n \in \mathbb{N}$. For any matrix M, $\|M\|_p$ denotes its p-induced norm, $1 \leq p \leq \infty$. $\mathcal{L}_2$ denotes the set of square integrable functions, and $\mathcal{L}_\infty$ denotes the set of bounded functions.

III. PROBLEM FORMULATION

The linear system under consideration satisfies,

Assumption 1: The linear system is known to be at most n dimensional, where $n \in \mathbb{N}$. ◇

The true system dynamics are given by

$$\dot{\grave{x}} = \grave{A}\grave{x} + \grave{B}u + \grave{D}\grave{w}; \qquad \grave{x}(0) = \grave{x}_0 \tag{1a}$$
$$y = \grave{C}\grave{x} + \grave{E}\grave{w} \tag{1b}$$

where $\grave{x}$ is the $\grave{n}$-dimensional state vector; $\grave{n} \in \mathbb{N}$; u is the scalar control input; y is the scalar system output; $\grave{w}$ is the $\grave{q}$-dimensional disturbance input, $\grave{q} \in \mathbb{N}$; all signals in the system are assumed to be continuous, i.e., in the space $\mathcal{C}$; and the matrices $\grave{A}$, $\grave{B}$, $\grave{C}$, $\grave{D}$, and $\grave{E}$ are generally unknown or partially unknown. The true system (1) satisfies the following assumption.

Assumption 2: The pair $(\grave{A}, \grave{C})$ is observable. The transfer function $H(s) = \grave{C}(sI_{\grave{n}} - \grave{A})^{-1}\grave{B}$ is known to have relative degree $r \in \mathbb{N}$, and is strictly minimum phase. The uncontrollable part (with respect to the control input u) of the unknown system is stable in the sense of Lyapunov. Any uncontrollable mode corresponding to an eigenvalue of the matrix $\grave{A}$ on the jw-axis are uncontrollable from the disturbance $\grave{w}$. ◇

Without loss of generality, we assume $\grave{n} = n$ ([1]).

By Assumption 2, there always exist a state transformation $\grave{x} = \grave{T}x$ and a disturbance transformation $w = \grave{M}\grave{w}$, such that the system can be written as

$$\dot{x} = Ax + (y\bar{A}_{211} + u\bar{A}_{212})\theta + Bu + Dw; x(0) = x_0 \tag{2a}$$
$$y = Cx + Ew \tag{2b}$$

where $\grave{T}$ is an unknown real invertible matrix; $\grave{M}$ is an unknown real $q \times \grave{q}$-dimensional matrix, $q \in \mathbb{N}$; $\theta \in \mathbb{R}^\sigma$ is the vector of unknown parameters of the system, $\sigma \in \mathbb{N}$; and the matrices A,

$\bar{A}_{211}$, $\bar{A}_{212}$, B, D, C, and E are known and admit the following structure: $A = (a_{ij})_{n\times n}$, $a_{i,i+1} = 1$, $a_{ij} = 0$, for $1 \le i \le r-1$ and $i+2 \le j \le n$; $\bar{A}_{212} = [\mathbf{0}_{\sigma\times(r-1)}\ \bar{A}'_{212\,0}\ \bar{A}'_{212r}]'$; $\bar{A}_{212\,0}$ is a row vector; $B = [\mathbf{0}_{1\times(r-1)}\ b_{p0}\ \cdots,\ b_{pn-r}]'$; $C = [1\ \mathbf{0}_{1\times(n-1)}]$; b_{pi}, $i = 1, \cdots, n-r$, are constants. Denote the elements of x by $[x_1\ \ldots\ x_n]'$. The equation (2) is called the design model which satisfies Assumptions 3–5 described below.

Assumption 3: Define $\zeta = (EE')^{-1/2} > 0$ and $L = DE'$. $\diamond$

Because of the structures of A, $\bar{A}_{212}$, and B, the high frequency gain of the transfer function $\grave{H}(s)$, b_0, is equal to $b_{p0} + \bar{A}_{212\,0}\theta$. The assumption on the parameter θ is given below.

Assumption 4: There exists a known smooth nonnegative radially unbounded strictly convex function $P : \mathbb{R}^\sigma \to \mathbb{R}$, such that θ belongs to the set $\Theta := \{\bar{\theta} \in \mathbb{R}^\sigma : P(\bar{\theta}) \le 1\}$. Furthermore, $\forall \bar{\theta} \in \Theta$, we have $\mathrm{sgn}(b_0)\,(b_{p0} + \bar{A}_{212\,0}\bar{\theta}) > 0$. $\diamond$

The following assumption is made on the reference trajectory y_d.

Assumption 5: The reference trajectory y_d is r times continuously differentiable. The signal y_d and the first r derivatives of y_d are available for feedback. Denote $Y_d := [y_d\ y_d^{(1)}\ \ldots\ y_d^{(r)}]'$ and $Y_{d0} = [y_d(0)\ y_d^{(1)}(0)\ \cdots\ y_d^{(r-1)}(0)]'$. $\diamond$

The control law is generated by $u(t) = \mu(y_{[0,t]}, Y_{d[0,t]})$. Furthermore, it must satisfy the following condition. For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}} := \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathbb{R}^r \times \mathcal{C}$, there must be a unique solution $\grave{x}_{[0,\infty)}$ for the closed-loop system, which results in a continuous control function $u_{[0,\infty)}$. The class of these admissible controllers is denoted by $\mathcal{M}_u$.

