2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Delay-Dependent Robust Stability and H_∞ Control for Jump Linear Systems with Delays

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***Abstract*— This paper considers robust stochastic stability, stabilization and H_∞ control problems for a class of jump linear systems with time delays. By taking some zero equations, neither model transformation nor bounding for cross terms is required to obtain the delay-dependent results, which are given in terms of linear matrix inequalities (LMIs). Maximum sizes of time delays are also studied for system stability. Furthermore, solvability conditions and corresponding H_∞ control laws are given which provide robust stabilization with a prescribed H_∞ disturbance attenuation level γ. Numerical examples show that our proposed methods are much less conservative than existing results.**

I. INTRODUCTION

Jump linear systems are a special class of hybrid systems with two components in their vector states: the modes and the states. The mode is described by a continuous Markovian process with a finite state space. The state in each mode is represented by a system of differential equations. This class of systems has the advantage of representing physical systems with abrupt variations adequately, e.g., solar thermal central receivers, economic systems, and so on. Therefore, a lot of attention has been paid to the stability analysis and controller synthesis for jump linear systems [1]-[5].

In process industries, time delays and parameter uncertainties always exist, which are sources of instability and oscillations and make the problems (e.g., stability, controller design, etc.) difficult to solve. So studies of the stability criteria and the performance for uncertain jump linear systems with delays are of theoretical and practical importance. The criteria can be generally classified into two categories: delay-independent and delay-dependent ones. Since delay-dependent criteria make use of information on the length of delays, they are less conservative than delay-independent ones, especially when the time delays are small. Thus much attention is paid on delay-dependent stability and stabilization recently [6]-[11]. In [6], delay-dependent stability conditions were obtained based on a first-order model transformation. Since additional eigenvalues are introduced, the transformed system is not equivalent to the original system. In [7], a neutral model transformation was presented, where no additional eigenvalues were needed. But an additional assumption is required to obtain the stability condition for the system. In [8], a new model transformation was introduced to guarantee the equivalence of the transformed system and the original system; it also obtained a less conservative inequality by introducing a free matrix and was further extended to a more general form in [9]. But [8] and [9] only replaced some delay terms $x(t-\tau)$ by the Leibniz-Newton formula to derive the stability condition. Since all delay terms affect the result, which one should be replaced is difficult to decide. Reference [10] combined a descriptor model transformation with Park and Moon's inequalities to yield a new transformed system; however, it was also based on the substitution for some $x(t-\tau)$, and did not entirely overcome the conservatism of the methods in [8], [9]. Reference [11] introduced some zero equations to reduce the conservatism induced by model transformations; therefore, it was least conservative. But in [11] only stability analysis was considered for a class of linear time-delay systems; the system performance analysis and controller synthesis were not considered.

In this paper, we focus on both delay-dependent stability analysis and H_∞ control synthesis for a class of jump linear time-delay systems. By introducing some zero equations, which are similar to those in [11], sufficient conditions for robust stochastic stability and stochastic stabilization are derived in the form of LMIs, where no model transformation is needed. Note that in computing the derivative of our Lyapunov functional which is different from [11], both the state and its derivative are maintained, by which substitution and bounding for cross terms are not needed. Thus the results obtained are less conservative and over-design is avoided to some extent.

Briefly, the paper is organized as follows. Section 2 introduces the problem definitions. Section 3 studies the robust stability and stabilization problems, giving solvability conditions and stabilizing controllers in terms of LMI's. Section 4 considers the robust H_∞ control problem – design a control law to stabilize the uncertain jump linear system with a prescribed disturbance attenuation level $\gamma > 0$, and presents an LMI solution. Section 5 provides several numerical examples to provide illustration of the effectiveness of our results and comparison with existing results. Finally, Section 6 offers some concluding remarks.

II. PROBLEM STATEMENT

Given a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ where Ω is the sample space, $\mathbb{F}$ is the algebra of events and $\mathbb{P}$ is the probability

This research was supported by NSERC and China 973 Program (2002CB312200).

