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Stabilization of a 3D Rigid Pendulum

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***Abstract*— We introduced models for a 3D pendulum, consisting of a rigid body that is supported at a frictionless pivot, in a 2004 CDC paper [1]. In that paper, several different classifications were given and models were developed for each classification. Control problems were posed based on these various models. This paper continues that line of research by studying stabilization problems for a reduced model of the 3D pendulum. Two different stabilization strategies are proposed. The first controller, based on angular velocity feedback only, asymptotically stabilizes the hanging equilibrium. The domain of attraction is shown to be almost global. The second controller, based on angular velocity and reduced attitude feedback, asymptotically stabilizes the inverted equilibrium, providing an almost global domain of attraction. Simulation results are provided to illustrate closed loop properties.**

I. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and in nonlinear control. Much of the published research treats 1D planar pendulum models or 2D spherical pendulum models or some multi-body version of these. In a recent paper [1], we summarized much of this published research, emphasizing papers that treat control issues. In addition, we introduced a new 3D pendulum model that, surprisingly, seems not to have been studied in the prior literature. Another overview of pendulum control problems was given in [2], which also provides motivation for the importance of such control problems. This paper continues that line of research by developing new control and stabilization results for a 3D rigid pendulum.

In [1], we introduced new 3D pendulum models, several of which give rise to pendulum control problems that have not been previously studied. In particular, one class of pendulum control problems posed was to stabilize the pendulum by bringing it to rest at a specified equilibrium attitude, with its dynamics considered on $TSO(3)$. Another class of pendulum control problems posed was to stabilize the pendulum by bringing it to rest at a specified reduced equilibrium attitude, with its dynamics considered to evolve on $TSO(3)/S^1$. A reduced attitude is the attitude of the pendulum, modulo rotations about the vertical. In other words, two attitudes have identical reduced attitudes if they differ only by a rotation about the vertical. In [1], control problems for 3D pendulums were introduced for control of the full attitude and for control of the reduced attitude.

In this paper, we study stabilization problems for a 3D rigid pendulum defined in terms of the reduced attitude. A 3D rigid pendulum is supported at a pivot. The pivot is assumed to be frictionless and inertially fixed. The rigid body is allowed to be either asymmetric or symmetric, and the location of its center of mass is distinct from the location of the pivot. Forces that arise from uniform and constant gravity act on the rigid body. For simplicity, three independent control moments are assumed to act on the rigid body.

We follow the development and notation introduced in [1]. In particular, the formulation of the model depends on construction of a Euclidean coordinate frame fixed to the rigid body with origin at the pivot and an inertial Euclidean coordinate frame with origin at the pivot. We assume that the inertial coordinate frame is selected so that the first two axes lie in a horizontal plane and the "positive" third axis points down. The relevant mathematical model is expressed in terms of the angular velocity vector of the rigid body and the reduced attitude vector of the rigid body. The reduced attitude vector is a unit vector in the direction of gravity, expressed in the body fixed coordinate frame. The control problems that are treated in this paper involve asymptotic stabilization of the hanging and the inverted equilibrium of the 3D pendulum.

The main contribution of this paper is its development of results for almost global asymptotic stabilization of an equilibrium of the 3D rigid pendulum. These results are new and interesting. The results are derived by using novel Lyapunov functions that are suited to the 3D pendulum problem. An important additional contribution is that the results are developed and stated in terms of a global representation of the reduced attitude. In particular, we avoid the use of Euler angles and other non-global attitude representations.

This work compares with [3], wherein PD control laws for systems evolving over Lie groups were proposed. In contrast with the PD-based laws in [3] that generally give a

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conservative domain of attraction, we provide almost global asymptotic stabilization results. Our approach for obtaining almost global asymptotic stabilization in a direct way, using a single nonlinear controller, should also be contrasted with the more traditional approach, introduced in [4], that utilizes a "swing-up" controller, a locally stabilizing controller, and a strategy for switching between the two controllers.

II. Uncontrolled 3D Pendulum

In this section we introduce the reduced model for the uncontrolled 3D pendulum and we summarize fundamental stability properties of the two equilibrium solutions. Consider the reduced attitude model for the 3D pendulum. The model, expressed in body frame coordinates, is given by

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma, \\ \dot{\Gamma} = \Gamma \times \omega, \end{cases} \tag{1}$$

where ω represents the angular velocity of the rigid body, Γ is a unit vector in the direction of gravity in the body reference frame, and ρ represents the constant vector from the pivot to the center of mass of the pendulum, in body frame coordinates. Since $SO(3)/S^1$ is diffeomorphic to S^2, it is sufficient to view the motion of the 3D pendulum as evolving on $\mathbb{R}^3 \times S^2$ according to (1).