The objective of the control design is to make the system output Cx to track the reference trajectory y_d asymptotically while attenuating the effect of the uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)})$, where the exogenous input $\grave{w}$ can be taken to be any open-loop time function, as in the case of H^∞-optimal control problems. For such uncertainty, we have $(x_0, \theta, w_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)})$ belongs to the set $\mathcal{W} = \mathbb{R}^n \times \Theta \times \mathcal{C} \times \mathbb{R}^r \times \mathcal{C}$. Then, a precise definition of the objective is further given below as that in [1].

Definition 1: A controller $\mu \in \mathcal{M}_u$ is said to achieve *disturbance attenuation level* γ if there exists a nonnegative function $l(t, \theta, x, y_{[0,t]}, Y_{d[0,t]})$ and a constant $l_0 \ge 0$, such that, for all $t_f \ge 0$, $\sup_{(x_0,\theta,\grave{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \le 0$, where

$$\begin{aligned} J_{\gamma t_f} &:= \int_0^{t_f} ((x_1(\tau) - y_d(\tau))^2 + l(\tau, \theta, x(\tau), y_{[0,\tau]}, Y_{d[0,\tau]}) \\ &\quad -\gamma^2|w(\tau)|^2)d\tau - \gamma^2|[\theta' - \breve{\theta}'_0\ x'_0 - \breve{x}'_0]'|^2_{\bar{Q}_0} - l_0 \quad (3) \end{aligned}$$

where $\breve{\theta}_0 \in \Theta$ is the initial guess of θ, $\breve{x}_0$ is the initial guess of x_0, $\bar{Q}_0 > 0$ is the weighting matrix, quantifying the level of confidence in the estimate $[\breve{\theta}'_0\ \breve{x}'_0]'$; $\bar{Q}_0^{-1}$ admits the structure $\begin{bmatrix} Q_0^{-1} & Q_0^{-1}\Phi'_0 \\ \Phi_0 Q_0^{-1} & \Pi_0 + \Phi_0 Q_0^{-1}\Phi'_0 \end{bmatrix}$, where $Q_0 > 0$ and $\Pi_0 > 0$ are $\sigma\times\sigma$- and $n \times n$-dimensional, respectively. $\diamond$

The definition above aims to guarantee that, for any $t_f \ge 0$, the squared $\mathcal{L}_2$ norm of the output tracking error $x_1 - y_d$ on $[0, t_f]$ is bounded by γ^2 times the squared $\mathcal{L}_2$ norm of the transformed disturbance input $w_{[0,t]}$ plus a constant that depends only on the initial condition of the system.

The problem formulated above can be brought into the framework of H^∞-optimal control with imperfect state measurements as in [1]. Let ξ denote the expanded state vector $\xi = [\theta'\ x']'$, which satisfies the following dynamics:

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ y\bar{A}_{211} + u\bar{A}_{212} & A \end{bmatrix}\xi + \begin{bmatrix} \mathbf{0}_{\sigma\times 1} \\ B \end{bmatrix} u + \begin{bmatrix} \mathbf{0}_{\sigma\times q} \\ D \end{bmatrix} w \\ &=: \bar{A}(y,u)\xi + \bar{B}u + \bar{D}w; \quad (4a) \\ y &= \begin{bmatrix} \mathbf{0}_{1\times\sigma} & C \end{bmatrix}\xi + Ew =: \bar{C}\xi + Ew \quad (4b) \end{aligned}$$

The worst-case optimization of the cost function (3) can be carried out in two steps with the following inequality:

$$\sup_{(x_0,\theta,\grave{w}_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\grave{\mathcal{W}}} J_{\gamma t_f} \le \sup_{Y_{d0}\in\mathbb{R}^r,\, y_{[0,\infty)}\in\mathcal{C},\, y^{(r)}_{d[0,\infty)}\in\mathcal{C}} \ \sup_{(x_0,\theta,w_{[0,\infty)},Y_{d0},y^{(r)}_{d[0,\infty)})\in\mathcal{W}|y_{[0,\infty)},y^{(r)}_{d[0,\infty)},Y_{d0}} J_{\gamma t_f} \quad (5)$$

The design procedure starts with the inner supremization, which can be interpreted as the evaluation of the worst-case performance for a known output waveform. As a function of the output, the control input waveform is independent of the actual disturbance input waveform, and can be viewed as an open-loop time function. This step is actually the estimation design step discussed next in Section IV. The outer supremization can be interpreted as the computation of the worst-case measurement waveform against a given control law, which is crucial for the determination of achievability of the control objective. This step is the control design step carried out in Section V. The design function $l(t, \theta, x, y_{[0,t]}, Y_{d[0,t]})$ is selected based on the same considerations as [1]: the existence of a solution to the problem; the ease of analysis of stability and robustness of the resulting closed-loop system.

This completes the formulation of the problem. Next, we turn to the estimation and control design in the next two sections.

IV. Estimation Design

In order to set up an appropriate basis for the discussion of control design in next section, we summarize the key results of the estimation design obtained in [1].