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0-7803-9098-9/05/$25.00 ©2005 AACC

measure defined on $\mathbb{F}$. $\{\eta_t, t \geq 0\}$ is a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $S = \{1, 2, \cdots, s\}$ with generator $\Lambda = (\lambda_{ij})$. The transition probability from mode i at time t to mode j at time $t + \Delta t$, $i, j \in S$ is

$$\Pr\left(\eta_{t+\Delta t} = j \middle| \eta_t = i\right) = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t) & i \neq j \\ 1 + \lambda_{ij}\Delta t + o(\Delta t) & i = j \end{cases} \tag{1}$$

where $\Delta t > 0$, $\lim_{\Delta t \to 0}(o(\Delta t)/\Delta t) = 0$ and the transition probability rates satisfy $\lambda_{ij} \geq 0$ for $i, j \in S$, $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{s} \lambda_{ij}$. We consider a class of stochastic uncertain systems over the space $(\Omega, \mathbb{F}, \mathbb{P})$ described by:

$$\begin{cases} \dot{x}(t) = [A(\eta_t) + \Delta A(\eta_t, t)]\, x(t) \\ \qquad\quad + [A_\tau(\eta_t) + \Delta A_\tau(\eta_t, t)]\, x(t-\tau) \\ \qquad\quad + [B(\eta_t) + \Delta B(\eta_t, t)]\, u(t) + B_w(\eta_t)\, w(t) \\ z(t) = \begin{bmatrix} C(\eta_t)\, x(t) + D_w(\eta_t)\, w(t) \\ C_\tau(\eta_t)\, x(t-\tau) \\ D(\eta_t)\, u(t) \end{bmatrix} \\ x(t) = \psi(t),\ -\tau \leq t \leq 0,\ \eta_0 = r_0 \end{cases} \tag{2}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $z(t) \in \Re^r$ is the system output, $w(t) \in \Re^q$ is the deterministic disturbance input which belongs to $L_2[0, \infty)$. $A(\eta_t)$, $A_\tau(\eta_t)$, $B_w(\eta_t)$, $B(\eta_t)$, $C(\eta_t)$, $C_\tau(\eta_t)$, $D_w(\eta_t)$ and $D(\eta_t)$ are known constant matrices of appropriate dimensions, $\Delta A(\eta_t, t)$, $\Delta A_\tau(\eta_t, t)$, $\Delta B(\eta_t, t)$ are unknown matrices which represent time-varying parametric uncertainties and are assumed to belong to certain bounded compact sets. τ is the constant time delay of the state in the system which satisfies $0 \leq \tau \leq h$. $\psi(t)$ is a vector-valued initial condition of the continuous state of the mode. For notational simplicity, in the sequel, for $\eta_k = i \in S$, we will denote $A(\eta_t)$ by A_i, $\Delta A(\eta_t, t)$ by ΔA_i, and so on.

The admissible parameter uncertainties are assumed to be of the following forms:

$$\begin{bmatrix} \Delta A_i & \Delta A_{\tau i} & \Delta B_i \end{bmatrix} = H_i \Delta_i \begin{bmatrix} E_{1i} & E_{2i} & E_{3i} \end{bmatrix} \tag{3}$$

Here H_i, E_{1i}, E_{2i} and E_{3i} are known real constant matrices with appropriate dimensions and the elements of Δ_i are Lebesgue measurable for any $\eta_t \in S$ satisfying

$$\Delta_i^T \Delta_i \leq I,\ \forall t \geq 0 \tag{4}$$

Let the Lyapunov-Krasovskii functional be

$$\begin{aligned} V(x, t, \eta_t) = x^T(t)\, P(\eta_t)\, x(t) + \int_{t-\tau}^{t} x^T(s)\, Q x(s)\, ds \\ + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(s)\, R \dot{x}(s)\, ds d\theta \end{aligned} \tag{5}$$

where $P(\eta_t)$, Q, R are positive definite symmetric unknown matrices. We introduce the following definitions.

Definition 1 The free jump system ($u(t) \equiv w(t) \equiv 0$) in (1)-(4) is said to be *robustly stochastically stable* if for all finite $\psi(t) \in \Re^n$ defined on $[-\tau, 0]$ and initial mode r_0, there exists a finite number $\tilde{\Xi}(\psi(\cdot), h, r_0) > 0$ such that

$$\lim_{N \to \infty} \left\{ \int_0^N E \left\| x(\psi, h,\ t) \right\|^2 dt \right\} < \tilde{\Xi}(\psi(\cdot), h, r_0) \tag{6}$$

holds for all admissible uncertainties satisfying (3)-(4), where E is the statistical expectation operator.

