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# Statistical Control for Smart Base-Isolated Buildings via Cost Cumulants and Output Feedback Paradigm

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***Abstract*— A derivation of an output feedback solution to the statistical control problem which minimizes a finite linear combination of first $k$ cost cumulants of the finite-horizon integral quadratic cost associated with a linear stochastic system, when the controller measures the noisy states is first presented. Of course for the $k = 1$ case, there has been the celebrated Kalman result. Also, there are already some results for infinite-horizon output-feedback risk sensitive control which minimizes a certain linear combination of denumerable cost cumulants. The contribution of this paper is to fill in the other values of $k \geq 2$ here, in hoping to complete the picture of linear $k$-cost-cumulant ($k$CC) control theory. The phase I benchmark problem for response control of smart base-isolated buildings then follows, in which the realistic 8-story, steel braced building representing mid rise buildings in the city of Los Angeles, California is protected by a linear elastometric isolation system under seven different earthquakes with fault normal and parallel components acting in two perpendicular directions. A group of nine non-dimensionalized evaluation criteria is used to assess the performance of both base isolation system and benchmark structure. Simulation results indicate that the active control of the augmenting devices at the isolation layer using output feedback statistical control paradigm offers broad improvement in structural performance over the baseline LQG design. Therefore, statistical control is well applicable to the protection of civil structures.**

## I. Introduction

Recent work reported at various American Control Conferences [4], [5], [6] has continued to show that the statistical control method using state feedback law is quite competitive with other modern control design paradigms for several benchmark problems of protecting building and bridge structures from earthquakes and winds. Due to effectiveness of response control offered by this statistical control, the need to address the output feedback and filtering in the extension of state feedback measurement indicates an exciting and challenging final step in this problem class. Pham [7] has successfully found an output feedback solution to the finite horizon $k$-cost-cumulant ($k$CC) control problem on which the following development is mostly based on. Consider a stochastic linear dynamical system modeled on $[t_0, t_F]$ with the initial condition $x(t_0) = x_0$

$$dx(t) = (A(t)x(t) + B(t)u(t))dt + G(t)dw(t)\,, \qquad (1)$$

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where the coefficients $A \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{n\times m})$, $G \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{n\times p})$; the initial state $x_0 \in \mathbb{R}^n$ is known; and the vector noise $w(t) \in \mathbb{R}^p$ is the p-dimensional stationary Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ over $[t_0, t_F]$ with the correlation of increments for all $\tau, \xi \in [t_0, t_F]$

$$E\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\} = W|\tau - \xi|,\ W \in \mathbb{R}^{p\times p}.$$

All observable outputs $y \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_F]; \mathbb{R}^r))$ are the linear functions of $x \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_F]; \mathbb{R}^n))$ by means of a linear transformation matrix $C \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{r\times n})$

$$dy(t) = C(t)x(t)dt + dv(t)\,, \qquad t \in [t_0, t_F]\,, \qquad (2)$$

with the measurement noise $v(t) \in \mathbb{R}^r$ modeled by another r-dimensional stationary Wiener process having the correlation of increments for all $\tau, \xi \in [t_0, t_F]$

$$E\left\{[v(\tau) - v(\xi)][v(\tau) - v(\xi)]^T\right\} = \Xi|\tau - \xi|\,, \quad \Xi \in \mathbb{R}^{r\times r}.$$

The system and measurement noise processes may be correlated by the relation and for all $\tau, \xi \in [t_0, t_F]$

$$E\left\{[w(\tau) - w(\xi)][v(\tau) - v(\xi)]^T\right\} = \Gamma|\tau - \xi|\,,$$

where $\Gamma \in \mathbb{R}^{p\times r}$ satisfies the inequality $W - \Gamma\Xi^{-1}\Gamma^T \geq 0$.

Further, the system (1) is associated with a finite-horizon integral quadratic form (IQF) random cost $J : [t_0, t_F] \times \mathbb{R}^n \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_F]; \mathbb{R}^m)) \mapsto \mathbb{R}^+$ such that

$$J(t_0, x_0; u) = \int_{t_0}^{t_F} \left[x^T(\tau)Q(\tau)x(\tau) + u^T(\tau)R(\tau)u(\tau)\right] d\tau + x^T(t_F)Q_F x(t_F)\,, \qquad (3)$$

where the state weighting symmetric matrix $Q \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{n\times n})$ is positive semidefinite as is the terminal penalty weighting symmetric matrix $Q_F \in \mathbb{R}^{n\times n}$. The control effort weighting symmetric matrix $R \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{m\times m})$ is positive definite.

In order to control the system, it is necessary and sufficient to know the probability density function of $x(t)$ conditioned on the accumulated observations $y(\tau)$ for $0 \leq \tau \leq t$. It is observed that the random process (1) is linear and Gaussian. We shall consider linear controllers. Thus, the conditional probability density function of $x(t)$ is Gaussian and can now be parameterized by the conditional mean $\widehat{x}(t)$ and covariance $\Sigma(t)$. Moreover, it is known that the conditional covariance $\Sigma(t)$ can be determined prior to any observations made and any control policies

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applied once the structure of the system and the second-order statistics of the noise processes are specified. It is then enough to conclude that the conditional mean $\widehat{x}(t)$ is the sufficient statistical information which should be taken into account when forming a candidate control policy. Note that under the system assumptions of linearity and Gaussianness, cost cumulants of the traditional IQF random cost are quadratic-affine in the initial state mean; they ensure that the information structure for optimal control law is linear. Control is allowed to be a function of state-estimate; that is, for some function $\widehat{\gamma}$, the class of linear feedback control laws is considered as follows.