Equating the vector field of (1) to zero, we obtain two different reduced equilibria. These correspond to the angular velocity, $\omega = 0$ and to the two configurations in S^2 given by $\Gamma_h = \frac{\rho}{\|\rho\|}$ and $\Gamma_i = -\frac{\rho}{\|\rho\|}$. The former is referred to as the hanging reduced equilibrium, while the latter is referred to as the inverted reduced equilibrium. Note that these two reduced equilibria correspond to equilibrium manifolds for the full dynamics of the 3D pendulum.

III. Control of 3D Pendulum

We study stabilization problems for the reduced attitude of a 3D pendulum. In this section, we introduce the control model, assuming full control actuation. In the following section, we study asymptotic stabilization of the hanging equilibrium, based on feedback of the angular velocity. A domain of attraction of the hanging equilibrium is provided. Subsequently, we study almost global asymptotic stabilization of the inverted equilibrium, based on feedback of angular velocity and the reduced attitude vector. By almost global asymptotic stabilization of an equilibrium, we mean that for every initial condition in the phase-space contained in the complement of a set of Lebesgue measure zero, the solution converges to this equilibrium. Thus, the domain of attraction of the equilibrium is the whole of the phase-space, excluding a set of Lebesgue measure zero.

As shown in [1], the control model for the fully actuated 3D pendulum is given by

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma + u, \\ \dot{\Gamma} = \Gamma \times \omega, \end{cases} \tag{2}$$

where, $\omega \in \mathbb{R}^3$, $\Gamma \in S^2$, and $u \in \mathbb{R}^3$.

IV. Asymptotic Stabilization of the Hanging Equilibrium

In this section, a simple controller is developed that makes the hanging equilibrium asymptotically stable. The controller is based on the observation that the control model given by equation (2) is input-output passive if the angular velocity is taken as the output. The total energy is the storage function. Since the total energy $\frac{1}{2}\,\omega^{\mathrm{T}} J\omega - mg\rho^{\mathrm{T}}\Gamma$, has a minimum at the hanging equilibrium $(0, \Gamma_h)$, a control law based on angular velocity feedback is suggested.

Let $\Psi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a smooth function such that

$$\epsilon_1 \|x\|^2 \leq x^{\mathrm{T}}\Psi(x) \leq \alpha(\|x\|), \quad \forall x \in \mathbb{R}^3, \tag{3}$$

where $\epsilon_1 > 0$, and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function. Thus, we propose a class of damping-injection based controllers, given by

$$u = -\Psi(\omega), \tag{4}$$

where $\Psi(\cdot)$ satisfies (3). We next show that the above family of controllers, which requires only angular velocity feedback, renders the hanging equilibrium of a 3D pendulum asymptotically stable, with a guaranteed domain of attraction.

Lemma 1: *Consider the fully actuated 3D pendulum given by (2). Let $\Psi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a smooth function satisfying (3) and choose controller as given in (4). Then, the hanging equilibrium is asymptotically stable. Furthermore, for every $\epsilon \in (0, 2mg\|\rho\|)$, all solutions of the closed-loop system given by (2) and (4), such that $(\omega(0), \Gamma(0)) \in \mathcal{H}$, where*

$$\mathcal{H} = \left\{ (\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^{\mathrm{T}} J\omega + \frac{1}{2} mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| - \epsilon \right\} \tag{5}$$

satisfy $(\omega(t), \Gamma(t)) \in \mathcal{H}$, $t \geq 0$, and $\lim_{t\to\infty} \omega(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.

Proof: Consider the closed loop system given by (2) and (4). We propose the following candidate Lyapunov function

$$V(\omega, \Gamma) = \frac{1}{2}\left[\omega^{\mathrm{T}} J\omega + mg\|\rho\| \, \|\Gamma - \Gamma_h\|^2\right]. \tag{6}$$

Note that the above Lyapunov function is positive definite on $\mathbb{R}^3 \times S^2$ and $V(0, \Gamma_h) = 0$. Furthermore, the derivative along a solution of the closed-loop given by (2) and (4) is

$$\begin{aligned}\dot{V}(\omega,\Gamma) &= \omega^{\mathrm{T}}(J\dot{\omega}) + mg\|\rho\|(\Gamma-\Gamma_h)^{\mathrm{T}}\dot{\Gamma},\\ &= \omega^{\mathrm{T}}(J\omega\times\omega + mg\rho\times\Gamma - \Psi(\omega))\\ &\quad + mg\|\rho\|(\Gamma-\Gamma_h)^{\mathrm{T}}(\Gamma\times\omega),\\ &= -\omega^{\mathrm{T}}\Psi(\omega) + mg\omega^{\mathrm{T}}(\rho\times\Gamma) - mg\rho^{\mathrm{T}}(\Gamma\times\omega),\\ &= -\omega^{\mathrm{T}}\Psi(\omega),\\ &\le -\epsilon_1\|\omega\|^2,\end{aligned}$$

where the last inequality follows from (3). Thus, $V(\cdot)$ is positive definite and $\dot{V}(\cdot)$ is negative semidefinite on $\mathbb{R}^3\times S^2$.