Definition 2 The system in (1)-(4) is said to be *robustly stochastically stable with disturbance attenuation level* $\gamma > 0$ if for all $w(t) \in L_2[0, \infty)$, the system is robustly stochastically stable and the response $\{z(t)\}$ under zero initial condition, i.e., $\psi = 0$, satisfies

$$E\left[\int_0^\infty z^T(t)\, z(t)\, dt\right] \leq \gamma^2 \left[\int_0^\infty w^T(t)\, w(t)\, dt\right] \tag{7}$$

Definition 3 The jump system in (1)-(4) is said to be *robust stochastically stabilizable with disturbance attenuation level* $\gamma > 0$ if there exists a state feedback control law

$$u(t) = K(\eta_t)\, x(t) \tag{8}$$

such that the resulting closed-loop system satisfies the inequality in (7).

In this paper we shall investigate techniques of robust stability, robust stabilization and robust H_∞ control which depend on the size of the time delay. Our purpose is to develop criteria for stochastic stability and stabilization of the system in (1)-(4), examine its robustness and design appropriate H_∞ state feedback controllers that guarantee stochastic stability with a prescribed performance γ.

III. ROBUST STABILITY AND STABILIZATION

In this section, we will consider the stability and stabilization of the system in (1)-(4) with $w(t) \equiv 0$. First we introduce the following zero equation which will be used in our main results:

$$\begin{aligned} \Xi_1 = 2\left[x^T(t)\, Y_i + x^T(t-\tau)\, T_i\right] \\ \times \left[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)\, ds\right] = 0 \end{aligned} \tag{9}$$

where Y_i and T_i are unknown constant matrices with appropriate dimensions. On the other hand, for any semi-positive-definite (SPD) matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$$

we have

$$\Xi_2 = h\xi^T(t)\, X \xi(t) - \int_{t-\tau}^{t} \xi^T(t)\, X \xi(t)\, ds \geq 0 \tag{10}$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau) \end{bmatrix}^T$. It is easy to see that the equation (9)-(10) are always satisfied.

Theorem 1: The free jump system in (1)-(4) is robustly stochastically stable for any constant time delay τ satisfying $0 \leq \tau \leq h$, if there exist $P_i = P_i^T > 0$, $Q = Q^T > 0$, $R > 0$, $\varepsilon_i > 0$, a symmetric SPD matrix $X \geq 0$ and appropriately dimensioned matrices M_{1i}, M_{2i}, M_{3i}, Y_i and T_i such that the following LMIs are satisfied:

$$\Theta_1 = \begin{bmatrix} X_{11} & X_{12} & Y_i \\ * & X_{22} & T_i \\ * & * & R \end{bmatrix} \geq 0 \tag{11}$$

$$\Theta_2 = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & M_{1i}H_i & \varepsilon_i^{-1}E_{1i}^T \\ * & \Pi_{22} & \Pi_{23} & M_{2i}H_i & \varepsilon_i^{-1}E_{2i}^T \\ * & * & \Pi_{33} & M_{3i}H_i & 0 \\ * & * & * & -\varepsilon_i^{-1}I & 0 \\ * & * & * & * & -\varepsilon_i^{-1}I \end{bmatrix} < 0 \quad (12)$$

where $*$ denotes blocks that are readily inferred by symmetry and

$$\begin{aligned} &\Pi_{11} = Q + \textstyle\sum_{j=1}^{s} \lambda_{ij} P_j + hX_{11} + M_{1i}A_i \\ &\qquad + A_i^T M_{1i}^T + Y_i + Y_i^T \\ &\Pi_{12} = hX_{12} + M_{1i}A_{\tau i} + A_i^T M_{2i}^T + T_i^T - Y_i \\ &\Pi_{13} = -M_{1i} + P_i + A_i^T M_{3i}^T \\ &\Pi_{22} = -Q + hX_{22} + M_{2i}A_{\tau i} + A_{\tau i}^T M_{2i}^T - T_i - T_i^T \\ &\Pi_{23} = -M_{2i} + A_{\tau i}^T M_{3i}^T \\ &\Pi_{33} = -M_{3i} - M_{3i}^T + hR \end{aligned} \quad (13)$$