*Definition 1.1:* Linear Feedback Control Laws.

$$u(t) = \widehat{\gamma}(t, \widehat{x}(t)) = K(t)\,\widehat{x}(t)\,, \qquad t \in [t_0, t_F]\,, \quad (4)$$

where $K \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{m\times n})$ is an admissible feedback gain.

Now the original system (1) can be augmented as follows

$$\begin{bmatrix} d\widehat{x} \\ d\widetilde{x} \end{bmatrix}(t) = \left( \begin{bmatrix} A & LC \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widetilde{x} \end{bmatrix}(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \right) dt + \begin{bmatrix} 0 & L \\ G & -L \end{bmatrix} \begin{bmatrix} dw \\ dv \end{bmatrix}(t)\,, \quad (5)$$

for all $t \in [t_0, t_F]$ and the initial condition

$$\begin{bmatrix} \widehat{x} \\ \widetilde{x} \end{bmatrix}(t_0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}.$$

In view of the control law (4), the augmented dynamical system (5) defined on $[t_0, t_F]$ can be rewritten as

$$dx_a(t) = F_a(t)x_a(t)dt + G_a(t)dw_a(t)\,, \quad (6)$$
$$x_a(t_0) = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix}.$$

The IQF random cost in (3) $J : [t_0, t_F] \times \mathbb{R}^n \times \mathcal{C}([t_0, t_F]; \mathbb{R}^{2n\times 2n}) \mapsto \mathbb{R}^+$, is rewritten as

$$J(t_0, x_0; N_a) = \int_{t_0}^{t_F} x_a^T(\tau)N_a(\tau)x_a(\tau)d\tau + x_a^T(t_F)Q_a^F x_a(t_F)\,. \quad (7)$$

Herein matrix coefficients in (6)-(7) are defined by

$$F_a = \begin{bmatrix} A + BK & LC \\ 0 & A - LC \end{bmatrix}, \quad G_a = \begin{bmatrix} 0 & L \\ G & -L \end{bmatrix},$$

$$Q_a^F = \begin{bmatrix} Q_F & Q_F \\ Q_F & Q_F \end{bmatrix}, \quad N_a = \begin{bmatrix} Q + K^TRK & Q \\ Q & Q \end{bmatrix},$$

and the augmented noise $w_a$ is the $(p+r)$-dimensional stationary Wiener process with correlation of increments $E\left\{[w_a(\tau) - w_a(\xi)][w_a(\tau) - w_a(\xi)]^T\right\} = W_a|\tau - \xi|$ and

$$dw_a = \begin{bmatrix} dw \\ dv \end{bmatrix}, \quad W_a = \begin{bmatrix} W & \Gamma \\ \Gamma^T & \Xi \end{bmatrix}.$$

Cost cumulants for any finite-horizon integral quadratic form cost (7) associated with the augmented system formulation (6) are then defined in the following theorem.

*Theorem 1.2:* Cumulants in Output-Feedback Problem. Let $F_a \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{2n\times 2n})$, $G_a \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{2n\times (p+r)})$, symmetric $N_a \in \mathcal{C}([t_0, t_F]; \mathbb{R}^{2n\times 2n})$ positive semidefinite, and symmetric $W_a \in \mathbb{R}^{(p+r)\times(p+r)}$ positive definite. Suppose that

$$dx_a(t) = F_a(t)x_a(t)dt + G_a(t)dw_a(t)\,,$$

where $x_a(t_0) = (x_0^T, 0)^T$. Then, for $k \in \mathbb{Z}^+$, the $kth$ cost cumulant of the random cost (7) is given by

$$\kappa_k = x_{a0}^T H_a(t_0, k)x_{a0} + D_a(t_0, k)\,, \quad (8)$$

where the symmetric matrix solutions $H_a(t_0, i) \in \mathcal{C}^1([t_0, t_F]; \mathbb{R}^{2n\times 2n})$ and $D_a(t_0, i) \in \mathcal{C}^1([t_0, t_F]; \mathbb{R})$ are evaluated at $\alpha = t_0$ of the cumulant-generating equations

$$\frac{d}{d\alpha}H_a(\alpha, 1) = -F_a^T(\alpha)H_a(\alpha, 1) - H_a(\alpha, 1)F_a(\alpha) - N_a(\alpha), \quad (9)$$

$$\frac{d}{d\alpha}H_a(\alpha, i) = -F_a^T(\alpha)H_a(\alpha, i) - H_a(\alpha, i)F_a(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_a(\alpha, j)G_a(\alpha)W_aG_a^T(\alpha)H_a(\alpha, i-j), \quad (10)$$

$$\frac{d}{d\alpha}D_a(\alpha, i) = -\mathrm{Tr}\left\{H_a(\alpha, i)G_a(\alpha)W_aG_a^T(\alpha)\right\}, \quad (11)$$

with the terminal conditions $H_a(t_F, 1) = Q_a^F$, $H_a(t_F, i) = 0$ for $2 \le i \le k$, and $D_a(t_F, i) = 0$ for $1 \le i \le k$.