Next, consider the sub-level set given by $\mathcal{H} = \{(\omega,\Gamma)\in\mathbb{R}^3\times S^2 : V(\omega,\Gamma)\le 2mg\|\rho\|-\epsilon\}$. Note that the compact set $\mathcal{H}$ contains the hanging equilibrium $(0,\Gamma_h)$. Since, $\dot{V}(\omega,\Gamma)\le 0$, all solutions such that $(\omega(0),\Gamma(0))\in\mathcal{H}$ satisfy $(\omega(t),\Gamma(t))\in\mathcal{H}$ for all $t\ge 0$. Thus, $\mathcal{H}$ is an invariant set for solutions of (2) and (4).

Furthermore, from LaSalle's invariant set theorem, we obtain that solutions satisfying $(\omega(0),\Gamma(0))\in\mathcal{H}$ converge to the largest invariant set in $\{(\omega,\Gamma)\in\mathcal{H} : \omega = 0\}$, that is $\rho\times\Gamma = 0$. Therefore, either $\Gamma(t)\to\frac{\rho}{\|\rho\|} = \Gamma_h$ or $\Gamma(t)\to -\frac{\rho}{\|\rho\|} = \Gamma_i$ as $t\to\infty$. Since $(0,\Gamma_i)\notin\mathcal{H}$, it follows that $\Gamma(t)\to\Gamma_h$ as $t\to\infty$. Thus, $(0,\Gamma_h)$ is an asymptotically stable equilibrium of the closed-loop system given by (2) and (4), with a domain of attraction $\mathcal{H}$. ■

Lemma 2: *Consider the fully actuated 3D pendulum given by (2). Let $\Psi:\mathbb{R}^3\mapsto\mathbb{R}^3$ be a smooth function satisfying (3) and choose controller as in (4). Then, all solutions of the closed-loop system given by (2) and (4), such that $\omega(0)\ne 0$ and $(\omega(0),\Gamma(0))\in\mathcal{A}$, where*

$$\mathcal{A} = \left\{(\omega,\Gamma)\in\mathbb{R}^3\times S^2 : \frac{1}{2}\omega^{\mathrm{T}}J\omega + \frac{1}{2}mg\|\rho\|\,\|\Gamma-\Gamma_h\|^2 = 2mg\|\rho\|\right\} \tag{7}$$

satisfy $\lim_{t\to\infty}\omega(t) = 0$ and $\lim_{t\to\infty}\Gamma(t) = \Gamma_h$.

Proof: Consider the closed loop system given by (2) and (4), and the Lyapunov function given in (6). As already shown, $\dot{V}(\omega,\Gamma) = -\omega^{\mathrm{T}}\Psi(\omega)$. Thus,

$$\begin{aligned}\ddot{V} &= -\dot{\omega}^{\mathrm{T}}\Psi(\omega) - \omega^{\mathrm{T}}\dot{\Psi}(\omega),\\ &= -\dot{\omega}^{\mathrm{T}}\left(\Psi(\omega) + \left[\frac{\partial\Psi}{\partial\omega}\right]^{\mathrm{T}}\omega\right),\end{aligned} \tag{8}$$

since $\Psi(\cdot)$ is smooth. Furthermore, for all $(\omega,\Gamma)\in\mathcal{A}$, ω is bounded and

$$\dot{\omega} = J^{-1}\left(J\omega\times\omega + mg\rho\times\Gamma - \Psi(\omega)\right)$$

is also bounded. Define

$$N = \sup_{(\omega,\Gamma)\in\mathcal{A}}\left\{\left\|\dot{\omega}^{\mathrm{T}}\left(\Psi(\omega) + \left[\frac{\partial\Psi}{\partial\omega}\right]^{\mathrm{T}}\omega\right)\right\|\right\} < \infty.$$

Next, since $(\omega(0),\Gamma(0))\in\mathcal{A}$, $V(\omega(0),\Gamma(0)) = 2mg\|\rho\|$ and $\|\ddot{V}(\omega(0),\Gamma(0))\|\le N$. Expanding $V(\omega(t),\Gamma(t))$ in a Taylor series expansion, we obtain

$$\begin{aligned}V(\omega(t),\Gamma(t)) &= 2mg\|\rho\| - \omega(0)^{\mathrm{T}}\Psi(\omega(0))t + R(t),\\ &\le 2mg\|\rho\| - \epsilon_1\|\omega(0)\|^2\,t + N\,t^2,\end{aligned} \tag{9}$$

since the remainder necessarily satisfies $\|R(t)\|\le Nt^2$.