Proof: The weak infinitesimal operator $\Im_t^x[\cdot]$ of the stochastic process $\{x(t), \eta_t, t \geq 0\}$, acting on $V(x, t, \eta_t)$ at the point $\{t, x, \ \eta_t = i\}$, is given by

$$\Im_t^x[V] = \frac{\partial V}{\partial t} + \dot{x}^T(t)\frac{\partial V}{\partial x}\bigg|_{\eta_t = i} + \sum_{j=1}^{s} \lambda_{ij} V(t, \ x, \ i, \ j)$$

Then we have

$$\begin{aligned} \Im_t^x[V] \;=\;& x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) \\ & + \tau\dot{x}(t)R\dot{x}(t) - \int_{t-\tau}^{t} \dot{x}(s)R\dot{x}(s)\,ds \\ & + \dot{x}^T(t)P_i x(t) + x^T(t)P_i\dot{x}(t) \\ & + \sum_{j=1}^{s} \lambda_{ij} x^T(t) P_j x(t) \end{aligned} \quad (14)$$

Introducing the following equation

$$\begin{aligned} \Xi_3 = 2&\left[x^T(t)M_{1i} + x^T(t-\tau)M_{2i} + \dot{x}(t)M_{3i}\right] \\ &\times[-\dot{x}(t) + (A_i + \Delta A_i(t))x(t) \\ &+ (A_{\tau i} + \Delta A_{\tau i}(t))x(t-\tau)] = 0 \end{aligned} \quad (15)$$

where M_{1i}, M_{2i}, and M_{3i} are unknown constant matrices with appropriate dimensions, it is obvious to see this free equation is obtained by the free jump system in (2). Adding Equations (9), (10) and (15) to Equation (14), we have the following inequality:

$$\begin{aligned} \Im_t^x[V] \;\leq\;& \bar{\xi}^T(t)\begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & \bar{\Pi}_{13} \\ * & \bar{\Pi}_{22} & \bar{\Pi}_{23} \\ * & * & \bar{\Pi}_{33} \end{bmatrix}\bar{\xi}(t) \\ & - \int_{t-\tau}^{t} \zeta^T(s)\Theta_1\zeta(s)\,ds \\ =\;& \bar{\xi}^T(t)\bar{\Theta}_2\bar{\xi}(t) - \int_{t-\tau}^{t} \zeta^T(s)\Theta_1\zeta(s)\,ds \end{aligned} \quad (16)$$

where $\bar{\xi}(t) = \begin{bmatrix} \xi^T(t) & \dot{x}^T(t) \end{bmatrix}^T$, $\zeta(s) = \begin{bmatrix} \xi^T(t) & \dot{x}^T(s) \end{bmatrix}^T$ and

$$\begin{aligned} \bar{\Pi}_{11} &= Q + \sum_{j=1}^{s} \lambda_{ij} P_j + hX_{11} + \left(A_i^T + \Delta A_i^T\right) M_{1i}^T \\ &\quad + M_{1i}(A_i + \Delta A_i) + Y_i + Y_i^T \\ \bar{\Pi}_{12} &= hX_{12} + M_{1i}(A_{\tau i} + \Delta A_{\tau i}) \\ &\quad + \left(A_{\tau i}^T + \Delta A_{\tau i}^T\right) M_{2i}^T + T_i^T - Y_i \\ \bar{\Pi}_{13} &= -M_{1i} + P_i + \left(A_i^T + \Delta A_i^T\right) M_{3i}^T \\ \bar{\Pi}_{22} &= -Q + hX_{22} + M_{2i}(A_{\tau i} + \Delta A_{\tau i}) \\ &\quad + \left(A_{\tau i}^T + \Delta A_{\tau i}^T\right) M_{2i}^T - T_i - T_i^T \\ \bar{\Pi}_{23} &= -M_{2i} + \left(A_{\tau i}^T + \Delta A_{\tau i}^T\right) M_{3i}^T \\ \bar{\Pi}_{33} &= -M_{3i} - M_{3i}^T + hR \end{aligned}$$