## II. Problem Statements

Let $2n \times 2n$ symmetric matrices $H_a(\alpha, i)$ and $G_a(\alpha)W_aG_a^T(\alpha)$ be partitioned as follows

$$H_a(\alpha, i) = \begin{bmatrix} H_{i,1}(\alpha) & H_{i,2}(\alpha) \\ H_{i,2}^T(\alpha) & H_{i,3}(\alpha) \end{bmatrix}, \quad 1 \le i \le k\,,$$

$$G_a(\alpha)W_aG_a^T(\alpha) = \begin{bmatrix} \Pi_1(\alpha) & \Pi_2(\alpha) \\ \Pi_2^T(\alpha) & \Pi_3(\alpha) \end{bmatrix},$$

where

$$\Pi_1(\alpha) = L(\alpha)\Xi L^T(\alpha)\,,$$
$$\Pi_2(\alpha) = G(\alpha)\Gamma L^T(\alpha) - L(\alpha)\Xi L^T(\alpha)\,,$$
$$\Pi_3(\alpha) = G(\alpha)WG^T(\alpha) - G(\alpha)\Gamma L^T(\alpha) - L(\alpha)\Gamma^T G^T(\alpha) + L(\alpha)\Xi L^T(\alpha)\,,$$

and use the fact that $x_{a0} = [x_0^T\, 0]^T$. For $1 \le i \le k$, it is often convenient to define the right members of the equations (9)-(10) as the mappings with the rules of action

$$\mathcal{F}_i : [t_0, t_F] \times (\mathbb{R}^{n\times n})^{3k} \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\mathcal{F}_{k+i} : [t_0, t_F] \times (\mathbb{R}^{n\times n})^{3k} \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\mathcal{F}_{2k+i} : [t_0, t_F] \times (\mathbb{R}^{n\times n})^{3k} \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\mathcal{G}_i : [t_0, t_F] \times (\mathbb{R}^{n\times n})^{3k} \mapsto \mathbb{R}$$

$$
\begin{aligned}
\mathcal{F}_1(\alpha,\mathcal{H},K) &= -[A(\alpha)+B(\alpha)K(\alpha)]^T\mathcal{H}_1(\alpha)\\
&\quad -\mathcal{H}_1(\alpha)[A(\alpha)+B(\alpha)K(\alpha)] - K^T(\alpha)R(\alpha)K(\alpha) - Q(\alpha),\\
\mathcal{F}_i(\alpha,\mathcal{H},K) &= -[A(\alpha)+B(\alpha)K(\alpha)]^T\mathcal{H}_i(\alpha)\\
&\quad -\mathcal{H}_i(\alpha)[A(\alpha)+B(\alpha)K(\alpha)]\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)\Pi_1(\alpha)\mathcal{H}_{i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)\Pi_2^T(\alpha)\mathcal{H}_{k+i-j}^T(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{k+i-j}^T(\alpha),\\
\mathcal{F}_{k+1}(\alpha,\mathcal{H},K) &= -[A(\alpha)+B(\alpha)K(\alpha)]^T\mathcal{H}_{k+1}(\alpha)\\
&\quad -\mathcal{H}_{k+1}(\alpha)[A(\alpha)-L(\alpha)C(\alpha)]\\
&\quad -\mathcal{H}_1(\alpha)L(\alpha)C(\alpha) - Q(\alpha),\\
\mathcal{F}_{k+i}(\alpha,\mathcal{H},K) &= -[A(\alpha)+B(\alpha)K(\alpha)]^T\mathcal{H}_{k+i}(\alpha)\\
&\quad -\mathcal{H}_{k+i}(\alpha)[A(\alpha)-L(\alpha)C(\alpha)] - \mathcal{H}_i(\alpha)L(\alpha)C(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)\Pi_2^T(\alpha)\mathcal{H}_{2k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}(\alpha),\\
\mathcal{F}_{2k+1}(\alpha,\mathcal{H},K) &= -[A(\alpha)-L(\alpha)C(\alpha)]^T\mathcal{H}_{2k+1}(\alpha)\\
&\quad -\mathcal{H}_{2k+1}(\alpha)[A(\alpha)-L(\alpha)C(\alpha)] - Q(\alpha)\\
&\quad -C^T(\alpha)L^T(\alpha)\mathcal{H}_{k+1}(\alpha) - \mathcal{H}_{k+1}^T(\alpha)L(\alpha)C(\alpha),\\
\mathcal{F}_{2k+i}(\alpha,\mathcal{H},K) &= -[A(\alpha)-L(\alpha)C(\alpha)]^T\mathcal{H}_{2k+i}(\alpha)\\
&\quad -\mathcal{H}_{2k+i}(\alpha)[A(\alpha)-L(\alpha)C(\alpha)]\\
&\quad -C^T(\alpha)L^T(\alpha)\mathcal{H}_{k+i}(\alpha) - \mathcal{H}_{k+i}^T(\alpha)L(\alpha)C(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}^T(\alpha)\Pi_1(\alpha)\mathcal{H}_{k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{2k+j}(\alpha)\Pi_2(\alpha)\mathcal{H}_{k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{k+j}^T(\alpha)\Pi_2^T(\alpha)\mathcal{H}_{2k+i-j}(\alpha)\\
&\quad -\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{2k+j}(\alpha)\Pi_3(\alpha)\mathcal{H}_{2k+i-j}(\alpha),\\
\mathcal{G}_i(\alpha,\mathcal{H}) &= -\mathrm{Tr}\{\mathcal{H}_i(\alpha)\Pi_1(\alpha)\} - \mathrm{Tr}\{\mathcal{H}_{k+i}(\alpha)\Pi_2(\alpha)\}\\
&\quad -\mathrm{Tr}\left\{\mathcal{H}_{k+i}^T(\alpha)\Pi_2^T(\alpha)\right\} - \mathrm{Tr}\{\mathcal{H}_{2k+i}(\alpha)\Pi_3(\alpha)\},
\end{aligned}
$$

where the components of $\mathcal{H}$ and $\mathcal{D}$ are defined by

$$
\begin{aligned}
\mathcal{H} &= (\mathcal{H}_1,\ldots,\mathcal{H}_k,\mathcal{H}_{k+1},\ldots,\mathcal{H}_{2k},\mathcal{H}_{2k+1},\ldots,\mathcal{H}_{3k}),\\
&= (H_{1,1},\ldots,H_{k,1},H_{1,2},\ldots,H_{k,2},H_{1,3},\ldots,H_{k,3}),\\
\mathcal{D} &= (\mathcal{D}_1,\ldots,\mathcal{D}_k) = (D_1,\ldots,D_k).
\end{aligned}
$$