Define

$$\bar{\epsilon} = \min\left(2mg\|\rho\|, 3\epsilon_1^2\frac{\|\omega(0)\|^4}{16N}\right).$$

It can be easily shown that for all $t\in[T_1,T_2]$,

$$t^2 - \epsilon_1\frac{\|\omega(0)\|^2}{N}t + \frac{\bar{\epsilon}}{N}\le 0,$$

where

$$T_1 = \epsilon_1\frac{\|\omega(0)\|^2}{2N}\left(1-\sqrt{1-\frac{4\bar{\epsilon}N}{\epsilon_1^2\|\omega(0)\|^4}}\right) > 0,$$

and $T_2 - T_1\ge\epsilon_1\frac{\|\omega(0)\|^2}{2N}$. Choose an $\epsilon\in(0,\bar{\epsilon})$. Then, for all $t\in[T_1,T_2]$, $V(\omega(t),\Gamma(t))\le 2mg\|\rho\|-\bar{\epsilon} < 2mg\|\rho\|-\epsilon$ and hence, $(\omega(t),\Gamma(t))\in\mathcal{H}$, where $\mathcal{H}$ is an invariant set given in Lemma 1. Thus, from Lemma 1, we obtain the result that $\omega(t)\to 0$ and $\Gamma(t)\to\Gamma_h$, as $t\to\infty$. ■

Theorem 1: *Consider the fully actuated 3D pendulum given by (2). Let $\Psi:\mathbb{R}^3\mapsto\mathbb{R}^3$ be a smooth function satisfying (3) and choose controller as in (4). Then, all solutions of the closed-loop system given by (2) and (4), such that $(\omega(0),\Gamma(0))\in\mathcal{N}\backslash\{(0,\Gamma_i)\}$, where*

$$\mathcal{N} = \left\{(\omega,\Gamma)\in\mathbb{R}^3\times S^2 : \frac{1}{2}\omega^{\mathrm{T}}J\omega + \frac{1}{2}mg\|\rho\|\,\|\Gamma-\Gamma_h\|^2\le 2mg\|\rho\|\right\} \tag{10}$$

satisfy $(\omega(t),\Gamma(t))\in\mathcal{N}$, $t\ge 0$, and $\lim_{t\to\infty}\omega(t) = 0$ and $\lim_{t\to\infty}\Gamma(t) = \Gamma_h$.

Proof: From Lemmas 1 and 2, we obtain the result that for every $\epsilon\in(0,2mg\|\rho\|)$ and $(\omega(0),\Gamma(0))\in\mathcal{H}\bigcup\mathcal{A}$, where $\mathcal{H}$ and $\mathcal{A}$ are as defined in Lemmas 1 and 2, $\omega(t)\to 0$ and $\Gamma(t)\to\Gamma_h$ as $t\to\infty$. Since, $\mathcal{N}$ can be written as

$$\mathcal{N} = \bigcup_{\epsilon\in(0,2mg\|\rho\|)}(\mathcal{H}\bigcup\mathcal{A}),$$

the result follows. ■

It is important to point that $\mathcal{N}\backslash\{(0,\Gamma_i)\}$ is an invariant set, but it is not the maximal domain of attraction of the hanging equilibrium. Any solution of (2) and (4), starting from an initial condition not in $\mathcal{N}\backslash\{(0,\Gamma_i)\}$, that does not pass through $(0,\Gamma_i)$, must eventually enter $\mathcal{N}\backslash\{(0,\Gamma_i)\}$. Hence, the maximal domain of attraction of the hanging equilibrium is a proper superset of the invariant set $\mathcal{N}\backslash\{(0,\Gamma_i)\}$.

In the subsequent theorem, we claim that for the control given in (4), the equilibrium $(0, \Gamma_h)$ of (2) and (4) is almost globally asymptotically stable.

Theorem 2: *Consider the fully actuated 3D pendulum described by (2). Let $\Psi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a smooth function satisfying (3) and $\Psi'(0)$ be positive definite and symmetric. Choose controller as given in (4). Let $\mathcal{M}$ denote the stable manifold of the equilibrium $(0, \Gamma_i)$. Then all solutions of the closed-loop system given by (2) and (4), such that $(\omega(0), \Gamma(0)) \in (TSO(3)/S^1)\backslash\mathcal{M}$, where $\mathcal{M}$ is a set of Lebesgue measure zero, satisfy $\lim_{t\to\infty} \omega(t) = 0$ and $\lim_{t\to\infty} \Gamma(t) = \Gamma_h$.*

Proof: Consider the closed loop system given by (2) and (4), and the Lyapunov function given in (6). Since S^2 is a compact set and the Lyapunov function $V(\omega, \Gamma)$ is quadratic in ω, each sublevel set of $V(\omega, \Gamma)$ is a compact set. Furthermore, since $\dot{V}(\omega, \Gamma) = -\omega^{\mathrm{T}}\Psi(\omega) \leq 0$, by LaSalle's invariant set theorem, all solutions converge to the largest invariant set in $\{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \dot{V}(\omega, \Gamma) = 0\}$.