By the Lyapunov stability theory, we know that the system is stochastically stable if there exist $\Theta_1 > 0$ and $\bar{\Theta}_2 < 0$ such that

$$\begin{aligned} \Im_t^x[V] &< \lambda_{\max}\left(\bar{\Theta}_2\right)\left\|\bar{\xi}(t)\right\|_2^2 - \lambda_{\min}(\Theta_1)\int_{t-\tau}^{t} \|\zeta(s)\|_2^2\,ds \\ &< 0 \end{aligned}$$

holds. Then by Lemma 1 [11] and Schur complements, $\bar{\Theta}_2 < 0$ can be easily obtained from the inequality (12). Thus the proof is completed. ■

If the system mode set $S = \{1\}$, the jump linear system is simplified into a general linear system. Then we have the following simplified result.

Corollary 1: The free system in (1)-(4) with $i \in S = \{1\}$ is robustly stable for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R > 0$, $\varepsilon > 0$, a symmetric SPD matrix $X \geq 0$ and appropriately dimensioned matrices M_1, M_2, M_3, Y and T such that the following LMIs are satisfied:

$$\Theta_0 = \begin{bmatrix} X_{11} & X_{12} & Y \\ * & X_{22} & T \\ * & * & R \end{bmatrix} \geq 0$$

$$\Theta_{00} = \begin{bmatrix} \Pi_{110} & \Pi_{120} & \Pi_{130} & M_1H & \varepsilon^{-1}E_1^T \\ * & \Pi_{220} & \Pi_{230} & M_2H & \varepsilon^{-1}E_2^T \\ * & * & \Pi_{330} & M_3H & 0 \\ * & * & * & -\varepsilon^{-1}I & 0 \\ * & * & * & * & -\varepsilon^{-1}I \end{bmatrix} < 0$$

where $*$ denotes blocks that are readily inferred by symmetry and

$$\begin{aligned} &\Pi_{110} = Q + hX_{11} + M_1A + A^TM_1^T + Y + Y^T \\ &\Pi_{120} = hX_{12} + M_1A_\tau + A^TM_2^T + T^T - Y \\ &\Pi_{130} = -M_1 + P + A^TM_3^T \\ &\Pi_{220} = -Q + hX_{22} + M_2A_\tau + A_\tau^TM_2^T - T - T^T \\ &\Pi_{230} = -M_2 + A_\tau^TM_3^T \\ &\Pi_{330} = -M_3 - M_3^T + hR \end{aligned}$$

Next we will present a solution to the robust stabilization problem for the system (1)-(4) with $w(t) \equiv 0$. In order to obtain an LMI solution, we have to restrict ourselves to the case of $M_{1i} = M_{2i} = M_{3i}$, $i \in S$ in the free equation in (15), where M_{1i}^{-1} exists. Then we have the following theorem.

Theorem 2: The jump system in (1)-(4) with $w(t) \equiv 0$ is robustly stochastically stabilizable for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $\tilde{P}_i = \tilde{P}_i^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon_i > 0$, a symmetric SPD matrix

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices $\tilde{M}_{1i}$, N_i, $\tilde{Y}_i$ and $\tilde{T}_i$ such that the following LMIs are true:

$$\tilde{\Theta}_1 = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} & \tilde{Y}_i \\ * & \tilde{X}_{22} & \tilde{T}_i \\ * & * & \tilde{R} \end{bmatrix} \geq 0 \tag{17}$$

$$\tilde{\Theta}_2 = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} & \varepsilon_i H_i & \tilde{M}_{1i}E_{1i}^T + N_i^T E_{3i}^T \\ * & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & \varepsilon_i H_i & \tilde{M}_{1i}E_{2i}^T \\ * & * & \tilde{\Pi}_{33} & \varepsilon_i H_i & 0 \\ * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & -\varepsilon_i I \end{bmatrix} < 0 \tag{18}$$