Hence, the product system of motion equations on $[t_0,t_F]$

$$
\begin{aligned}
&\mathcal{F}_1\times\cdots\times\mathcal{F}_{3k}:[t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^{m\times n}\mapsto(\mathbb{R}^{n\times n})^{3k}\\
&\mathcal{G}_1\times\cdots\times\mathcal{G}_k:[t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\mapsto\mathbb{R}^k
\end{aligned}
$$

in the output-feedback $k$CC control is rewritten as

$$
\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha)), \qquad \mathcal{H}(t_F)=\mathcal{H}_F, \quad (12)
$$

$$
\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H}(\alpha)), \qquad \mathcal{D}(t_F)=\mathcal{D}_F, \quad (13)
$$

under obvious definitions

$$
\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_{3k}, \qquad \mathcal{G}=\mathcal{G}_1\times\cdots\times\mathcal{G}_k,
$$

together with the terminal value conditions

$$
\begin{aligned}
\mathcal{H}_F &= Q_F\times\underbrace{0\times\cdots\times0}_{\text{(k-1)-times}}\times Q_F\times\underbrace{0\times\cdots\times0}_{\text{(k-1)-times}}\\
&\quad\times Q_F\times\underbrace{0\times\cdots\times0}_{\text{(k-1)-times}}, \qquad \mathcal{D}_F=\underbrace{0\times\cdots\times0}_{\text{k-times}}.
\end{aligned}
$$

**Remark.** Observe that the solutions of the equations (12)-(13) depend on the admissible control gain $K$. In the sequel and elsewhere, when we wish to make this dependence clear we shall use the notations $\mathcal{H}(\alpha,K)$ and $\mathcal{D}(\alpha,K)$ to denote the solution trajectories of the dynamics (12)-(13) with the given $K$ feedback gain.

*Definition 2.1:* $k$CC Performance Index.
Fix $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\geq 0\}_{i=1}^k$ with $\mu_1>0$. For given initial data $x_0$, the performance index of output feedback $k$CC control problems

$$
\widehat{\phi_0}:[t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k\mapsto\mathbb{R}^+
$$

is given by

$$
\begin{aligned}
\widehat{\phi_0}\,(t_0,\mathcal{H}(t_0,K),\mathcal{D}(t_0,K)) &= \sum_{i=1}^k\mu_i\kappa_i(K),\\
&= x_0^T\sum_{i=1}^k\mu_i\mathcal{H}_i(t_0,K)x_0 + \sum_{i=1}^k\mu_i\mathcal{D}_i(t_0,K), \quad (14)
\end{aligned}
$$

where the real constants $\mu_i$ represent extra degrees of control design freedom associated with additional cost cumulants included in the performance index.
The development in the sequel is motivated by the excellent treatment in [1] and is intended to follow it closely. Let the terminal time $t_F$ and states $(\mathcal{H}_F,\mathcal{D}_F)$ be given. Then the other end condition involved the initial time $t_0$ and state pair $(\mathcal{H}_0,\mathcal{D}_0)$ are specified by a target set requirement.

*Definition 2.2:* Target Set.
$(t_0,\mathcal{H}_0,\mathcal{D}_0)\in\widehat{\mathcal{M}}$, where the target set $\widehat{\mathcal{M}}$ is a closed subset of $[t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k$.

*Definition 2.3:* Admissible Feedback Gains.
Let the compact subset $\overline{K} \subset \mathbb{R}^{m\times n}$ be the set of allowable gain values. For given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, let $\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}$ be the class of $\mathcal{C}([t_0,t_F];\mathbb{R}^{m\times n})$ with values $K(\cdot) \in \overline{K}$ for which the performance index 2.1 is finite and trajectory solutions to the dynamic equations of motion

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\,, \qquad \mathcal{H}(t_F) = \mathcal{H}_F\,, \quad (15)$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H}(\alpha))\,, \qquad \mathcal{D}(t_F) = \mathcal{D}_F\,, \quad (16)$$

reach $(t_0,\mathcal{H}_0,\mathcal{D}_0) \in \widehat{\mathcal{M}}$.
The finite-horizon output-feedback $k$CC optimization problem is to minimize the performance index over all control gains $K = K(\cdot)$ in the admissible $\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}$.

*Definition 2.4:* $k$CC Optimization.
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then the finite-horizon output-feedback $k$CC control optimization is given by

$$\min_{K(\cdot)\in\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}} \widehat{\phi_0}\left(t_0,\mathcal{H}(t_0,K),\mathcal{D}(t_0,K)\right)$$

subject to the equations of motion, for all $\alpha \in [t_0,t_F]$

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\,, \qquad \mathcal{H}(t_F) = \mathcal{H}_F\,,$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H}(\alpha))\,, \qquad \mathcal{D}(t_F) = \mathcal{D}_F\,.$$

To embed the stated optimization into a larger optimal control problem, we shall then denote the terminal time and states by $(\varepsilon,\mathcal{Y},\mathcal{Z})$ rather than $(t_F,\mathcal{H}_F,\mathcal{D}_F)$. Thus the value of this optimization problem is now considered depending on its terminal value conditions. Ultimately, it leads to the definition of a value function.