Given Y_{d0}, the measurement waveform $y_{[0,\infty)}$, and the reference trajectory $y^{(r)}_{d[0,\infty)}$, then the control waveform $u_{[0,\infty)}$ is also known. The cost function we consider in this step is

$$\begin{aligned} J_{i\gamma t_f} = \int_0^{t_f} \Big(& |x_1(\tau) - y_d(\tau)|^2 + |\xi(\tau) - \hat{\xi}(\tau)|^2_{\bar{Q}(\tau, y_{[0,\tau]}, Y_{d[0,\tau]})} \\ & -\gamma^2|w(\tau)|^2 + 2\,(\xi - l_1(\tau, y_{[0,\tau]}, Y_{d[0,\tau]})' \\ & \cdot l_2(\tau, y_{[0,\tau]}, Y_{d[0,\tau]}) \Big) d\tau - \gamma^2|[\theta' - \breve{\theta}'_0\ x'_0 - \breve{x}'_0]'|^2_{\bar{Q}_0} \quad (6) \end{aligned}$$

where $\hat{\xi}$ is the worst-case estimate for the expanded state ξ to be designed later; l_1 and l_2 are functions to be introduced in this section; the term $2(\xi - l_1)'l_2$ is added to incorporate a soft-projection algorithm, which keeps $\breve{\theta}$ within a vicinity of Θ. $\bar{Q}$ is the nonnegative-definite weighting function which exhibits a special structure given by

$$\begin{aligned} \bar{Q}(t, y_{[0,t]}, Y_{d[0,t]}) &= (\bar{\Sigma}(t))^{-1} \begin{bmatrix} \mathbf{0}_{\sigma\times\sigma} & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \Delta \end{bmatrix} (\bar{\Sigma}(t))^{-1} \\ &\quad + \begin{bmatrix} \epsilon(t)\,(\Phi(t))'C'\,(\gamma^2\zeta^2 - 1)C\Phi(t) & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix} \\ &= \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix} \gamma^4\Pi^{-1}\Delta\Pi^{-1} \begin{bmatrix} -(\Phi(t))' \\ I_n \end{bmatrix}' \\ &\quad + \begin{bmatrix} \epsilon(t)\,(\Phi(t))'\,C'\,(\gamma^2\zeta^2 - 1)C\Phi(t) & \mathbf{0}_{\sigma\times n} \\ \mathbf{0}_{n\times\sigma} & \mathbf{0}_{n\times n} \end{bmatrix} \end{aligned}$$

where $\bar{\Sigma}$ is the worst-case covariance matrix defined later in (7f); Φ and Π are defined in (7e) and (7a), respectively, $\Delta = \gamma^{-2}\beta_\Delta\Pi + \Delta_1$ with the constant $\beta_\Delta \ge 0$ and the constant matrix $\Delta_1 > 0$; ϵ is a scalar function defined by $\epsilon(t) = \mathrm{Tr}((\Sigma(t))^{-1})/K_c, \forall t \in [0, t_f]$

or by $\epsilon(t) = 1$, $\forall t \in [0, t_f]$, $K_c \geq \gamma^2 \text{Tr}(Q_0)$ is a constant; and the matrix Σ will play the role of worst-case covariance matrix of the parameter estimation error. With the choice of $\epsilon \equiv 1$, we observe, from (7b) and (7c), Σ and s_Σ remains constant matrices on $[0, \infty)$, which simplifies the controller structure by $\frac{\sigma(\sigma+1)}{2} + 1$ integrators. In the following, we will considers $\bar{Q}$ as a function $\bar{Q} : \mathbb{R}^{n \times \sigma} \times \mathbb{R} \to \mathbb{R}^{(n+\sigma)\times(n+\sigma)}$, $\bar{Q}(\Phi, s_\Sigma)$.

On the basis of Assumption 4, a soft projection algorithm is applied. Define $\rho := \inf\{P(\bar{\theta}) \mid \bar{\theta} \in R^\sigma \text{ and } b_{p0} + \bar{A}_{212\,0}\bar{\theta} = 0\}$ then, $\rho > 1$. Fix any $\rho_o \in (1, \rho)$. The design will try to guarantee that the estimate $\check{\theta}$ lies in the open set $\Theta_o := \{\bar{\theta} \in \mathbb{R}^\sigma \mid P(\bar{\theta}) < \rho_o\}$, then, we have that the estimate $\check{b}_0 := b_{p0} + \bar{A}_{212\,0}\check{\theta}$ is bounded away from 0. This soft projection algorithm is incorporated into the cost function by setting l_1 to be $\check{\xi}$ and l_2 to be $[-P_r(\check{\theta})' \; \mathbf{0}_{1\times n}]'$, where

$$\begin{aligned} P_r(\check{\theta}) \quad &:= \quad \begin{cases} \frac{\exp((1-P(\check{\theta}))^{-1})}{(\rho_o - P(\check{\theta}))^3} \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)'; & \forall \check{\theta} \in \Theta_o \backslash \Theta \\ \mathbf{0}_{\sigma\times 1}; & \forall \check{\theta} \in \Theta \end{cases} \\ &=: \quad p_r(\check{\theta}) \left(\frac{\partial P}{\partial \theta}(\check{\theta})\right)' \end{aligned}$$

It is obvious that $P_r(\check{\theta})$ and $p_r(\check{\theta})$ are smooth on the set Θ_o, and $(\theta - \check{\theta})' P_r(\check{\theta}) \leq 0$, $\forall \check{\theta} \in \Theta_o$.