where

$$\begin{aligned} \tilde{\Pi}_{11} &= \tilde{Q} + \sum_{j=1}^{s} \lambda_{ij}\tilde{P}_j + h\tilde{X}_{11} + A_i\tilde{M}_{1i}^T + \tilde{M}_{1i}A_i^T \\ &\quad + \tilde{Y}_i + \tilde{Y}_i^T + B_iN_i + N_i^T B_i^T \\ \tilde{\Pi}_{12} &= h\tilde{X}_{12} + A_{\tau i}\tilde{M}_{1i}^T + \tilde{M}_{1i}A_i^T + \tilde{T}_i^T - \tilde{Y}_i + N_i^T B_i^T \\ \tilde{\Pi}_{13} &= -\tilde{M}_{1i}^T + \tilde{P}_i + \tilde{M}_{1i}A_i^T + N_i^T B_i^T \\ \tilde{\Pi}_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_{\tau i}\tilde{M}_{1i}^T + \tilde{M}_{1i}A_{\tau i}^T - \tilde{T}_i - \tilde{T}_i^T \\ \tilde{\Pi}_{23} &= -\tilde{M}_{1i} + \tilde{M}_{1i}A_{\tau i}^T \\ \tilde{\Pi}_{33} &= -\tilde{M}_{1i} - \tilde{M}_{1i}^T + h\tilde{R} \end{aligned}$$

In this case, the controller law is given by

$$K_i = N_i\tilde{M}_{1i}^{-T} \tag{19}$$

Proof: With the memoryless state feedback control law $u(t) = K_i x(t)$, where the matrix $K_i \in \Re^{m\times n}$ is to be designed, the resulting closed-loop system becomes

$$\dot{x}(t) = \left(\bar{A}_i + \Delta\bar{A}_i\right)x(t) + \left[A_{\tau i} + \Delta A_{\tau i}\right]x(t-\tau)$$

where $\bar{A}_i = A_i + B_iK_i$, $\Delta\bar{A}_i = \Delta A_i + \Delta B_iK_i$. Hence, the result follows immediately by applying the proof of Theorem 1, representing A_i by $\bar{A}_i$, ΔA_i by $\Delta\bar{A}_i$, pre-multiplying and post-multiplying the resulting inequality (12) by diag$\left\{ M_{1i}^{-1} \quad M_{1i}^{-1} \quad M_{1i}^{-1} \quad \varepsilon_i I \quad \varepsilon_i I \right\}$ and its corresponding transpose respectively, and setting

$$\begin{aligned} &\tilde{Q} = M_{1i}^{-1}QM_{1i}^{-T},\ \tilde{R} = M_{1i}^{-1}RM_{1i}^{-T},\ \tilde{P}_i = M_{1i}^{-1}P_iM_{1i}^{-T} \\ &\tilde{Y}_i = M_{1i}^{-1}Y_iM_{1i}^{-T}, \tilde{T}_i = M_{1i}^{-1}T_iM_{1i}^{-T}, \tilde{X}_{11} = M_{1i}^{-1}X_{11}M_{1i}^{-T} \\ &\tilde{X}_{12} = M_{1i}^{-1}X_{12}M_{1i}^{-T}, \tilde{X}_{22} = M_{1i}^{-1}X_{22}M_{1i}^{-T}, N_i = K_iM_{1i}^T \end{aligned}$$

Thus the proof is completed. ■

Remark 1: Theorems 1 and 2 provide delay-dependent conditions for robust stability and robust stabilization of uncertain time-delayed jump linear systems. Corollary 1 simplifies the results to a general linear system. These results do not need any system transformation, do not require any parameter tuning, and can be tested numerically very efficiently by using standard LMI techniques. Note that in Theorem 2, we restrict the results to the case of $M_{1i} = M_{2i} = M_{3i}$, $i \in S$, which are the free weighting matrices used to express the relationship of the terms $\dot{x}(t)$, $x(t)$ and $x(t-\tau)$ in the free equation. The zero term (15) is inserted into the derivative of the Lyapunov functional so that the LMIs, which determine the stability of the system, does not include any terms containing the product of the Lyapunov matrices and the system matrices. Moreover, the Leibniz-Newton formula (9) is also employed to make the criterion delay-dependent. Another advantage is that the problem of finding the largest τ in the context of Theorems 1 and 2 can be computed by solving the following quasi-convex optimization problem in $X_{11}, X_{12}, X_{22}, P_i$, Q, R, ε_i, M_{1i}, M_{2i}, M_{3i}, Y_i, T_i and h:

$$\begin{aligned} &\max \quad h > 0 \\ &\text{s.t.} \begin{cases} X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0, P_i = P_i^T > 0, \\ Q = Q^T > 0, R > 0, \varepsilon_i > 0 \text{ and} \\ \text{inequality (11)-(12) or inequality (17)-(18)} \end{cases} \end{aligned} \tag{20}$$

Note that the above problem has the form of a generalized eigenvalue problem and can be solved efficiently using the LMI algorithm "GEVP" [5].