*Definition 2.5:* Value Function.
Suppose that $(\varepsilon,\mathcal{Y},\mathcal{Z}) \in [t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k$ is given and fixed. Then the value function $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is defined by

$$\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) = \inf_{K(\cdot)\in\widehat{\mathcal{K}}_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}} \widehat{\phi_0}\left(t_0,\mathcal{H}(t_0,K),\mathcal{D}(t_0,K)\right)\,.$$

Conventionally, set $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) = \infty$ when $\widehat{\mathcal{K}}_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$ is empty. To avoid notational cumbersome, we are suppressing the dependence of trajectory solutions on $K$ in the proofs. To construct functions $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$–the candidates for value functions, we first need the concept of a reachable set.

*Definition 2.6:* Reachable Set.
Let the *reachable set* $\widehat{\mathcal{Q}_1}$ be defined as follows

$$\left\{(\varepsilon,\mathcal{Y},\mathcal{Z}) \in [t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k : \widehat{\mathcal{K}}_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu} \neq 0\right\}\,.$$

**Remark.** The definition says that $\widehat{\mathcal{Q}_1}$ contains a set of points $(\varepsilon,\mathcal{Y},\mathcal{Z})$ from which it is possible to reach the target set $\widehat{\mathcal{M}}$ with some trajectory pairs corresponding to a continuous control gain.

*Theorem 2.7:* HJB Equation for $k$CC Control.
Let $(\varepsilon,\mathcal{Y},\mathcal{Z})$ be any interior point of the reachable set $\widehat{\mathcal{Q}_1}$ at which the scalar-valued function $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is differentiable. If there is an optimal control gain $K^*$ in $\widehat{\mathcal{K}}_{\varepsilon,\mathcal{Y},\mathcal{Z};\mu}$, then the partial differential equation of dynamic programming

$$0 = \min_{K\in\overline{K}}\left\{\frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{F}(\varepsilon,\mathcal{Y},K)) + \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\cdot\mathrm{vec}(\mathcal{G}(\varepsilon,\mathcal{Y})) + \frac{\partial}{\partial\varepsilon}\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})\right\} \quad (17)$$

is satisfied and $\mathrm{vec}(\cdot)$ the vectorizing operator of enclosed entities. The minimum in (17) is achieved by the left limit $K^*(\varepsilon)^-$ of the optimal control gain at $\varepsilon$.

*Definition 2.8:* Refined Reachable Set.
Let the admissible control gain $K$ be a function

$$K = K(\alpha,\mathcal{H},\mathcal{D})\,.$$

from a subset $\widehat{\mathcal{Q}}$ of $[t_0,t_F]\times(\mathbb{R}^{n\times n})^{3k}\times\mathbb{R}^k$ into $\overline{K}$ such that for each $(\varepsilon,\mathcal{Y},\mathcal{Z})$ in $\widehat{\mathcal{Q}}$ there is a unique solution pair $\mathcal{H}(\alpha;\varepsilon,\mathcal{Y})$ and $\mathcal{D}(\alpha;\varepsilon,\mathcal{Z})$ of the equations of motion

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H},K(\alpha,\mathcal{H},\mathcal{D}))\,, \quad \frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H})\,,$$

on $t_0(\varepsilon,\mathcal{Y},\mathcal{Z}) \leq \alpha \leq \varepsilon$ with $\mathcal{H}(\varepsilon;\varepsilon,\mathcal{Y}) = \mathcal{Y}$ and $\mathcal{D}(\varepsilon;\varepsilon,\mathcal{Z}) = \mathcal{Z}$, such that $(\alpha,\mathcal{H}(\alpha;\varepsilon,\mathcal{Y}),\mathcal{D}(\alpha;\varepsilon,\mathcal{Z})) \in \widehat{\mathcal{Q}}$ for $t_0(\varepsilon,\mathcal{Y},\mathcal{Z}) \leq \alpha \leq \varepsilon$ and $(t_0(\varepsilon,\mathcal{Y},\mathcal{Z}),\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y}),\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z})) \in \widehat{\mathcal{M}}$.

**Remark.** Clearly, $\widehat{\mathcal{Q}}$ is a subset of the reachable set $\widehat{\mathcal{Q}_1}$. Whenever the $k$CC control problem formulated for any set of terminal conditions $(\varepsilon,\mathcal{Y},\mathcal{Z})$ in the reachable set $\widehat{\mathcal{Q}}$ admits an optimal feedback control gain, then the equality $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z}) = \widehat{\phi_0}\left(t_0(\varepsilon,\mathcal{Y},\mathcal{Z}),\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y}),\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z})\right)$ holds between the performance index and the value function.