The largest such invariant set is given by $\{(0, \Gamma_h)\} \bigcup \{(0, \Gamma_i)\}$. Hence it is sufficient to show that the stable manifold of the inverted equilibrium $(0, \Gamma_i)$, has dimension less than the dimension of $TSO(3)/S^1$ i.e. five, since all other solutions converge to the hanging equilibrium $(0, \Gamma_h)$ by invariant set theorem.

Using linearization, it can be shown that the equilibrium $(0, \Gamma_i)$ of the closed loop given by (2) and (4) is unstable and hyperbolic with nontrivial stable and unstable manifolds. Hence, from Theorem 3.2.1 in [5], it follows that the dimension of the stable manifold is less than five, so that the domain of attraction in Theorem 2 is almost global in $TSO(3)/S^1$. ■

We briefly compare the conclusions in Theorem 1 and Theorem 2. Theorem 1 provides an explicit description of a compact domain of attraction of the hanging equilibrium; however that domain of attraction is not maximal. In contrast, Theorem 2 shows that the maximal domain of attraction consists of all points in $TSO(3)/S^1$ that are not in the stable manifold $\mathcal{M}$ of the inverted equilibrium. The geometry of the stable manifold $\mathcal{M}$ of the inverted equilibrium may be complicated; it depends on the model parameters and the specific nonlinear controller.

V. Asymptotic Stabilization of the Inverted Equilibrium

In the last section, we proposed a family of controllers based on angular velocity feedback that asymptotically stabilizes the hanging equilibrium, with a guaranteed domain of attraction. This motivates us to study the problem of almost global stabilization using angular velocity and reduced attitude feedback. The development in this section is easily modified to provide an almost globally stabilizing controller for the hanging equilibrium; in this case improved response performance can be obtained when compared with the feedback controller (3).

However, the specific focus of this section is to develop an almost globally stabilizing controller for the inverted equilibrium. It should be noted that there exists a topological obstruction in designing a continuous time-invariant controller for global stabilization. This is due to the fact that the configuration manifold $SO(3)/S^1 \simeq S^2$ is a compact manifold [6]. Thus, we propose a controller that almost globally asymptotically stabilizes the reduced inverted equilibrium.

Let $\Phi : [0, 1) \mapsto \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$ and $\Phi(x) \to \infty$ as $x \to 1$. Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function satisfying (3). We propose a class of controllers given by

$$u = -\left[\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_i + mg\rho\right] \times \Gamma - \Psi(\omega). \tag{11}$$

Theorem 3: *Consider the fully actuated system given by (2). Choose a controller as given in (11). Then $(0, \Gamma_i)$ is an equilibrium of the closed loop system (2) and (11) that is almost globally asymptotically stable with domain of attraction $\mathbb{R}^3 \times (S^2\backslash\{\Gamma_h\})$.*

Proof: Consider the closed-loop system given by (2) and (11) and the candidate Lyapunov function

$$V(\omega, \Gamma) = \frac{1}{2}\omega^{\mathrm{T}} J\omega + 2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right). \tag{12}$$

Note that the Lyapunov function is positive definite on $\mathbb{R}^3 \times S^2$ and $V(0, \Gamma_i) = 0$. Furthermore, every sub-level set of the Lyapunov function in $\mathbb{R}^3 \times S^2$ is compact, and the closed-loop vector field given by (2) and (11) has only one equilibrium in each sub-level set, namely $(0, \Gamma_i)$.

Computing the derivative of the Lyapunov function along the solution of (2) and (11) yields

$$\begin{aligned}
\dot{V}(\omega, \Gamma) &= \omega^{\mathrm{T}} J\dot{\omega} + \Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_i^{\mathrm{T}}\dot{\Gamma}, \\
&= \omega^{\mathrm{T}}(J\omega \times \omega + mg\rho \times \Gamma - \Psi(\omega) - mg\rho \times \Gamma) \\
&\quad -\omega^{\mathrm{T}}\left(\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)(\Gamma_i \times \Gamma)\right) \\
&\quad + \Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(\Gamma_i^{\mathrm{T}}\Gamma - 1)\Gamma_i^{\mathrm{T}}(\Gamma \times \omega), \\
&= \Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)[-\omega^{\mathrm{T}}(\Gamma_i \times \Gamma) \\
&\qquad + \Gamma_i^{\mathrm{T}}(\Gamma \times \omega)](\Gamma_i^{\mathrm{T}}\Gamma - 1) - \omega^{\mathrm{T}}\Psi(\omega), \\
&= -\omega^{\mathrm{T}}\Psi(\omega), \\
&\leq -\epsilon_1\|\omega\|^2.
\end{aligned}$$

