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H_∞ Fixed-Lag Smoothing for Descriptor Systems

Haoqian Wang, Huanshui Zhang, and Guangren Duan

***Abstract*— In this paper, we study the finite horizon H_∞ fixed-lag smoothing problem for linear descriptor systems. The key approach applied for deriving the H_∞ fixed-lag smoother is the re-organization innovation analysis in Krein space. Under the Krein space, the H_∞ fixed-lag smoothing is converted into an H_2 estimation problem for the system with current and delayed measurements. By using innovation re-organization approach, the H_2 estimation with delayed measurements is transformed into a problem of delay-free measurements and thus a simple solution to H_∞ smoothing is given.**

I. Introduction

The state estimation for linear descriptor systems has received much attention in the last few years (see [1], [2], [3], [4], [5] and references therein). As is well known, there are mainly two approaches to the linear estimation. One is the linear minimum variance which is termed as H_2 estimation. The other is the H_∞ performance index which is to find an estimator such that the energy gain between input noises (including system and measurement noises) and estimation error is bounded by a prescribed level [2].

In the H_2 contents, the optimal estimation for the descriptor systems have been well studied (see [1], [3], [4], [5]). Reference [1] studies the filtering problem for the descriptor systems through the standard decomposition and Kalman filtering formulation. By applying Maximum Likehood approach, Nikoukhah [3] presents a recursive descriptor Kalman filter and a general formulation for a discrete-time linear estimation without involving any transformation. Reference [4] considers a time-invariant descriptor system, a recursive estimate algorithm is obtained, and the convergence and stability of this estimation is investigated in [5].

Under H_∞ performance, since the estimator is insensitive to the exact knowledge of the statistics of the noise signals, a lot of interests in the filtering estimation for descriptor systems have been received, see [6], [7] and references therein. In [6], the H_∞ optimal filtering for descriptor system is investigated by using Linear Matrix Inequality (LMI). Note that only filtering is considered and existence condition for the filter is only sufficient. In [7], the infinite-horizon H_∞ estimation for the descriptor systems have been studied where both the filtering and smoothing are considered. The steady-state estimator is calculated by performing one J-spectral factorization, and calculation of the J-spectral factorization for the case of smoothing in [7] involves state augmentation, which leads very much computational cost. As for the normal systems [8], the H_∞ fixed-lag smoothing for descriptor systems is a very complicated problem and remains challenging. There have been a considerable amount of researches on the problem of fixed-lag smoothing for normal systems (see [9], [10], [6], [11]).

The purpose of the present paper is to present a simple method for the finite horizon H_∞ fixed-lag smoothing problem for the descriptor system. Firstly, the H_∞ fixed-lag smoothing problem is converted into H_2 estimation for the descriptor system with current and time delayed observations in Krein space. Secondly, we solve the H_2 estimation problem with measurement delays by applying re-organized innovation approach.

The following symbols are used throughout this paper. δ_{ij} is Kronecker delta function and T(superscript) denotes the transpose, $k_l \triangleq k-l$.

Notation 1: Whenever the Krein space elements [12] and the Euclidean space elements satisfy the same set of constraints, we shall denote them by the same letters with the former identified by bold faces and the latter by normal faces.

II. Problem Statement

Consider a linear time-invariant system described by the following discrete-time model:

$$Mx(k+1) = \Phi x(k) + \Gamma u(k) \quad (1)$$

$$y(k) = Hx(k) + v(k) \quad (2)$$

$$z(k) = Lx(k) \quad (3)$$

where $x(k) \in \mathbb{R}^n, y(k) \in \mathbb{R}^m, z(k) \in \mathbb{R}^p, u(k) \in \mathbb{R}^r$ and $v(k) \in \mathbb{R}^m$ represent the state, measurement output, the signal to be estimated, input noise, measurement noise, respectively. The initial state $x(0)$ may depend on $u(0)$ as well as a finite number of future values of the input noise $u(k)$, due to the non-causality of the system. It is assumed that the input and measurement noises are bounded deterministic signals.

Assumption 1: The system (1) is regular, i.e., exists s such that $\det(sM - \Phi) \not\equiv 0$.

Under Assumption 1, there exist two nonsingular matrices P_1 and Q_1 such that

$$Q_1 M P_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & M_1 \end{bmatrix}, Q_1 \Phi P_1 = \begin{bmatrix} \Phi_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \quad (4)$$

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H.Q. Wang is with Center of Control Theory and Guidance Technology and Shenzhen Graduate School, Harbin Institute of Technology, P.R. China. wangyizhai@hotmail.com

H.S. Zhang is with Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen, 518055, P.R. China h_s_zhang@hit.edu.cn

G.R. Duan is with Center of Control Theory and Guidance Technology, Harbin Institute of Technology, P.R. China. Grduan@iee.org

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where $n_1+n_2=n, M_1$ is a nilpotent matrix with index λ_0, i.e. $M_1^{\lambda_0}=0, M_1^{\lambda_0-1}\neq 0$. Then system (1)–(3) is restricted system equivalent (RSE) to the following system.

$$x_1(k+1)=\Phi_1 x_1(k)+\Gamma_1 u(k), x_1(0) \quad (5)$$

$$M_1 x_2(k+1)=x_2(k)+\Gamma_2 u(k) \quad (6)$$

$$y(k)=H_1 x_1(k)+H_2 x_2(k)+v(k) \quad (7)$$

$$z(k)=L_1 x_1(k)+L_2 x_2(k) \quad (8)$$

where $x_1(k)\in\mathbb{R}^{n_1}, x_2(k)\in\mathbb{R}^{n_2}$, and $\begin{bmatrix} x_1(k) \\ x_2(k)\end{bmatrix} = P_1^{-1}x(k), \begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}=Q_1\Gamma, \begin{bmatrix} H_1 & H_2 \\ L_1 & L_2\end{bmatrix}=\begin{bmatrix} H \\ L\end{bmatrix}P_1$.