Then the identifier dynamics are summarized as follows, which is the result of [1].

$$(A - \zeta^2 LC + \beta_\Delta/2I_n)\Pi + \Pi\,(A - \zeta^2 LC + \beta_\Delta/2I_n)' \\ -\Pi C'\,(\zeta^2 - \gamma^{-2}) C\Pi + DD' - \zeta^2 LL' + \gamma^2 \Delta_1 = \mathbf{0} \tag{7a}$$

$$\dot{\Sigma} = -(1-\epsilon)\Sigma\Phi'C'\,(\gamma^2\zeta^2 - 1)C\Phi\Sigma; \;\; \Sigma(0) = \gamma^{-2}Q_0^{-1} \tag{7b}$$

$$\dot{s}_\Sigma = (\gamma^2\zeta^2 - 1)\,(1-\epsilon)C\Phi\Phi'C'; \;\; s_\Sigma(0) = \gamma^2 \text{Tr}(Q_0) \tag{7c}$$

$$A_f = A - \zeta^2 LC - \Pi C'C\,(\zeta^2 - \gamma^{-2}) \tag{7d}$$

$$\dot{\Phi} = A_f\Phi + y\bar{A}_{211} + u\bar{A}_{212}; \quad \Phi(0) = \Phi_0 \tag{7e}$$

$$(\bar{\Sigma}(t))^{-1} = \begin{bmatrix} (\Sigma(t))^{-1} + \gamma^2(\Phi(t))'\Pi^{-1}\Phi(t) & -\gamma^2(\Phi(t))'\Pi^{-1} \\ -\gamma^2\Pi^{-1}\Phi(t) & \gamma^2\Pi^{-1} \end{bmatrix}$$

$$\bar{\Sigma}(t) = \begin{bmatrix} \Sigma(t) & \Sigma(t)\,(\Phi(t))' \\ \Phi(t)\Sigma(t) & \gamma^{-2}\Pi + \Phi(t)\Sigma(t)\,(\Phi(t))' \end{bmatrix}; \tag{7f}$$

$$\dot{\check{\theta}} = -\Sigma P_r(\check{\theta}) - \Sigma\Phi'C'\,(y_d - C\check{x}) - \begin{bmatrix} \Sigma & \Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c \\ +\gamma^2\zeta^2\Sigma\Phi'C'\,(y - C\check{x}); \;\; \check{\theta}(0) = \check{\theta}_0 \tag{7g}$$

$$\dot{\check{x}} = -\Phi\Sigma P_r(\check{\theta}) + A\check{x} - (\gamma^{-2}\Pi + \Phi\Sigma\Phi')C'\,(y_d - C\check{x}) \\ -\begin{bmatrix} \Phi\Sigma & \gamma^{-2}\Pi + \Phi\Sigma\Phi' \end{bmatrix} \bar{Q}\xi_c + Bu + (y\bar{A}_{211} + u\bar{A}_{212})\check{\theta} \\ +\zeta^2\,(\Pi C' + \gamma^2\Phi\Sigma\Phi'C' + L)\,(y - C\check{x}); \;\; \check{x}(0) = \check{x}_0 \tag{7h}$$

$$W(\xi, \check{\xi}, \bar{\Sigma}) = |\xi - \check{\xi}|^2_{\bar{\Sigma}^{-1}} = \gamma^2|x - \check{x} - \Phi\,(\theta - \check{\theta})|^2_{\Pi^{-1}} \\ +|\theta - \check{\theta}|^2_{\Sigma^{-1}}; \;\; W : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \to \mathbb{R} \tag{7i}$$

$$\dot{W}(\xi, \check{\xi}, \hat{\xi}, \bar{\Sigma}, y_d, u, w) = -|x_1 - y_d|^2 + 2(\theta - \check{\theta})'P_r(\check{\theta}) \\ -\gamma^4|x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} - \gamma^2\zeta^2|y - C\check{x}|^2 \\ -\epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi'C'C\Phi} - \gamma^2|w - w_*(\xi, \check{\xi}, \bar{\Sigma}, w)|^2 \\ +\gamma^2|w|^2 + |C\check{x} - y_d|^2 + |\xi_c|^2_{\bar{Q}(\Phi, s_\Sigma)} \tag{7j}$$

where $s_\Sigma(t) := \text{Tr}((\Sigma(t))^{-1})$, which is introduced to avoid the computation of $(\Sigma(t))^{-1}$ on line; Φ is a filtered signal of y and u; $\check{\theta}$ is an estimate for θ; $\check{x}$ is an estimate for x; W is the value function for the estimation design step, which is defined when $\Sigma > 0$; $\xi_c = \hat{\xi} - \check{\xi}$ and $\check{\xi} = [\check{\theta}' \; \check{x}']'$; $w_* : \mathbb{R}^{n+\sigma} \times \mathbb{R}^{n+\sigma} \times \mathcal{S}_{+(n+\sigma)} \times \mathbb{R}^q \to \mathbb{R}^q$ denotes the worst-case disturbance for the estimation step, (7j) holds as long as $\Sigma > 0$ and $\check{\theta} \in \Theta_o$.

Lemma 1: Consider the dynamic equation (7b) for the covariance matrix Σ. Let $\gamma \geq \zeta^{-1}$, $Q_0 > 0$, with $\epsilon = K_c^{-1}s_\Sigma$ and $K_c \geq \gamma^2\text{Tr}(Q_0)$. Then, the matrix Σ is upper and lower bounded: $K_c^{-1}I_\sigma \leq \Sigma(t) \leq \Sigma(0)$, $\gamma^2\text{Tr}(Q_0) \leq \text{Tr}((\Sigma(t))^{-1}) \leq K_c$, $\forall t \in [0, t_f]$, whenever it exists on $[0, t_f]$ and Φ is continuous on $[0, t_f]$.