*Theorem 2.9:* Differentiability of Value Function.
Let $K^*(\alpha,\mathcal{H},\mathcal{D})$ be an optimal feedback gain and let $t_0(\varepsilon,\mathcal{Y},\mathcal{Z})$ and $(\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y}),\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z}))$ be the target time and target states for the trajectories of

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha,\mathcal{H},K^*(\alpha,\mathcal{H},\mathcal{D})), \quad \frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha,\mathcal{H}),$$

with the terminal condition $(\varepsilon,\mathcal{Y},\mathcal{Z})$. Then $\mathcal{V}(\varepsilon,\mathcal{Y},\mathcal{Z})$ is differentiable at each point at which $t_0(\varepsilon,\mathcal{Y},\mathcal{Z})$ and $\mathcal{H}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Y})$ and $\mathcal{D}(t_0(\varepsilon,\mathcal{Y},\mathcal{Z});\varepsilon,\mathcal{Z})$ are differentiable with respect to $(\varepsilon,\mathcal{Y},\mathcal{Z})$.

*Theorem 2.10:* Verification Theorem.
Fix $k \in \mathbb{Z}^+$ and let $\mathcal{W}(\varepsilon,\mathcal{Y},\mathcal{Z})$ be a continuously differentiable solution of the HJB equation (17) which satisfies the boundary condition and $(t_0,\mathcal{H}_0,\mathcal{D}_0) \in \widehat{\mathcal{M}}$

$$\mathcal{W}(t_0,\mathcal{H}_0,\mathcal{D}_0) = \widehat{\phi_0}\left(t_0,\mathcal{H}_0,\mathcal{D}_0\right)\,. \quad (18)$$

Let $(t_F,\mathcal{H}_F,\mathcal{D}_F)$ be a point of $\widehat{\mathcal{Q}}$, $K$ a control gain in $\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}$ and $\mathcal{H}$, $\mathcal{D}$ the corresponding solutions of the equations of motion

$$\frac{d}{d\alpha}\mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad \mathcal{H}(t_F) = \mathcal{H}_F,$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha)), \qquad \mathcal{D}(t_F) = \mathcal{D}_F.$$

Then $\mathcal{W}(\alpha, \mathcal{H}(\alpha), \mathcal{D}(\alpha))$ is a non-increasing function of $\alpha$. If $K^*$ is a control gain in $\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}$ defined on $[t_0, t_F]$ with the corresponding solutions, $\mathcal{H}^*$ and $\mathcal{D}^*$ of the preceding equations such that for $\alpha \in [t_0, t_F]$

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \\ &+ \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Y})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \mathrm{vec}(\mathcal{F}(\alpha, \mathcal{H}^*(\alpha), K^*(\alpha))) \\ &+ \frac{\partial}{\partial\, \mathrm{vec}(\mathcal{Z})}\mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \mathrm{vec}(\mathcal{G}(\alpha, \mathcal{H}^*(\alpha))), \end{aligned} \tag{19}$$

then $K^*$ is an optimal control gain in $\widehat{\mathcal{K}}_{t_F,\mathcal{H}_F,\mathcal{D}_F;\mu}$ and

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z}), \tag{20}$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}, \mathcal{Z})$ is the value function.

## III. Output Feedback Control Solution

What follows is an optimal output-feedback solution to the finite-horizon $k$CC control problem by means of the dynamic programming approach. Typically, we would like to parameterize the terminal time and states of the optimization problem by $(\varepsilon, \mathcal{Y}, \mathcal{Z})$ in place of $(t_F, \mathcal{H}_F, \mathcal{D}_F)$. Hence, for $\varepsilon \in [t_0, t_F]$, the states of the system (15)-(16) defined on the interval $[t_0, \varepsilon]$ have the terminal values denoted by $\mathcal{H}(\varepsilon) = \mathcal{Y}$ and $\mathcal{D}(\varepsilon) = \mathcal{Z}$. Observe that the performance index (14) is quadratic affine in fixed $x_0$. Therefore, a candidate value function is guessed of the form

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \mathcal{Z}) = x_0^T \sum_{i=1}^{k} \mu_i(\mathcal{Y}_i + \mathcal{E}_i(\varepsilon))x_0 + \sum_{i=1}^{k} \mu_i(\mathcal{Z}_i + \mathcal{T}_i(\varepsilon))$$

where $\mathcal{E}_i \in \mathcal{C}^1([t_0, t_F]; \mathbb{R}^{n\times n})$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0, t_F]; \mathbb{R})$ are to be determined. Because allotted space is not available for a detailed analysis of this candidate function, the final result which comes from [7] is given instead.

*Theorem 3.1:* Output-Feedback $k$CC Control Solution. Let (1)-(3) be defined as before. For the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the optimal output-feedback gain $K^*(\alpha)$ that minimizes $\widehat{\phi_0}(t_0, \mathcal{H}(t_0, K), \mathcal{D}(t_0, K))$ is given by

$$K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r \mathcal{H}_r^*(\alpha); \; \widehat{\mu}_r = \frac{\mu_i}{\mu_1}, \tag{21}$$

where $\{\mathcal{H}_r(\alpha) \geq 0\}_{r=1}^k$ are unique symmetric solutions of the backward-in-time control design equations with the previously specified terminal conditions