From (6), we have

$$x_2(k)=-\Gamma_2 u(k)-M_1\Gamma_2 u(k+1)-\cdots - M_1^{\lambda_0-1}\Gamma_2 u(k+\lambda_0-1) \quad (9)$$

It can be seen that the present state $x_2(k)$ depends on the future input noises $u(k),\ldots,u(k+\lambda_0-1)$, which implies the non-causality of the descriptor system. In fact, a difference between discrete-time descriptor systems and normal ones is that descriptor systems do not always have solutions for any initial condition $x(0)$. This point can be easily seen from (9). And in equation (9), let $k=0$,

$$x(0)=P_1\begin{bmatrix} x_1(0) \\ -\sum_{i=0}^{\lambda_0-1} M_1^i\Gamma_2 u(i)\end{bmatrix}$$

where $x_1(0)$ is the initial value of (5). This gives

$$\begin{bmatrix} 0 & I\end{bmatrix} P_1^{-1}x(0)=-\sum_{i=0}^{\lambda_0-1} M_1^i\Gamma_2 u(i) \quad (10)$$

which is the admissible initial condition satisfied by the initial state $x(0)$. Denote the set of all admissible initial values of the state $x(0)$:

$$I_0=\{x\mid x\in\mathbb{R}^n, \begin{bmatrix} 0 & I\end{bmatrix} P_1^{-1}x(0)=-\sum_{i=0}^{\lambda_0-1} M_1^i\Gamma_2 u(i)\}$$

Then the H_∞ fixed-lag smoothing problem of the descriptor systems can be formulated as:

Problem 1: Given a scalar $\gamma>0$, an integer $l>0$ and the observations $\{y(i)\}_{i=0}^k$, find an estimate $\check{z}(k_l\mid k)$ of $z(k_l)$, if exist, such that the following inequality is satisfied

$$\sup_{(x_0,u,v)\neq 0}\frac{\sum_{k=l}^{N}[\check{z}(k_l\mid k)-z(k_l)]^T[\check{z}(k_l\mid k)-z(k_l)]}{\|x_1(0)\|_{\Pi_1^{-1}}+\sum_{k=0}^{N+\lambda_0-1}u^T(k)u(k)+\sum_{k=0}^{N}v^T(k)v(k)}<\gamma^2 \quad (11)$$

where $\|x_1(0)\|_{\Pi_1^{-1}}=x_1^T(0)\Pi_1^{-1}x_1(0)$, and λ_0 is the nilpotent index, $x_1(0)$ is as in (5) and Π_1 is a given positive definite matrix which reflects the relative uncertainty of the initial state $x_1(0)$ to the input and measurement noises.

III. Preliminary

Recall [13], we define that

$$J_{l,N}\triangleq x_1^T(0)\Pi_1^{-1}x_1(0)+\sum_{k=0}^{N+\lambda_0-1}u^T(k)u(k) + \sum_{k=0}^{N}v^T(k)v(k)-\gamma^{-2}\sum_{k=l}^{N}v_z^T(k)v_z(k) \quad (12)$$

where

$$v_z(k)=\check{z}(k_l\mid k)-Lx(k_l), k_l=k-l \quad (13)$$

and $v_z(k)=0, k<l$. Then the H_∞ estimation problem is equivalent to that $J_{l,N}(x_1(0),u^{N+\lambda_0};y_s^N)$ has a minimum over $\{x_1(0),u^{N+\lambda_0}\}$ and the estimator is such that the minimum is positive [14], [7]. Further, equation (12) can be rewritten as

$$J_{l,N}=\begin{bmatrix} x_1(0)\\ u^{N+\lambda_0}\\ v_s^N\end{bmatrix}^T\begin{bmatrix}\Pi_1 & 0 & 0\\ 0 & R_u^{N+\lambda_0} & 0\\ 0 & 0 & R_{v_s}^N\end{bmatrix}^{-1}\begin{bmatrix} x_1(0)\\ u^{N+\lambda_0}\\ v_s^N\end{bmatrix}$$

where

$$u^{N+\lambda_0}=\mathrm{col}\{u(0),u(1),\ldots,u(N+\lambda_0-1)\} \quad (14)$$

$$R_u^{N+\lambda_0}=\overbrace{Q_u\oplus\cdots\oplus Q_u}^{N+\lambda_0 \text{ blocks}}, Q_u=I_r \quad (15)$$

and

$$v_s^N=\mathrm{col}\{v_s(0),v_s(1),\ldots,v_s(N)\} \quad (16)$$

$$v_s(k)=\begin{cases} v(k), & 0\le k<l\\ \begin{bmatrix} v(k)\\ v_z(k-l)\end{bmatrix}, & k\ge l\end{cases} \quad (17)$$

$$R_{v_s}^N=Q_{v_s}(0)\oplus\cdots\oplus Q_{v_s}(N) \quad (18)$$

$$Q_{v_s}(k)=\begin{cases} I_m, & 0\le k<l\\ \begin{bmatrix} I_m & 0\\ 0 & -\gamma^2 I_p\end{bmatrix}, & k\ge l\end{cases} \quad (19)$$