We list Assumptions 7 and 8 in [1] below.

Assumption 6: If the matrix $A - \zeta^2 LC$ is Hurwitz, then choose the desired disturbance attenuation level $\gamma \geq \zeta^{-1}$. In case $\gamma = \zeta^{-1}$, choose $\beta_\Delta \geq 0$ such that $A - \zeta^2 LC + \beta_\Delta/2I_n$ is Hurwitz. If the matrix $A - \zeta^2 LC$ is not Hurwitz, then choose the desired disturbance attenuation level $\gamma > \zeta^{-1}$. $\diamond$

Assumption 7: The matrix Π_0 is chosen as the unique positive-definite solution to the algebraic Riccati equation (7a) $\diamond$

Assumption 6 makes the observations that the quantity ζ^{-1} is the ultimate lower bound on the achievable performance level for the adaptive system using the proposed method. Assumption 7 sets Π to a constant and simplifies the identifier. Under Assumption 7, the matrix A_f is Hurwitz.

To simplify the estimator dynamics, we may generate Φ by the following $3n$-dimensional prefiltering system for y and u: $\dot{\eta} = A_f\eta + p_n y$, $\eta(0) = \eta_0$; $\dot{\lambda} = A_f\lambda + p_n u$, $\lambda(0) = \lambda_0$; $\dot{\lambda}_o = A_f\lambda_o$, $\lambda_o(0) = p_n$, where p_n is an n-dimensional vector such that (A_f, p_n) is controllable. Then, Φ is given by

$$\Phi = \begin{bmatrix} A_f^{n-1}\eta & \ldots & A_f\eta & \eta \end{bmatrix} M_f^{-1}\bar{A}_{211} + \begin{bmatrix} A_f^{n-1}\lambda & \ldots & A_f\lambda & \lambda \end{bmatrix} \\ \cdot M_f^{-1}\bar{A}_{212} + \begin{bmatrix} A_f^{n-1}\lambda_o & \cdots & A_f\lambda_o & \lambda_o \end{bmatrix} M_f^{-1}\Phi_{o0} \tag{8}$$

where the matrix $M_f = \begin{bmatrix} A_f^{n-1}p_n & \ldots & A_f p_n & p_n \end{bmatrix}$, and $\eta_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}^n$, and $\Phi_{o0} \in \mathbb{R}^{n\times\sigma}$ are such that (8) holds at $t = 0$. This completes the summary of the identification design of [1]. We now turn to the controller design step in the next section.

V. Control Design

We describe in this section the reduced-order controller design for the uncertain system. By (5) and (7j), we have

$$\sup_{(x_0, \theta, \dot{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \mathcal{W}} J_{\gamma t_f} \leq \sup_{Y_{d0} \in \mathbb{R}^r, y_{[0,\infty)} \in \mathcal{C}, y^{(r)}_{d[0,\infty)} \in \mathcal{C}} \\ \int_0^{t_f} (|C\check{x}(\tau) - y_d(\tau)|^2 + |\xi_c(\tau)|^2_{\bar{Q}(\Phi(\tau), s_\Sigma(\tau))} \\ +\check{l}(t, y_{[0,t]}, Y_{d[0,t]}) - \gamma^2\zeta^2|y(\tau) - C\check{x}(\tau)|^2)d\tau - l_0 \tag{9}$$

where $\check{l} = l - |\xi - \hat{\xi}|^2_{\bar{Q}} + 2(\theta - \check{\theta})'P_r(\check{\theta})$, and (9) holds if all closed-loop signals exist, $\Sigma(t) > 0$, and $\check{\theta}(t) \in \Theta_o$, on $[0, t_f]$. The control design aims to guarantee that the supremum is less than or equal to zero for all measurement waveforms. A new variable $v := \zeta\,(y - C\check{x})$ is introduced, instead of considering y as the maximizing variable. The control design problem now is an H^∞-optimal control problem with full information measurements.

We apply the integrator backstepping methodology to design the control law u. Express the elements of $\check{x}$ as $\begin{bmatrix} \check{x}_1 & \cdots & \check{x}_n \end{bmatrix}'$. Express row vectors of Φ as $\begin{bmatrix} \Phi_1' & \cdots & \Phi_n' \end{bmatrix}'$. In view of a non-negative definite weighting term on ξ_c in (9), the design for ξ_c will be carried out last. Therefore, we, for convenience, set ξ_c to be zero in the backstepping procedure. We start the backstepping procedure directly from Step 1 to stabilize $\check{x}_1$ by viewing $\check{x}_2$ as the virtual control. Then, we view $\check{x}_3$ as the virtual control to stabilize $\check{x}_1$ and $\check{x}_2$. Continue in this fashion, until step r, where u appears in the dynamics of $\check{x}_r$, we complete the design for u.