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_1^*(\alpha) &= -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_1^*(\alpha) \\ &\quad - \mathcal{H}_1^*(\alpha)[A(\alpha) + B(\alpha)K^*(\alpha)] \\ &\quad - K^{*T}(\alpha)R(\alpha)K^*(\alpha) - Q(\alpha), \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_r^*(\alpha) &= -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_r^*(\alpha) \\ &\quad - \mathcal{H}_r^*(\alpha)[A(\alpha) + B(\alpha)K^*(\alpha)] \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)\Pi_1\mathcal{H}_{r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^*(\alpha)\Pi_2\mathcal{H}_{r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)\Pi_2^T\mathcal{H}_{k+r-s}^{*T}(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^*(\alpha)\Pi_3\mathcal{H}_{k+r-s}^{*T}(\alpha), \end{aligned} \tag{23}$$

and of the backward-in-time auxiliary equations

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{k+1}^*(\alpha) &= -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_{k+1}^*(\alpha) \\ &\quad - \mathcal{H}_{k+1}^*(\alpha)[A(\alpha) - L(\alpha)C(\alpha)] \\ &\quad - \mathcal{H}_1^*(\alpha)L(\alpha)C(\alpha) - Q(\alpha), \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{k+r}^*(\alpha) &= -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_{k+r}^*(\alpha) \\ &\quad - \mathcal{H}_{k+r}^*(\alpha)[A(\alpha) - L(\alpha)C(\alpha)] - \mathcal{H}_r^*(\alpha)L(\alpha)C(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)\Pi_1\mathcal{H}_{k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^*(\alpha)\Pi_2\mathcal{H}_{k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\alpha)\Pi_2^T\mathcal{H}_{2k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^*(\alpha)\Pi_3\mathcal{H}_{2k+r-s}^*(\alpha), \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{2k+1}^*(\alpha) &= -[A(\alpha) - L(\alpha)C(\alpha)]^T \mathcal{H}_{2k+1}^*(\alpha) \\ &\quad - \mathcal{H}_{2k+1}^*(\alpha)[A(\alpha) - L(\alpha)C(\alpha)] - Q(\alpha) \\ &\quad - C^T(\alpha)L^T(\alpha)\mathcal{H}_{k+1}^*(\alpha) - \mathcal{H}_{k+1}^{*T}(\alpha)L(\alpha)C(\alpha), \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{H}_{2k+r}^*(\alpha) &= -[A(\alpha) - L(\alpha)C(\alpha)]^T \mathcal{H}_{2k+r}^*(\alpha) \\ &\quad - \mathcal{H}_{2k+r}^*(\alpha)[A(\alpha) - L(\alpha)C(\alpha)] \\ &\quad - C^T(\alpha)L^T(\alpha)\mathcal{H}_{k+r}^*(\alpha) - \mathcal{H}_{k+r}^{*T}(\alpha)L(\alpha)C(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^{*T}(\alpha)\Pi_1\mathcal{H}_{k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{2k+s}^*(\alpha)\Pi_2\mathcal{H}_{k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{k+s}^{*T}(\alpha)\Pi_2^T\mathcal{H}_{2k+r-s}^*(\alpha) \\ &\quad - \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_{2k+s}^*(\alpha)\Pi_3\mathcal{H}_{2k+r-s}^*(\alpha), \end{aligned} \tag{27}$$

while the Kalman gain $L(t) = \left[\Sigma(t)C^T(t) + G(t)\Gamma\right]\Xi^{-1}$

is solved forwardly in time by the filtering equation

$$\frac{d}{dt}\Sigma(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + G(t)WG^T(t)$$
$$-[\Sigma(t)C^T(t) + G(t)\Gamma]\Xi^{-1}[C(t)\Sigma(t) + \Gamma^T G^T(t)] \,, \quad (28)$$

with the initial condition $\Sigma(t_0) = 0$.

## IV. EVALUATION OF BASE-ISOLATED BENCHMARK

The application for output feedback statistical control paradigm is the first phase of benchmark problem for response control of base-isolated buildings. The benchmark structure is a 8-story, steel-braced building of 82.4 m long and 54.3 m wide whose L-shaped floor plan is symmetric in the x-direction but has asymmetry in the y-direction. Hence, the three-dimensional model is needed. With the basic idea of decoupling the horizontal movement of the structure from the ground movement during the earthquake, the structure is protected by the critical 3% damping linear elastometric isolation system which is then augmented with 16 additional, variable damping active or semi-active control devices, 8 in the x-direction and 8 in the y-direction. A collection of Newhall, Slymar, El Centro, Rinaldi, Kobe, Jiji and Erzinkan earthquakes is chosen for their special dynamic characteristics which can give quite a good dynamic test for the protected building model. The detailed development of the benchmark is available from [2] and [3].

The statistical controller design makes use of the baseline LQG design parameters together with its actuators and sensors setup. Numerical simulations have been carried out to study the effects produced by the cost cumulants, as their features are incorporated into the controller. By choosing the additional degree of design freedom $\mu_2 = 2.7 \times 10^{-9}$, the steady-state 2CC controller using Kalman state estimate demonstrates how higher authority controllers can give certain improvements in the structural performance as the reader may judge from tables of results.

Tables II and III contain the peak and RMS responses for the base-isolated building with controlled damping devices attached to elastometric isolation systems using statistical control method in which the first two cost cumulants and output feedback scheme are employed. Bold-faced values represent the responses produced by the output feedback 2CC controller which are greater than those of the baseline LQG controller. Of the 63 entries involved in Tables II and III, there are respectively 37 and 41 show significant improvements over the baseline LQG case. In view of structural and base responses, the statistical controller outperforms the baseline LQG on both RMS absolute floor acceleration and peak/RMS base displacements. Moreover, the statistical control method also offers a comparable level of performance in the peak inter-story drift and absolute floor acceleration over the spectrum of bi-directional earthquakes. From the tables, we can observe some trade-offs present in the statistical controller design. There are some increases in the peak base and structure shear for the fault-parallel (FP-X) components of Newhall, Rinaldi and Kobe earthquakes. Across the earthquake spectrum, the statistical controller tends to use more peak force J6 and total energies J9 when comparing to those of the baseline LQG. Finally, the performance of statistical control on corner drifts is quite better than that of the baseline LQG. Refer to Table I.