Thus, $\dot{V}(\omega, \Gamma)$ is negative semidefinite on $\mathbb{R}^3 \times S^2$ and hence, all solutions remain in the compact sub-level set given by $\mathcal{K} = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : V(\omega, \Gamma) \leq V(\omega(0), \Gamma(0))\}$. Note that $\mathcal{K} \subset \mathbb{R}^3 \times (S^2 \backslash \{\Gamma_h\})$.

Next, from LaSalle's invariant set theorem, for any arbitrary initial condition, the solutions converge to the largest invariant set in $\{(\omega, \Gamma) \in \mathcal{K} : \omega = 0\}$. Thus, from the second equation in (2), we obtain that Γ is constant and from the first equation in (2), we obtain that either $\Gamma_i^{\mathrm{T}}\Gamma = 1$ or $\Gamma_i \times \Gamma = 0$, or both. Therefore, either $\Gamma(t) \to \Gamma_i$ or $\Gamma(t) \to \Gamma_h$ as $t \to \infty$. However, $(0, \Gamma_h)$ is not contained in $\mathcal{K}$, hence, $\Gamma(t) \to \Gamma_i$ as $t \to \infty$. ∎

It should be noted that the controller (11) is not defined at $\Gamma = \Gamma_h \in S^2$ and also $\|u\| \to \infty$ as $\Gamma \to \Gamma_h$. For all other initial conditions in $\mathbb{R}^3 \times (S^2 \backslash \{\Gamma_h\})$, the controller guarantees that the solution of the closed-loop, given by (2) and (11), asymptotically converges to $(0, \Gamma_i)$.

It is insightful to view the class of controllers in (11) as providing a combination of potential shaping, represented by the function $\Phi(\cdot)$ and the term containing gravity, and damping injection, represented by the function $\Psi(\cdot)$. Furthermore, it may be noted that the argument of the potential function $\Phi(\cdot)$ is proportional to the cosine of the angle between Γ and Γ_i. This yields the closed loop property that if $\omega(0) = 0$, then for all $t \geq 0$, the angle between $\Gamma(t)$ and Γ_i is bounded above by the angle between $\Gamma(0)$ and Γ_i.

Corollary 1: *Consider the fully actuated system given by (2) with controller as in (11). Furthermore, let $\omega(0) = 0$ and $\Gamma(0) \neq \Gamma_h$. Then, for all $t \geq 0$,*

$$\measuredangle(\Gamma_i, \Gamma(t)) \leq \measuredangle(\Gamma_i, \Gamma(0)).$$

Proof: Consider the candidate Lyapunov function (12), for the closed-loop system. As already shown, $\dot{V}(\omega, \Gamma) = -\omega^{\mathrm{T}}\Psi(\omega)$. Thus, since $\dot{V}(\omega, \Gamma)$ is negative semidefinite, $V(\omega(t), \Gamma(t)) \leq V(\omega(0), \Gamma(0))$. Thus substituting $\omega(0) = 0$ in (12), we obtain the result that for all $t \geq 0$,

$$\frac{1}{2}\omega(t)^{\mathrm{T}}J\omega(t) + 2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma(t) - 1)^2\right) \leq 2\Phi\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma(0) - 1)^2\right).$$

Since, the kinetic energy term is strictly non-negative and $\Phi(\cdot)$ is a monotonic function, we obtain

$$|\Gamma_i^{\mathrm{T}}\Gamma(t) - 1| \leq |\Gamma_i^{\mathrm{T}}\Gamma(0) - 1|, \qquad t \geq 0.$$

Thus,

$$\Gamma_i^{\mathrm{T}}\Gamma(t) \geq \Gamma_i^{\mathrm{T}}\Gamma(0), \qquad t \geq 0,$$

which implies that

$$\cos\left(\measuredangle(\Gamma_i, \Gamma(t))\right) \geq \cos\left(\measuredangle(\Gamma_i, \Gamma(0))\right), \ t \geq 0.$$

Since $\measuredangle(\Gamma_i, \Gamma(t)) \in [0, \pi)$ and $\cos(\cdot)$ is non-increasing in $[0, \pi)$, the result follows. ∎