As a result of the backstepping design, we have obtained X_{jo}, X_{ja}, X_{jd}, $\mathcal{D}_{jo}$, $\mathcal{D}_{ja}$, $\mathcal{D}_{jd}$, $\gamma_j \in \mathbb{R}$, $\alpha_0 : \mathcal{D}_{1o} \to \mathbb{R}$, $\beta_j : \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}$, $j = 1, \cdots, r$, $\alpha_j : \mathcal{D}_{jo} \times$

$\mathcal{D}_{ja} \times \mathcal{D}_{jd} \to \mathbb{R}$, $j = 1, \cdots, r-1$, $\nu_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$, $V : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \to \mathbb{R}$, and $\mu : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}$ such that $X_{j+1\,o} = [X'_{jo}\ X_{ja}\ X'_{jd}]'$, $\mathcal{D}_{j+1\,o} = \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd}$, $j = 1, \ldots, r-1$, α_j, $j = 0, \cdots, r-1$, ν_r, V, and μ are smooth, $\gamma_j > 0$, $\beta_j(X_{jo}, X_{ja}, X_{jd}) \geq c_{\beta_j} > 0$, $\forall (X_{jo}, X_{ja}, X_{jd}) \in \mathcal{D}_{jo} \times \mathcal{D}_{ja} \times \mathcal{D}_{jd}$, $X_{ja} = \check{x}_j$, $X_{jo} = [y_d\ \check{\theta}'\ \overleftarrow{\Sigma}'\ s_\Sigma\ \check{x}_1\ y_d^{(1)}\ \Phi_1\ \cdots\ \check{x}_{j-1}\ y_d^{(j-1)}\ \Phi_{j-1}]'$, $\mathcal{D}_{jo} := \{X_{jo} \mid \check{\theta} \in \Theta_o\}$, $j = 1, \cdots, r$, $X_{rd} = [y_d^{(r)}\ \Phi_r\ \check{x}_{r+1}\ \cdots\ \check{x}_n\ \Phi_{r+1}\ \cdots\ \Phi_n]'$, $\mathcal{D}_{rd} = \{X_{rd}\}$, and

$$u = \mu(X_{ro}, X_{ra}, X_{rd}) \tag{10}$$

$$V(X_{ro}, X_{ra}) = \sum_{j=1}^{r} \gamma_j \left(\check{x}_j - \alpha_{j-1}(X_{jo})\right)^2$$

$$\begin{aligned}
&\dot{V}(X_{ro}, X_{ra}, X_{rd}, u, v)|_{u=\mu(X_{ro}, X_{ra}, X_{rd})} = -(\check{x}_1 - \alpha_0(X_{1o}))^2 \\
&-\sum_{j=1}^{r} \beta_j \left(\check{x}_j - \alpha_{j-1}(X_{jo})\right)^2 - \gamma^2 \left(v - \nu_r(X_{ro}, X_{ra}, X_{rd})\right)^2 \\
&+\gamma^2 v^2;\ \forall (X_{ro}, X_{rd}) \in \mathcal{D}_{ro} \times \mathcal{D}_{rd}, \quad \forall X_{ra} \in \mathcal{D}_{ra}, \quad \forall v \in \mathbb{R}
\end{aligned}$$

where $\xi_c = 0$ in the above. The closed-loop system admits the state vector

$$X := \left[\, \theta'\ x'\ X'_{ro}\ X_{ra}\ \Phi_r\ \check{x}_{r+1}\ \cdots\ \check{x}_n\ \Phi_{r+1}\ \cdots\ \Phi_n \,\right]'$$

which belongs to the set $\mathcal{D} := \{X \mid \Sigma > 0, s_\Sigma > 0, \check{\theta} \in \Theta_o\}$. Based on the value functions of the estimation design and control design, we obtain value function for the closed-loop system: $U = V + W$, where $U : \mathcal{D} \to \mathbb{R}$ is smooth. Then,

$$\begin{aligned}
&\dot{U}(X, y_d^{(r)}, \hat{\xi}, w) = -|x_1 - y_d|^2 - \epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} \\
&-\gamma^4 |x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} + 2(\theta - \check{\theta})' P_r(\check{\theta}) \\
&+|\xi_c + \frac{1}{2}\varsigma_r(X_{ro}, X_{ra}, X_{rd})|^2_{\bar{Q}(\Phi, s_\Sigma)} - \sum_{j=1}^{r} \beta_j z_j^2 + \gamma^2 |w|^2 \\
&-\gamma^2 |w - w_{opt}(X, y_d^{(r)})|^2 - \frac{1}{4}|\varsigma_r(X_{ro}, X_{ra}, X_{rd})|^2_{\bar{Q}(\Phi, s_\Sigma)} \\
&\forall X \in \mathcal{D}, \quad \forall y_d^{(r)} \in \mathbb{R}, \quad \forall \hat{\xi} \in \mathbb{R}^{\sigma+n}, \quad \forall w \in \mathbb{R}^q
\end{aligned} \tag{11}$$

where the worst-case disturbance $w_{opt} : \mathcal{D} \times \mathbb{R} \to \mathbb{R}^q$ is

$$\begin{aligned}
w_{opt}(X, y_d^{(r)}) \quad &= \quad \zeta E' \nu_r + \gamma^{-2}\,(I_q - \zeta^2 E' E)\bar{D}' \bar{\Sigma}^{-1}\,(\xi - \check{\xi}) \\
&\quad\ + \zeta^2 E' C\,(\check{x} - x)
\end{aligned}$$

and $\varsigma_r : \mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^{n+\sigma}$ is smooth.