Furthermore, putting together (2), (3) and (13) we have

$$y_s(k)=\begin{cases} y(k), & 0\le k<l\\ \begin{bmatrix} y(k)\\ \check{z}(k_l\mid k)\end{bmatrix}, & k\ge l\end{cases} = \begin{cases} Hx(k)+v_s(k), & 0\le k<l\\ \begin{bmatrix} H & 0\\ 0 & L\end{bmatrix}\begin{bmatrix} x(k)\\ x(k_l)\end{bmatrix}+v_s(k), & k\ge l\end{cases} \quad (20)$$

It follows that

$$\begin{bmatrix} x_1(0)\\ u^{N+\lambda_0}\\ y_s^N\end{bmatrix}=\Psi\begin{bmatrix} x_1(0)\\ u^{N+\lambda_0}\\ v_s^N\end{bmatrix}$$

$$\Psi=\begin{bmatrix} I_n & 0 & 0\\ 0 & I_{r(N+1)} & 0\\ \Psi_2 & \Psi_1 & \Psi_0\end{bmatrix}$$

where

$$y_s^N=\mathrm{col}\{y_s(0),\ldots,y_s(N)\} \quad (21)$$

and Ψ is invertible. Thus, it is easy to know that (see also [7])

$$J_{l,N} = \begin{bmatrix} x_1(0) \\ u^{N+\lambda_0} \\ y_s^N \end{bmatrix}^T \times \left(\Psi \begin{bmatrix} \Pi_1 & 0 & 0 \\ 0 & R_u^{N+\lambda_0} & 0 \\ 0 & 0 & R_{v_s}^N \end{bmatrix} \Psi^T \right)^{-1} \begin{bmatrix} x_1(0) \\ u^{N+\lambda_0} \\ y_s^N \end{bmatrix} \tag{22}$$

On the other hand, based on the state space model (1), (2) and (13), the following stochastic descriptor system is introduced

$$M\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma\mathbf{u}(k) \tag{23}$$
$$\mathbf{y}(k) = H\mathbf{x}(k) + \mathbf{v}(k) \tag{24}$$
$$\check{\mathbf{z}}(k_l \mid k) = L\mathbf{x}(k_l) + \mathbf{v}_z(k) \tag{25}$$

where $\mathbf{u}(k), \mathbf{v}(k)$ and $\mathbf{v}_z(k)$, in bold faces, are mutually uncorrelated white noises, $\langle \mathbf{u}(k), \mathbf{u}(j) \rangle = Q_u\delta_{kj}, \langle \mathbf{v}(k), \mathbf{v}(j) \rangle = Q_v\delta_{kj}$, and $\langle \mathbf{v}_z(k), \mathbf{v}_z(j) \rangle = Q_{v_z}\delta_{kj}$,while

$$Q_u = I_r, Q_v = I_m, Q_{v_z} = -\gamma^2 I_p \tag{26}$$

It is easy to know that $J_{l,N}$ in (22) can be rewritten as

$$J_{l,N} = \begin{bmatrix} x_1(0) \\ u^{N+\lambda_0} \\ y_s^N \end{bmatrix}^T \left\langle \begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{u}^{N+\lambda_0} \\ \mathbf{y}_s^N \end{bmatrix}, \begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{u}^{N+\lambda_0} \\ \mathbf{y}_s^N \end{bmatrix} \right\rangle^{-1} \begin{bmatrix} x_1(0) \\ u^{N+\lambda_0} \\ y_s^N \end{bmatrix}$$

where $\mathbf{x}_1(0)$, $\mathbf{u}^{N+\lambda_0}$, and $\mathbf{y}_s^N$ in bold faces are from Krein space (23)–(25) and have the same forms as (14), (21). Since the covariance of $\mathbf{v}_z(k)$ is negative, $\check{\mathbf{z}}(k_l \mid k)$ is regarded as one 'fictitious' observation at time k. Now we recall the discussion [14], it is easy to know that the H_∞ fixed-lag smoother $\check{z}(k_l \mid k)$ that is to be sought in this paper is in fact an H_2 filtering problem for the associated system with current and delayed measurements as (23)-(25).