VI. Simulation Results

In the previous section, we introduced a class of controllers that guarantee almost global stabilization of the inverted equilibrium of the reduced system (2). In this section, we present simulation results for a specific controller selected from the family of controllers given by (11). We choose

$$\Phi(x) = -k\ln(1 - x),$$

where $k > 0$, and $\Psi(x) = Px$, where P is a positive definite matrix. The resulting control law (11) is given by

$$u = -P\omega - mg\rho \times \Gamma + k\frac{(\Gamma_i^{\mathrm{T}}\Gamma - 1)}{1 - \frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}(\Gamma \times \Gamma_i).$$

Consider the model (2) where $m = 140$ kg, $\rho = (0, 0, 0.5)^{\mathrm{T}}$ m and $J = diag(3, 40, 50)$ kg-m^2. Let u be the controller (11) where $P = 2.5I_3$ and $k = 5$. The following figures describe the evolution of the closed loop system (2) and (11). The initial conditions are $\omega(0) = (1, 3, 1)^{\mathrm{T}}$ rad/s and $\Gamma(0) = (0.5, 0.7071, 0.5)^{\mathrm{T}}$.

Simulation results in Figures 1 and 2 show that $\omega(t) \to 0$, and $\Gamma(t) \to \Gamma_i$ as $t \to \infty$. This is also clearly seen from the plot of the angle between $\Gamma(t)$ and Γ_i in Figure 3. The path that the center of mass of the 3D pendulum follows in the inertial frame in 3D is shown in Figure 4.

For the case, $\omega(0) = 0$, we only show the plot of the angle between $\Gamma(t)$ and Γ_i in Figure 5. The gains were chosen to be $P = 2I_3$ and $k = 10$. As expected from Corollary 1, the angle between $\Gamma(t)$ and Γ_i remains bounded above by the angle between $\Gamma(0)$ and Γ_i which is 120 deg. This is in contrast to Figure 3, where the excursion in angle exceeds 120 deg.

VII. Conclusions

In [1], we introduced models for a 3D pendulum, consisting of a rigid body that is supported at a frictionless pivot and we posed various control problems. In this paper we study stabilization problems for a reduced model of the 3D rigid pendulum. We propose two different controllers for stabilization of the 3D pendulum. The first controller, based on angular velocity feedback only, asymptotically stabilizes the hanging equilibrium. Furthermore, it is also shown that the domain of attraction is almost global in $TSO(3)/S^1$, i.e. the domain of attraction is the whole phase-space, except for a set of Lebesgue measure zero. The second controller, based on angular velocity and reduced attitude feedback, asymptotically stabilizes the inverted equilibrium providing an almost global domain of attraction. Simulation results are provided to illustrate the closed loop properties.

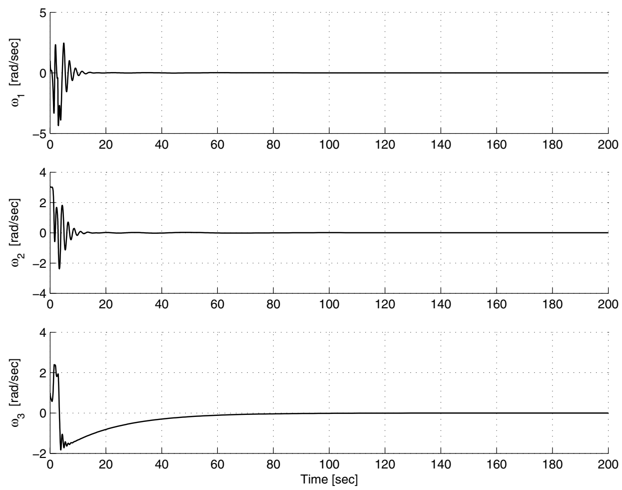

Fig. 1. Evolution of the angular velocity of the 3D pendulum in the body frame.

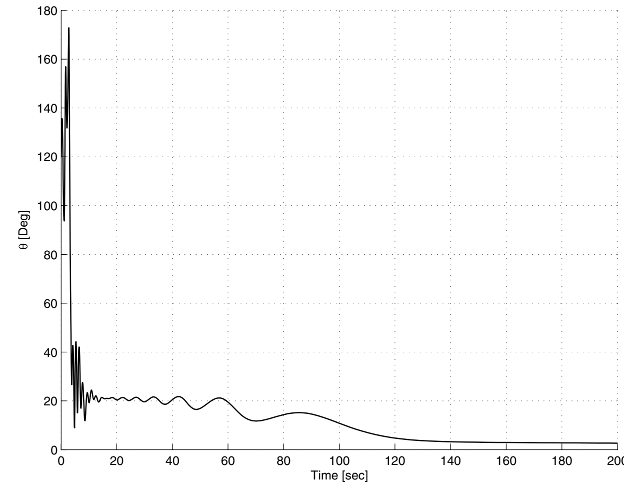

Fig. 3. Evolution of the angle between the reduced attitude vector $\Gamma(t)$ and the desired reduced attitude vector Γ_i for $\omega(0) = (1, 3, 1)^{\mathrm{T}}$ rad/sec.