The design of ξ_c is based on two considerations: (1) to guarantee $[0\ \gamma^{-2}\Pi]\bar{Q}\xi_c$ is bounded, which is applied later in the proof of Theorem 1; (2) to guarantee the terms $|\xi_c + \frac{1}{2}\varsigma_r|^2_{\bar{Q}} - \frac{1}{4}|\varsigma_r|^2_{\bar{Q}} + |\xi_c|^2_{\bar{Q}}$ is non-positive. Then, we choose ξ_c to be

$$\begin{aligned}
\xi_c &:= \mu_\xi(X_{ro}, X_{ra}, X_{rd}) \\
&= -\frac{1}{2}\varepsilon_0(X_{ro}, X_{ra}, X_{rd})\varsigma_r(X_{ro}, X_{ra}, X_{rd})
\end{aligned} \tag{12}$$

where

$$\varepsilon_0 := \frac{|k_c|}{\sqrt{k_c^2 + |[\mathbf{0}\ \gamma^{-2}\Pi]\bar{Q}\varsigma_r|^2 + 1}}$$

and $k_c \in \mathbb{R}$ is a constant. Clearly, μ_ξ: $\mathcal{D}_{ro} \times \mathcal{D}_{ra} \times \mathcal{D}_{rd} \to \mathbb{R}^{\sigma+n}$ is a smooth function. We observe that $\xi_c \equiv 0$ when $k_c = 0$. This special case will result in simpler control system structure.

This completes the design for the entire closed-loop system, which involves state X. We shall turn to study the robustness and tracking properties of the closed-loop system.

VI. Main Result

In this section, we investigate the robustness and tracking properties of the closed-loop system with the reduced-order controller. The closed-loop system dynamics can be expressed as

$$\dot{X} = F(X, y_d^{(r)}) + G(X)\grave{M}\grave{w};\ X(0) = X_0 \tag{13}$$

where the function F and G are smooth mappings of $\mathcal{D} \times \mathbb{R}$ and $\mathcal{D}$, respectively; and the initial state X_0 satisfies $X_0 \in \mathcal{D}_0 := \{X_0 \in \mathcal{D} \mid \theta \in \Theta,\ \check{\theta}_0 \in \Theta,\ \Sigma(0) = \gamma^{-2}Q_0^{-1} > 0,\ s_\Sigma(0) = \gamma^2\mathrm{Tr}(Q_0) \leq K_c\}$. As described in [1], the value function U satisfies the Hamilton-Jacobi-Isaacs equation

$$\begin{aligned}
&\frac{\partial U}{\partial X}(X)F(X, y_d^{(r)}) + \frac{1}{4\gamma^2}\frac{\partial U}{\partial X}(X)G(X)\,(G(X))'\left(\frac{\partial U}{\partial X}(X)\right)' \\
&+Q(X, y_d^{(r)}) = 0;\ \forall X \in \mathcal{D}, \forall y_d^{(r)} \in \mathbb{R}
\end{aligned} \tag{14}$$

where $Q : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is smooth and given by

$$\begin{aligned}
Q(X, y_d^{(r)}) = &|x_1 - y_d|^2 + \gamma^4 |x - \hat{x} - \Phi\,(\theta - \hat{\theta})|^2_{\Pi^{-1}\Delta\Pi^{-1}} \\
&+\epsilon\,(\gamma^2\zeta^2 - 1)|\theta - \hat{\theta}|^2_{\Phi' C' C \Phi} - 2(\theta - \check{\theta})' P_r(\check{\theta}) \\
&-(\varepsilon_0^2 - 2\varepsilon_0)|\varsigma_r|^2_{\bar{Q}}/4 + \sum_{j=1}^{r} \beta_j z_j^2
\end{aligned}$$

The stability of the closed-loop system can not be deduced directly from the value function U, which is not a positive-definite function for the closed-loop system. The following theorem will state the stability properties of the closed-loop system, it shows that the closed-loop system admits strong robustness properties.

Theorem 1: Consider the robust adaptive control problem formulated in Section III, with Assumptions 1–7 holding. Then, the robust adaptive controller μ defined by (10), with ξ_c given by (12), achieves the following strong robustness properties.

1) Given $c_w \geq 0$ and $c_d \geq 0$, there exists a constant $c_c \geq 0$ and a compact set $\Theta_c \subset \Theta_o$, such that, for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $|x_0| \leq c_w$, $|\grave{w}(t)| \leq c_w$, $|Y_d(t)| \leq c_d$, $\forall t \in [0, \infty)$, all closed-loop state variables x, $\check{x}$, $\check{\theta}$, Σ, s_Σ, and Φ are bounded as follows: $\forall t \in [0, \infty)$, $|x(t)| \leq c_c$, $|\check{x}(t)| \leq c_c$, $\check{\theta}(t) \in \Theta_c$, $|\overrightarrow{\Phi(t)}| \leq c_c$, $K_c^{-1} I_\sigma \leq \Sigma(t) \leq \gamma^{-2}Q_0^{-1}$, $\gamma^2\mathrm{Tr}(Q_0) \leq s_\Sigma(t) \leq K_c$. Therefore, there is a compact set $S \subset \mathcal{D}$ such that $X(t) \in S$, $\forall t \in [0, \infty)$. Hence, there exists a constant $c_u \geq 0$ such that $|u(t)| \leq c_u$, $|\hat{\xi}(t)| \leq c_u$, $|\eta(t)| \leq c_u$, $|\lambda(t)| \leq c_u$, and $|\lambda_o(t)| \leq c_u$, $\forall t \in [0, \infty)$.
2) The controller $\mu \in \mathcal{M}_u$ and achieves disturbance attenuation level γ for any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$.
3) For any uncertainty $(x_0, \theta, \grave{w}_{[0,\infty)}, Y_{d0}, y^{(r)}_{d[0,\infty)}) \in \grave{\mathcal{W}}$ with $\grave{w}_{[0,\infty)} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $Y_{d[0,\infty)} \in \mathcal{L}_\infty$, $\lim_{t\to\infty}(x_1(t) - y_d(t)) = 0$.