IV. H_∞ Fixed-Lag Smoothing for Descriptor Systems

From [15], it is easy to know that the descriptor system (23)–(25) is RSE to the following Krein-space stochastic system

$$\bar{\mathbf{x}}(k+1) = \bar{\Phi}\bar{\mathbf{x}}(k) + \bar{\Gamma}\bar{\mathbf{u}}(k) \tag{27}$$
$$\mathbf{y}(k) = \bar{H}\bar{\mathbf{x}}(k) + \mathbf{v}(k) \tag{28}$$
$$\check{\mathbf{z}}(k_l \mid k) = \bar{L}\bar{\mathbf{x}}(k_l) + \mathbf{v}_z(k) \tag{29}$$

where

$$\bar{\mathbf{u}}(k) = \mathbf{u}(k+\lambda_0-1) \tag{30}$$
$$\bar{\mathbf{x}}(k) = \begin{bmatrix} \mathbf{x}_1^T(k) & \mathbf{x}_2^T(k) & \mathbf{x}_3^T(k) \end{bmatrix}^T \tag{31}$$
$$\mathbf{x}_3(k) = \begin{bmatrix} \mathbf{u}^T(k) & \mathbf{u}^T(k+1) & \cdots & \mathbf{u}^T(k+\lambda_0) \end{bmatrix}^T \tag{32}$$

$$\bar{\Phi} = \begin{bmatrix} \Phi_1 & 0 & \Phi_2 \\ 0 & 0 & \Phi_3 \\ 0 & 0 & \Phi_4 \end{bmatrix} \tag{33}$$

Φ_1 is defined in (4) and

$$\Phi_2 = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times r(\lambda_0+1)}$$
$$\Phi_3 = \begin{bmatrix} 0 & -\Gamma_2 & -M_1\Gamma_2 & \cdots & -M_1^{\lambda_0-1}\Gamma_2 \end{bmatrix}$$
$$\Phi_4 = \begin{bmatrix} 0 & I_r & 0 & \cdots & 0 \\ 0 & 0 & I_r & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{r(\lambda_0+1)\times r(\lambda_0+1)}$$
$$\bar{\Gamma} = \begin{bmatrix} 0 & 0 & \Gamma_3^T \end{bmatrix}^T \in \mathbb{R}^{(n+(\lambda_0+1))\times r}$$
$$\Gamma_3 = \begin{bmatrix} 0 & 0 & \cdots & I_r \end{bmatrix}^T \in \mathbb{R}^{r(\lambda_0+1)\times r}$$
$$\bar{H} = \begin{bmatrix} H_1 & H_2 & 0 \end{bmatrix} \in \mathbb{R}^{m\times(n+(\lambda_0+1))}$$
$$\bar{L} = \begin{bmatrix} L_1 & L_2 & 0 \end{bmatrix} \in \mathbb{R}^{s\times(n+(\lambda_0+1))}$$

and

$$\bar{L}\bar{\mathbf{x}}(k) = L\mathbf{x}(k) = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \end{bmatrix} \tag{34}$$

with $\mathbf{u}(k), \mathbf{v}(k), \mathbf{v}_z(k)$ having the same definition as in (26). Based on the state space model (27), combining (28) with (29), we have

$$\bar{\mathbf{x}}(k+1) = \bar{\Phi}\bar{\mathbf{x}}(k) + \bar{\Gamma}\bar{\mathbf{u}}(k) \tag{35}$$
$$\mathbf{y}_s(k) = \begin{cases} \bar{H}\bar{\mathbf{x}}(k) + \mathbf{v}_s(k), \ 0 \le k < l \\ \begin{bmatrix} \bar{H}\bar{\mathbf{x}}(k) \\ \bar{L}\bar{\mathbf{x}}(k-l) \end{bmatrix} + \mathbf{v}_s(k), k \ge l \end{cases} \tag{36}$$

with $\mathbf{v}_s(k)$ having the similar definition as in (17). Now we introduce the innovation of $\mathbf{y}_s(k)$ as

$$\mathbf{w}_s(k) \triangleq \begin{bmatrix} \mathbf{w}_{s_y}(k) \\ \mathbf{w}_{s_z}(k) \end{bmatrix} = \mathbf{y}_s(k) - \hat{\mathbf{y}}_s(k \mid k-1) \tag{37}$$

where $\hat{\mathbf{y}}_s(k \mid k-1)$ is obtained from the Krein space projection of $\mathbf{y}_s(k)$ onto the following linear space $\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^{k-1}$. It is obvious that $\mathbf{w}_s(k)$ is a white noise sequence with the covariance matrix $Q_{w_s}(k)$, i.e., $\langle \mathbf{w}_s(k), \mathbf{w}_s(j) \rangle = Q_{w_s}(k)\delta_{kj}$.

It will be seen that the innovation covariance matrix plays an important role in the H_∞ estimation. In fact, it is used to check if the H_∞ estimator $\check{\mathbf{z}}(k-l \mid k)$ exists or not. Since $\mathbf{y}_s(k)$ contains delay, we shall calculate the covariance matrix $Q_{w_s}(k)$ by applying re-organization innovation analysis.