TABLE I

CORNER DRIFTS (NORMALIZED BY UNCONTROLLED VALUES)

| Quakes | | | Output Feedback 2CC Control | |
|---|---|---|---|---|
| Newhall | FP-X | FN-X | **0.8895** | 0.7104 |
| Sylmar | FP-X | FN-X | 1.0848 | 0.7551 |
| ElCentro | FP-X | FN-X | 0.5935 | 0.8628 |
| Rinaldi | FP-X | FN-X | 0.8732 | 0.9900 |
| Kobe | FP-X | FN-X | 0.8400 | 0.9805 |
| Jiji | FP-X | FN-X | 0.7389 | 0.7609 |
| Erzinkan | FP-X | FN-X | 0.7085 | 0.7971 |

TABLE II

RESULTS FOR OUTPUT FEEDBACK 2CC CONTROL (FP-X AND FN-Y)

| Quakes | J1 | J2 | J3 | J4 | J5 | J6 | J7 | J8 | J9 |
|---|---|---|---|---|---|---|---|---|---|
| Newhall | **0.8925** | **0.9226** | 0.7693 | **0.9473** | **0.9480** | **0.2117** | 0.5731 | 0.7797 | **0.5860** |
| Slymar | 0.8693 | 0.8900 | 0.8718 | 0.8023 | 0.9468 | **0.3945** | 0.6305 | **0.7946** | **0.7143** |
| ElCentro | 0.8755 | 0.8484 | 0.6105 | 0.7256 | 0.8184 | **0.2621** | 0.5410 | 0.5125 | **0.5214** |
| Rinaldi | **0.9837** | **0.9842** | 0.7428 | **0.9730** | **0.9932** | **0.2937** | 0.5223 | 0.6369 | **0.5926** |
| Kobe | **0.9193** | **0.9192** | 0.6688 | 0.7592 | **0.9730** | **0.1741** | 0.5459 | 0.4613 | **0.4793** |
| Jiji | 0.8528 | 0.8540 | 0.7908 | 0.8596 | 0.8719 | **0.2452** | 0.6845 | 0.7355 | **0.4293** |
| Erzinkan | 0.9422 | 0.9610 | 0.5618 | 0.8075 | 0.9237 | **0.3526** | 0.5153 | 0.6520 | **0.5902** |

TABLE III

RESULTS FOR OUTPUT FEEDBACK 2CC CONTROL (FN-X AND FP-Y)

| Quakes | J1 | J2 | J3 | J4 | J5 | J6 | J7 | J8 | J9 |
|---|---|---|---|---|---|---|---|---|---|
| Newhall | 0.7758 | 0.8021 | 0.6452 | 0.7249 | 0.7810 | **0.3219** | 0.6571 | 0.6395 | **0.6104** |
| Slymar | 0.7963 | 0.7764 | 0.6579 | 0.7321 | 0.7380 | **0.4077** | 0.4571 | 0.6472 | **0.7802** |
| ElCentro | 0.9812 | 0.9826 | 0.7409 | **0.9845** | 0.9836 | **0.2097** | 0.6606 | 0.6541 | **0.4427** |
| Rinaldi | 0.9037 | 0.9494 | 0.6531 | **0.9783** | **0.9835** | **0.2483** | 0.3907 | 0.3616 | **0.5973** |
| Kobe | **1.0081** | **0.9991** | 0.6968 | **0.9860** | **0.9832** | **0.1659** | 0.7181 | 0.7088 | **0.4099** |
| Jiji | 0.7142 | 0.7144 | 0.6917 | 0.7097 | 0.7225 | **0.3116** | 0.6157 | 0.5105 | **0.5306** |
| Erzinkan | 0.8569 | 0.8643 | 0.4829 | 0.8943 | 0.9255 | **0.3879** | 0.4135 | 0.4863 | **0.7021** |

## REFERENCES


[1] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control.* New York: Springer-Verlag, 1975

[2] S. Narasimhan, S. Nagarajaiah, E. A. Johnson, H. P. Gavin, "Phase I Smart Base Isolated Benchmarking Building: Part I–Problem Definition", Special Issue on Smart Isolated Benchmark Building, *Journal of Structural Control and Health Monitoring*, July 2004

[3] S. Nagarajaiah, S. Narasimhan, "Phase I Smart Base Isolated Benchmarking Building: Part II–Sample Controllers for Linear Isolation System", Special Issue on Smart Isolated Benchmark Building, *Journal of Structural Control and Health Monitoring*, July 2004

[4] K. D. Pham, S. R. Liberty, M. K. Sain, and B. F. Spencer, Jr., "Generalized Risk Sensitive Building Control: Protecting Civil Structures with Multiple Cost Cumulants," *Proceedings American Control Conference*, pp. 500-504, San Diego, California, June 28-30, 1999

[5] K. D. Pham, M. K. Sain, and S. R. Liberty, "Robust Cost-Cumulants Based Algorithm for Second and Third Generation Structural Control Benchmarks, " *Proceedings American Control Conference* , pp. 3070-3075, Anchorage, Alaska, May 08-10, 2002

[6] K. D. Pham, M. K. Sain, and S. R. Liberty, "Infinite Horizon Robustly Stable Seismic Protection of Cable-Stayed Bridges Using Cost Cumulants," *Proceedings American Control Conference,* pp. 691-696, Boston, Massachusetts, June 30-July 03, 2004

[7] K. D. Pham, "Statistical Control Paradigms for Structural Vibration Suppression," Ph.D. Dissertation, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, May 2004