VII. Example

Consider a circuit shown in Figure 1(a). The resistance R is 1Ω. The capacitor C and the inductor L are linear and time invariant, $L = 1H$. v_i is a dependent voltage source. v_e is an unknown sinusoidal voltage source. v_{w1} is an unmeasured exogenous voltage disturbance; i_s is an unmeasured exogenous current disturbance. v_o is the voltage output. Our objective is to achieve the desired voltage output $v_o - v_{w1}$ by adjusting v_i.

The simulation results are shown in Figures 1(b), (c), and (d). The tracking error converge to zero and the parameter estimates converge to the true value, which is consistent with our theoretical

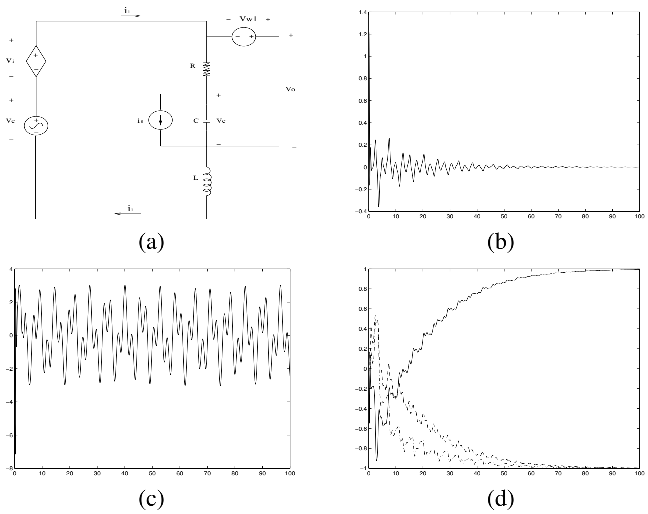

Fig. 1. System response for reduced-order controller with no arbitrarily varying disturbance $\dot{w}$.
(a) RLC circuit; (b) Tracking error; (c) Control input; (d)Parameter estimates.

findings. The control input is bounded by 8. The transient and steady-state performance are comparable with those of the full-order controller. Based on a couple of simulation examples, the reduced-order controller does not appear to perform better than the full-order controller.

VIII. Conclusions

In this paper, we have studied the reduced-order adaptive control design for SISO linear systems with noisy output measurements. The main contribution of this paper is that the controller structure is simplified by n integrators without any additional assumptions or sacrifice in the strong robustness properties of the closed-loop system, in comparison with the full-order controller designed in [1]. With the same class of systems, the same assumptions and the same formulation and approach to the problem as [1], the adaptive control design is carried out in two steps, first step is the estimation design step, second step is the controller design step. The estimation design results are completely the same as [1]. The main difference between this paper and [1] lies in the controller design step. At this step, although we still employ the backstepping methodology as [1], we start the controller design from step 1, without first stabilizing η dynamics as step 0 does in [1]. Then, there is no need to generate η_d dynamics for η to track, the dynamic order of the controller is thereby reduced by n. The step 0 of [1] may be skipped and does not affect the robustness of the closed-loop system because the dynamics of $\tilde{x} - \Phi\tilde{\theta}$ admits desired structure which may be substituted for the $\tilde{\eta}$ dynamics. On the other hand, the dynamics of $\tilde{x} - \Phi\tilde{\theta}$ have an undesirable feature that they depend also on $\hat{\xi}$. Then, the trade-off for the order-reduction is that $\hat{\xi}$ may not be set to the optimal choice. Also, the lack of step 0 leads to the disappearance of a nonpositive drift term related to η dynamics in the derivative of the closed-loop value function. All of these trade-offs may be responsible for the degradation of the closed-loop responses after order reduction, which has been observed in a simulation example. However, as mentioned above, the reduced-order controller is shown to guarantee exactly the same strong robustness properties as [1], namely, all closed-loop signals are bounded when the disturbance input is bounded and reference trajectory and its derivatives up to rth order are bounded; the controller achieves the desired disturbance attenuation level, whose ultimate lower bound is the noise intensity in the measurement channel; the noiseless output asymptotically tracks the reference trajectory when the disturbance input is bounded and of finite energy and the reference trajectory and its derivatives up to rth order are bounded.

This order-reduction scheme proposed in this paper may be applied to SISO linear systems with partly measured disturbances, and to SISO linear systems with repeated noisy measurements, these works are straightforward and easy to achieve, which will not be pursued in the near future. Another order-reduction scheme under consideration is to reduce the dynamic order of the controller by $n - 1$ integrators. The main difference between this order-reduction scheme and the one described in this paper is that $n-1$ order-reduction scheme allows the optimal selection of $\hat{\xi}$.

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