A. Re-organized Innovation and Riccati Equation

1) Re-organized Innovation Sequence: As is well known, given the measurement sequence $\{\mathbf{y}_s(i)\}_{i=0}^k$, the optimal state estimator $\hat{\bar{\mathbf{x}}}(k-l \mid k)$ for system (35)–(36) is the projection of $\bar{\mathbf{x}}(k-l)$ on the linear space spanned by the measurement sequence, denoted by $\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^k$ [16]. First, observe from (28) and (36) that for $0 \le k < l$,

$$\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^k = \mathcal{L}\{\mathbf{y}(i)\}_{i=0}^k \tag{38}$$

For $k \ge l$, it is easy to know that the linear space $\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^k$ can be re-organized as

$$\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^k = \mathcal{L}\left\{\{\mathbf{y}_f(i)\}_{i=0}^{k-l}, \mathbf{y}(k-l+1), \ldots, \mathbf{y}(k)\right\} \tag{39}$$

where

$$\mathbf{y}_f(i) \triangleq \begin{bmatrix} \mathbf{y}(i) \\ \check{\mathbf{z}}(i \mid i+l) \end{bmatrix} = \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix} \bar{\mathbf{x}}(i) + \mathbf{v}_f(i) \tag{40}$$

with

$$\mathbf{v}_f(i) = \begin{bmatrix} \mathbf{v}(i) \\ \mathbf{v}_z(i+l) \end{bmatrix}, Q_{v_f} = \begin{bmatrix} I_m & 0 \\ 0 & -\gamma^2 I_p \end{bmatrix} \tag{41}$$

Definition 1: $\hat{\boldsymbol{\zeta}}(j \mid k+i, k), i \ge 0$ is defined as the projection of $\boldsymbol{\zeta}(j)$ onto $\mathcal{L}_f(k+i,k)$, where for $i > 0$

$$\mathcal{L}_f(k+i,k) = \mathcal{L}\left\{\{\mathbf{y}_f(i)\}_{i=0}^k, \mathbf{y}(k+1), \ldots, \mathbf{y}(k+i)\right\} \tag{42}$$

and for $i = 0$

$$\mathcal{L}_f(k,k) = \mathcal{L}\left\{\{\mathbf{y}_f(i)\}_{i=0}^k\right\} \tag{43}$$

Now, we introduce another innovation associated with the re-organized observations $\mathbf{y}(k+i)$ and $\mathbf{y}_f(k)$ as

$$\mathbf{w}(k+i,k) = \mathbf{y}(k+i) - \hat{\mathbf{y}}(k+i \mid k+i-1,k), i > 0 \tag{44}$$

$$\mathbf{w}(k,k) = \mathbf{y}_f(k) - \hat{\mathbf{y}}_f(k \mid k-1, k-1) \tag{45}$$

where $\hat{\mathbf{y}}_f(0 \mid -1,-1) = \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix} \hat{\bar{\mathbf{x}}}(0 \mid -1,-1) = 0$, and $\hat{\mathbf{y}}(k+i \mid k+i-1,k)$ is the projection of $\mathbf{y}(k+i)$ on the linear space of $\mathcal{L}_f(k+i-1,k)$ and $\hat{\mathbf{y}}_f(k \mid k-1,k-1)$ is the projection of $\mathbf{y}_f(k)$ on the linear space $\mathcal{L}_f(k-1,k-1)$. It is clear that $\mathbf{w}(k,k)$ is the standard Kalman filtering innovation sequence of the observation $\mathbf{y}_f(k)$. Then we have the following relationships:

$$\mathbf{w}(k+i,k) = \bar{H}\mathbf{e}(k+i,k) + \mathbf{v}(k+i), i > 0 \tag{46}$$

$$\mathbf{w}(k,k) = \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix} \mathbf{e}(k,k) + \mathbf{v}_f(k) \tag{47}$$

where

$$\mathbf{e}(k+i,k) = \bar{\mathbf{x}}(k+i) - \hat{\bar{\mathbf{x}}}(k+i \mid k+i-1,k), i > 0 \tag{48}$$

$$\mathbf{e}(k,k) = \bar{\mathbf{x}}(k) - \hat{\bar{\mathbf{x}}}(k \mid k-1,k-1) \tag{49}$$

It is clear that $\mathbf{e}(k+1,k) = \mathbf{e}(k+1,k+1)$. The following lemma shows that $\{\mathbf{w}(\cdot,\cdot)\}$ is in fact the innovation sequence.

Lemma 1: $\{\mathbf{w}(0,0), \ldots, \mathbf{w}(k_l,k_l), \mathbf{w}(k_l+1,k_l), \ldots, \mathbf{w}(k,k_l)\}$ is the mutually uncorrelated innovation sequence which spans the same linear space as $\mathcal{L}_f(k,k_l)$ or equivalently $\mathcal{L}\{\mathbf{y}_s(0), \ldots, \mathbf{y}_s(k)\}$.

Proof: The proof is similar to [17]. ■

Remark 1: The re-organized innovation is different from the standard Kalman filtering innovation given by (37), however, they span the same linear space as $\mathcal{L}\{\mathbf{y}_s(0), \ldots, \mathbf{y}_s(k)\}$.

2) Riccati Equation: From (46) and (47), the innovation covariance matrix

$$Q_w(k+i,k) \triangleq \langle \mathbf{w}(k+i,k), \mathbf{w}(k+i,k) \rangle, i \ge 0$$

is given by

$$Q_w(k+i,k) = \begin{cases} \bar{H}P(k+i,k)\bar{H}^T + Q_v, & i > 0 \\ \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix} P(k,k) \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix}^T + Q_{v_f}, & i = 0 \end{cases} \tag{50}$$

where Q_v and Q_{v_f} have similar definition as in (26) and (41) and

$$P(k+i,k) \triangleq \left\langle \mathbf{e}(k+i,k), \mathbf{e}^T(k+i,k) \right\rangle, i \ge 0 \tag{51}$$

is the covariance matrix of the one-step ahead state estimation error, which can be computed using the lemma below [17].