IV. ROBUST H_∞ CONTROL

In this section, we will focus on the design of a delay-dependent robust H_∞ controller for the system in (1)-(4). In order to solve this problem, we first consider the problem of robust H_∞ performance analysis for the unforced system, namely $u(t) \equiv 0$. Assume the initial condition is zero; then we have the following theorems.

Theorem 3: Given a scalar $h > 0$, the system in (1)-(4) is robustly stochastically stable with disturbance attenuation γ for any constant time delay τ satisfying $0 \leq \tau \leq h$, if there exist $P_i = P_i^T > 0$, $Q = Q^T > 0$, $R > 0$, a symmetric SPD matrix $X \geq 0$ and appropriately dimensioned matrices M_{1i}, M_{2i}, M_{3i}, Y_i and T_i such that the inequality in (11) and the following LMI are satisfied:

$$\begin{bmatrix} \hat{\Pi}_{11} & \Pi_{12} & \Pi_{13} & M_{1i}H_i & \varepsilon_i^{-1}E_{1i}^T & \hat{\Pi}_{16} \\ * & \hat{\Pi}_{22} & \Pi_{23} & M_{2i}H_i & \varepsilon_i^{-1}E_{2i}^T & M_{2i}B_{wi} \\ * & * & \Pi_{33} & M_{3i}H_i & 0 & M_{3i}B_{wi} \\ * & * & * & -\varepsilon_i^{-1}I & 0 & 0 \\ * & * & * & * & -\varepsilon_i^{-1}I & 0 \\ * & * & * & * & * & \hat{\Pi}_{66} \end{bmatrix} < 0 \tag{21}$$

where $*$ denotes blocks that are readily inferred by symmetry, and

$$\hat{\Pi}_{11} = \Pi_{11} + C_i^TC_i, \quad \hat{\Pi}_{16} = C_i^TD_{wi} + M_{1i}B_{wi}$$
$$\hat{\Pi}_{22} = \Pi_{22} + C_{\tau i}^TC_{\tau i}, \quad \hat{\Pi}_{66} = -\gamma^2 I + D_{wi}^TD_{wi}$$

Proof: The proof is similar to the proof of Theorem 1; hence omitted. ■

Theorem 4: Given a scalar $h > 0$, the system in (1)-(4) is robustly stochastically stabilizable with disturbance attenuation γ for any constant time delay τ satisfying $0 \leq$

$\tau \leq h$ if there exist $\tilde{P}_i = \tilde{P}_i^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon_i > 0$, a symmetric SPD matrix

$$\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0$$

and appropriately dimensioned matrices $\tilde{M}_{1i}$, N_i, $\tilde{Y}_i$ and $\tilde{T}_i$ such that the inequality in (17) and the following LMI are satisfied:

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \varepsilon_i H_i & \Phi_{16} & \Phi_{17} \\ * & \Phi_{22} & \Phi_{23} & B_w & \varepsilon_i H_i & \tilde{M}_{1i} E_{2i}^T & \Phi_{27} \\ * & * & \Phi_{33} & B_w & \varepsilon_i H_i & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_i I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{22}$$