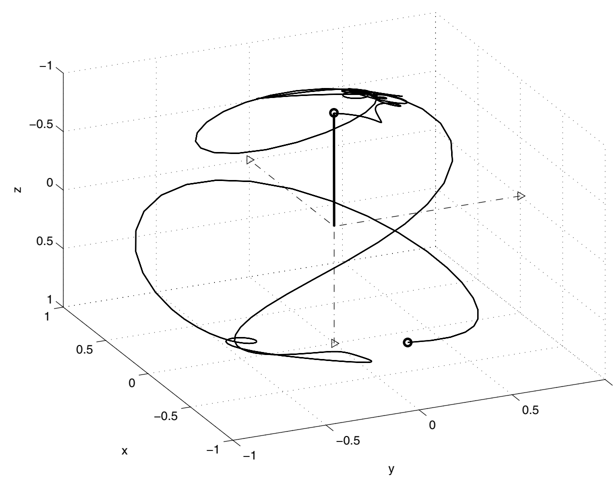

Fig. 4. Swing-up motion of the vector between the pivot and the center of mass of the 3D pendulum in the inertial frame.

REFERENCES

[1] J. Shen et. al., "Dynamics and Control of a 3D Pendulum" *Proceedings of the 43rd IEEE Conference on Decision & Control*, Bahamas, December, 2004, 323-328.

[2] K. Furuta, "Control of Pendulum: From Super Mechano-System to Human Adaptive Mechatronics," *Proceedings of* 42^{nd} *IEEE Conference on Decision and Control,* Maui, Hawaii, December, 2003, 1498-1507.

[3] F. Bullo and R. M. Murray, "Tracking for fully actuated mechanical systems: A geometric framework," *Automatica*, 35(1), 1999, 17-34.

[4] K.J. Astrom and K. Furuta, "Swinging-up a Pendulum by Energy Control," *Proceedings of the IFAC Congress,* Vol. E, 1996, 37-42.

[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Springer-Verlag New York, 1983.

[6] S. P. Bhat and D. S. Bernstein, "A Topological Obstruction to Global Asymptotic Stabilization of Rotational Motion and the Unwinding Phenomenon," *Proceedings of the American Control Conference,* June 1998, 2785 - 2789.

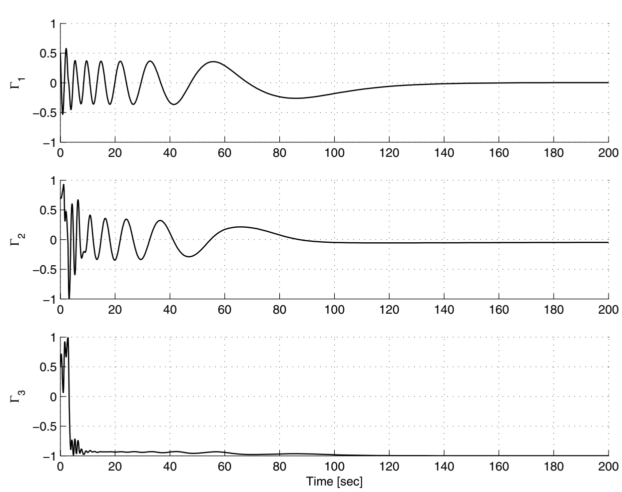

Fig. 2. Evolution of the components of the direction of gravity Γ in the body frame.

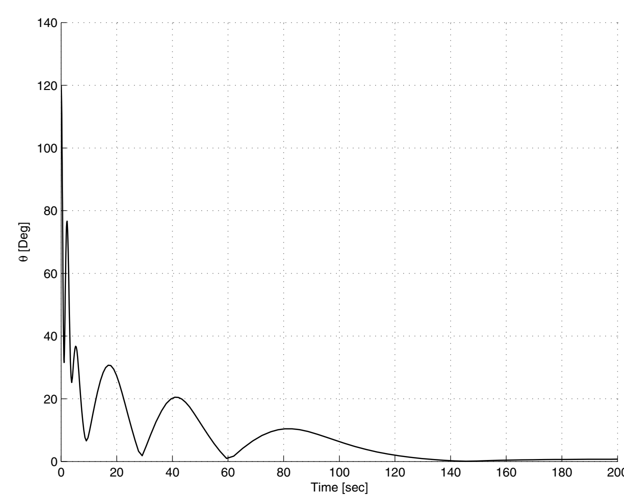

Fig. 5. Evolution of the angle between the reduced attitude vector $\Gamma(t)$ and the desired reduced attitude vector Γ_i for $\omega(0) = 0$ rad/sec.