Lemma 2: The covariance matrix $P(j+1,k)$ for $j = k+1, k+2, \ldots$ can be calculated recursively as

$$P(j+1,k) = \bar{\Phi}P(j,k)\bar{\Phi}^T + \bar{\Gamma}\bar{\Gamma}^T - \bar{\Phi}P(j,k)\bar{H}^T Q_w^{-1}(j,k)\bar{H}P(j,k)\bar{\Phi}^T \tag{52}$$

where the covariance matrix $P(k+1,k+1)$ is the solution of the following standard Riccati equation

$$P(k+1,k+1) = \bar{\Phi}P(k,k)\bar{\Phi}^T + \bar{\Gamma}\bar{\Gamma}^T - \bar{\Phi}P(k,k)\begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix}^T Q_w^{-1}(k,k)\begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix} P(k,k)\bar{\Phi}^T \tag{53}$$

where $Q_w(j,k)$ is as in (50).

Remark 2: The initial value $P(0,0)$ in (53) is as $P(0,0) = \langle \bar{\mathbf{x}}(0), \bar{\mathbf{x}}(0) \rangle$ with $\bar{\mathbf{x}}(0) = \begin{bmatrix} \mathbf{x}_1^T(0) & \mathbf{x}_2^T(0) & \mathbf{x}_3^T(0) \end{bmatrix}^T$.

B. Innovation Covariance Matrix $Q_{w_s}(.)$

Definition 2:

$$R_{k+i,k}^{k+j} \triangleq \left\langle \bar{\mathbf{x}}(k+j), \mathbf{e}^T(k+i,k) \right\rangle, i \ge 0 \tag{54}$$

is the cross-covariance matrix of the state $\bar{\mathbf{x}}(k+j)$ and the state estimation error $\mathbf{e}(k+i,k)$. And

$$K_{k+i,k}^{k+j} \triangleq \left\langle \bar{\mathbf{x}}(k+j), \mathbf{w}^T(k+i,k) \right\rangle Q_w^{-1}(k+i,k), i \ge 0 \tag{55}$$

is the gain matrix of the innovation $\mathbf{w}(k+i,k)$ in the estimation of the state $\bar{\mathbf{x}}(k+j)$, where $\mathbf{e}(k+i,k)$ is as in (48) and $\mathbf{w}(k+i,k)$ is as in (46).

It follows easily from the above definition that

$$K_{k+i,k}^{k+j} = R_{k+i,k}^{k+j} \bar{H}^T Q_w^{-1}(k+i,k), i > 0 \quad (56)$$

$$K_{k+i,k}^{k+j} = R_{k,k}^{k+j} \begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix}^T Q_w^{-1}(k,k), i = 0 \quad (57)$$

where $K_{k,k}^{k+1}$ is the Kalman filtering gain matrix. Using the projection formula and taking the above definition into consideration,

$$\hat{\bar{\mathbf{x}}}(k+j \mid k+i,k) = \hat{\bar{\mathbf{x}}}(k+j \mid k+i-1,k) + K_{k+i,k}^{k+j}\mathbf{w}(k+i,k) \quad (58)$$

and

$$\hat{\bar{\mathbf{x}}}(k+j \mid k,k) = \hat{\bar{\mathbf{x}}}(k+j \mid k-1,k-1) + K_{k,k}^{k+j}\mathbf{w}(k,k) \quad (59)$$

Furthermore, the cross-covariance matrix $R_{k+i,k}^{k+j}$ for $i \geq 0$ can be computed as [15]:

$$R_{k+i,k}^{k+j} = \begin{cases} P(k+j,k)A^T(k+j,k)\cdots A^T(k+j-1,k), i \geq j \\ \bar{\Phi}^{i-j+2}P(k+i,k), i < j \end{cases} \quad (60)$$

or be calculated through the recursion form

$$R_{k+i,k}^{k+j} = \begin{cases} R_{k+i-1,k}^{k+j}A^T(k+i-1,k), R_{k+j,k}^{k+j} = P(k+j,k), i \geq j \\ \bar{\Phi}R_{k+i,k}^{k+j-1}, R_{k+i,k}^{k+i} = P(k+i,k), i < j \end{cases} \quad (61)$$

where $A(k+i,k), i > 0$ is given by

$$A(k+i,k) = \bar{\Phi}\{I_n - P(k+i,k)\bar{H}^T Q_w^{-1}(k+i,k)\bar{H}\}. \quad (62)$$

For $i \leq 0$, we let $P(k+i,k) = P(k+i,k+i)$ and $A(k+i,k) = A(k+i,k+i)$, where the matrix $A(k+i,k+i)$ is given by

$$A(k+i,k+i) = \bar{\Phi} \times \left(I_n - P(k+i,k+i)\begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix}^T Q_w^{-1}(k+i,k+i)\begin{bmatrix} \bar{H} \\ \bar{L} \end{bmatrix}\right) \quad (63)$$

Theorem 1: The innovation covariance matrix $Q_{w_s}(k)$ is given by

$$Q_{w_s}(k) = \begin{cases} \bar{H}P(k,k_l-1)\bar{H}^T + Q_v, \ 0 \leq k < l \\ \begin{bmatrix} \bar{H}P(k,k_l-1)\bar{H}^T & \bar{H}[R_{k,k_l-1}^{k_l}]^T\bar{L}^T \\ \bar{L}R_{k,k_l-1}^{k_l}\bar{H}^T & \bar{L}\mathcal{P}(k_l)\bar{L}^T + Q_{v_z} \end{bmatrix} \\ \text{for } k \geq l \end{cases} \quad (64)$$

where

$$\mathcal{P}(k_l) = P(k,k_l-1) - \sum_{i=0}^{l-1} R_{k_l+i,k_l-1}^{k_l}\bar{H}^T Q_w^{-1}(k_l+i,k_l-1)\bar{H}[R_{k_l+i,k_l-1}^{k_l}]^T \quad (65)$$

$Q_w(\cdot)$ and $P(\cdot,\cdot)$ are as in (50), (52) and (53) , and $R_{k_l+i,k_l-1}^{k_l}, i = 0,1,\ldots,l$ are calculated recursively via (60), the covariance matrices Q_v, Q_{v_z}, and Q_{v_f} are as given in (26) and (41).

Proof: For $k \geq l$, the innovation is

$$\mathbf{w}_s(k) = \mathbf{y}_s(k) - \hat{\mathbf{y}}_s(k \mid k-1) = \begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{L} \end{bmatrix}\begin{bmatrix} \bar{\mathbf{x}}(k) - \hat{\bar{\mathbf{x}}}(k \mid k-1) \\ \bar{\mathbf{x}}(k_l) - \hat{\bar{\mathbf{x}}}(k_l \mid k-1) \end{bmatrix} + \mathbf{v}_s(k) \quad (66)$$

where $\hat{\mathbf{y}}_s(k \mid k-1)$ and $\hat{\bar{\mathbf{x}}}(k \mid k-1)$ are the projections of $\mathbf{y}_s(k)$ and $\bar{\mathbf{x}}(k)$ on the linear space of $\mathcal{L}\{\mathbf{y}_s(i)\}_{i=0}^{k-1}$, respectively. Note that

$$\hat{\bar{\mathbf{x}}}(k \mid k-1) = \hat{\bar{\mathbf{x}}}(k \mid k-1, k_l-1) \quad (67)$$

$$\hat{\bar{\mathbf{x}}}(k_l \mid k-1) = \hat{\bar{\mathbf{x}}}(k_l \mid k-1, k_l-1) \quad (68)$$

we have

$$\mathbf{w}_s(k) = \begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{L} \end{bmatrix}\begin{bmatrix} \mathbf{e}(k,k_l-1) \\ \boldsymbol{\eta}(k) \end{bmatrix} + \mathbf{v}_s(k) \quad (69)$$

where

$$\boldsymbol{\eta}(k) = \bar{\mathbf{x}}(k_l) - \hat{\bar{\mathbf{x}}}(k_l \mid k-1) = \mathbf{e}(k_l,k_l-1) - \sum_{i=0}^{l-1} K_{k_l+i,k_l-1}^{k_l}\mathbf{w}(k_l+i,k_l-1) \quad (70)$$

Then the innovation covariance matrix $Q_{w_s}(k)$ is given by

$$Q_{w_s}(k) = \langle \mathbf{w}_s(k), \mathbf{w}_s(k) \rangle = \begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{L} \end{bmatrix} S(k,k_l-1) \begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{L} \end{bmatrix}^T + Q_{v_s}(k) \quad (71)$$

where

$$S(k,k_l-1) = \begin{bmatrix} \langle \mathbf{e}(k,k_l-1), \mathbf{e}(k,k_l-1) \rangle & \langle \mathbf{e}(k,k_l-1), \boldsymbol{\eta}(k) \rangle \\ \langle \boldsymbol{\eta}(k), \mathbf{e}(k,k_l-1) \rangle & \langle \boldsymbol{\eta}(k), \boldsymbol{\eta}(k) \rangle \end{bmatrix}$$

Observe (51), we have $\langle \mathbf{e}(k,k_l-1), \mathbf{e}(k,k_l-1) \rangle = P(k,k_l-1)$. Also, by considering the fact that $\boldsymbol{\eta}(k)$ is uncorrelated with $\mathbf{w}(k_l+i,k_l-1), i = 0,1,\ldots,l-1$, it follows that

$$\langle \boldsymbol{\eta}(k), \mathbf{e}(k,k_l-1) \rangle = \langle \mathbf{e}(k_l,k_l-1), \mathbf{e}(k,k_l-1) \rangle = R_{k,k_l-1}^{k_l}$$

and

$$\langle \boldsymbol{\eta}(k), \boldsymbol{\eta}(k) \rangle = P(k_l,k_l-1) - \sum_{i=0}^{l-1} K_{k_l+i,k_l-1}^{k_l} \times Q_w(k_l+i,k_l-1)\left[K_{k_l+i,k_l-1}^{k_l}\right]^T$$

Thus, from (71), the innovation covariance matrix $Q_{w_s}(k)$ for $k \geq l$ is given by (64). For $k < l$, since the innovation is given directly as

$$\mathbf{w}_s(k) = \mathbf{y}(k) - \hat{\mathbf{y}}(k \mid k-1) = \bar{H}\left[\bar{\mathbf{x}}(k) - \hat{\bar{\mathbf{x}}}(k \mid k-1)\right] + \mathbf{v}(k) \quad (72)$$

then it's covariance matrix $Q_{w_s}(k)$ is given by (64). ■

C. Main Results

Theorem 2: Consider the system (1)-(3) and the associated performance criterion (11). Suppose Assumption 1 is satisfied and the recursion (53) is bounded, we have
1). For a given scalar $\gamma > 0$, an H_∞ estimator $\check{\mathbf{z}}(k_l \mid k)$ that achieves (11) exists if and only if for each $k = l, l+1, \ldots N, Q_{w_s}(k)$ and $Q_{v_s}(k)$ have the same inertia, where $Q_{w_s}(k)$ is given in Theorem 1 and $Q_{v_s}(k)$ is given in (19).
2). The possible central estimator $\check{\mathbf{z}}(k_l \mid k)$ is given by

$$\check{\mathbf{z}}(k_l \mid k) = L\hat{\mathbf{x}}(k_l \mid k, k_l - 1) = \bar{L}\hat{\bar{\mathbf{x}}}(k_l \mid k, k_l - 1)$$

where $\hat{\bar{\mathbf{x}}}(k_l \mid k, k_l - 1)$ is the projection of $\bar{\mathbf{x}}(k_l)$ on the linear space $\mathcal{L}_f(k, k_l - 1), k > l$ and is given as follows

$$\begin{aligned}&\hat{\bar{\mathbf{x}}}(k_l \mid k, k_l - 1) = \hat{\bar{\mathbf{x}}}(k_l \mid k_l - 1, k_l - 1)\\&+\sum\nolimits_{i=0}^{l} R^{k_l}_{k_l+i,k_l-1}\bar{H}^T Q_w^{-1}(k_l + i, k_l - 1)\\&\times[\mathbf{y}(k_l + i) - \bar{H}\hat{\bar{\mathbf{x}}}(k_l + i \mid k_l + i - 1, k_l - 1)]\end{aligned}$$

while $\hat{\bar{\mathbf{x}}}(k_l + i \mid k_l + i - 1, k_l - 1)$ is calculated recursively as

$$\begin{aligned}&\hat{\bar{\mathbf{x}}}(k_l + i + 1 \mid k + i, k_l - 1) = \bar{\Phi}\hat{\bar{\mathbf{x}}}(k_l + i \mid k + i - 1, k_l - 1)\\&+\bar{\Phi}P(k_l + i, k_l - 1)\bar{H}^T Q_w^{-1}(k_l + i, k_l - 1)\\&\times[\mathbf{y}(k_l + i) - \bar{H}\hat{\bar{\mathbf{x}}}(k_l + i \mid k_l + i - 1, k_l - 1)]\end{aligned}$$

with the initial value $\hat{\bar{\mathbf{x}}}(k_l \mid k_l - 1, k_l - 1)$ is given by the Kalman filter, which is computed recursively as

$$\begin{aligned}&\hat{\bar{\mathbf{x}}}(k_l \mid k_l - 1, k_l - 1) = \bar{\Phi}\hat{\bar{\mathbf{x}}}(k_l - 1 \mid k_l - 2, k_l - 2)\\&+\bar{\Phi}P(k_l - 1, k_l - 1)\begin{bmatrix}\bar{H}\\ \bar{L}\end{bmatrix}^T Q_w^{-1}(k_l - 1, k_l - 1)\\&\times\left[\mathbf{y}_f(k_l - 1) - \begin{bmatrix}\bar{H}\\ \bar{L}\end{bmatrix}\hat{\bar{\mathbf{x}}}(k_l - 1 \mid k_l - 2, k_l - 2)\right]\end{aligned}$$

with the initial value $\hat{\bar{\mathbf{x}}}(0 \mid -1, -1) = 0$, $\mathbf{y}_f(i)$ is defined in (40), and all covariance matrices as given by (26) and (41).

Proof: It is straightforward to prove 1). And 2) can be obtained similar to [17], [8] for system (27)–(29). ∎

Remark 3: By applying re-organization innovation analysis, a new approach to H_∞ fixed-lag smoothing is presented. The obtained results are completely different from our previous work reported in [7] in which a steady-state smoother is investigated by using J-spectral factorization. Moreover, the proposed H_∞ fixed-lag smoother is simpler for calculation compared with the results in [7] as we do not require the state augmentation for the "lag" of the smoothing.

V. Conclusion

In this paper, we have investigated the finite horizon H_∞ fixed-lag smoothing for discrete time-varying descriptor systems. An efficient solution, which is new to our knowledge, has been presented. The smoother is given in terms of two RDEs. The approach applied in this paper is the reorganization innovation analysis and the generalized Kalman filtering in Krein space.

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