where

$$\begin{aligned}
\Phi_{11} &= \tilde{Q} + \sum_{j=1}^{s} \lambda_{ij} \tilde{P}_j + h\tilde{X}_{11} + A_i \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i^T \\
&\quad + N_i^T B_i^T + B_i N_i + \tilde{Y}_i + \tilde{Y}_i^T \\
\Phi_{12} &= h\tilde{X}_{12} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_i + N_i^T B_i^T + \tilde{T}_i^T - \tilde{Y}_i \\
\Phi_{13} &= -\tilde{M}_{1i}^T + \tilde{P}_i + \tilde{M}_{1i} A_i^T + N_i^T B_i^T \\
\Phi_{14} &= B_w + \tilde{M}_{1i} C_i^T D_{wi} \\
\Phi_{16} &= \tilde{M}_{1i} E_{1i}^T + N_i^T E_{3i}^T \\
\Phi_{17} &= \begin{bmatrix} N_i^T D_i^T & \tilde{M}_{1i} C_i^T & 0 \end{bmatrix} \\
\Phi_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_{\tau i} \tilde{M}_{1i}^T + \tilde{M}_{1i} A_{\tau i}^T - \tilde{T}_i - \tilde{T}_i^T \\
\Phi_{23} &= -\tilde{M}_{1i}^T + \tilde{M}_{1i} A_{\tau i}^T \\
\Phi_{27} &= \begin{bmatrix} 0 & 0 & \tilde{M}_{1i} C_{\tau i}^T \end{bmatrix} \\
\Phi_{33} &= -\tilde{M}_{1i} - \tilde{M}_{1i}^T + h\tilde{R} \\
\Phi_{44} &= -\gamma^2 I + D_{wi}^T D_{wi}
\end{aligned}$$

Moreover, the controller gain can be given by

$$K_i = N_i \tilde{M}_{1i}^{-T} \tag{23}$$

Proof: The proof is similar to the proof of Theorem 2; hence omitted. ■

Remark 2: Theorems 3 and 4 provide delay-dependent methods for robust H_∞ analysis and robust H_∞ synthesis, respectively, for a class of uncertain linear time-delayed jump systems. Note that using the methods of Theorems 3 and 4, the problems of finding the largest h for a given γ, or the smallest γ for a given h can be easily solved without the need of explicitly tuning any parameters. For instance, the smallest γ for a given h obtainable from Theorem 4 can be determined by solving the following convex optimization problem:

$$\begin{aligned} &\min \quad \gamma^2 \\ &s.t. \begin{cases} \tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix} \geq 0, \tilde{P}_i = \tilde{P}_i^T > 0, \\ \tilde{Q} = \tilde{Q}^T > 0, \tilde{R} > 0, \varepsilon_i > 0 \\ \text{and inequalities (17) and (22)} \end{cases} \end{aligned}$$

These results can also be reduced to the case of linear time-delay systems. For example, for the following simplified linear system

$$\begin{cases} \dot{x}(t) = [A + \Delta A(t)] x(t) + [A_\tau + \Delta A_\tau(t)] x(t-\tau) \\ \qquad\quad + Bu(t) + B_w w(t) \\ z(t) = col\{Cx(t),\ Du(t)\} \end{cases} \tag{24}$$

the simplified results for robust stabilizability can be given as below.

Corollary 2: Given a scalar $h > 0$, the simplified system in (24) is robustly stabilizable with disturbance attenuation γ for any constant time delay τ satisfying $0 \leq \tau \leq h$ if there exist $\tilde{P} = \tilde{P}^T > 0$, $\tilde{Q} = \tilde{Q}^T > 0$, $\tilde{R} > 0$, $\varepsilon > 0$, a symmetric SPD matrix $\tilde{X} \geq 0$ and appropriately dimensioned matrices $\tilde{M}_1$, N, $\tilde{Y}$ and $\tilde{T}$ such that the inequality in (17) and the following LMI are true:

$$\begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \bar{\Phi}_{13} & B_w & \varepsilon H & \tilde{M}_1 E_1^T & \bar{\Phi}_{17} \\ * & \bar{\Phi}_{22} & \bar{\Phi}_{23} & B_w & \varepsilon H & \tilde{M}_1 E_2^T & 0 \\ * & * & \bar{\Phi}_{33} & B_w & \varepsilon H & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0$$

where

$$\begin{aligned}
\bar{\Phi}_{11} &= \tilde{Q} + h\tilde{X}_{11} + A\tilde{M}_1^T + \tilde{M}_1 A^T + N^T B^T \\
&\quad + BN + \tilde{Y} + \tilde{Y}^T \\
\bar{\Phi}_{12} &= h\tilde{X}_{12} + A_\tau \tilde{M}_1^T + \tilde{M}_1 A + N^T B^T + \tilde{T}^T - \tilde{Y} \\
\bar{\Phi}_{13} &= -\tilde{M}_1^T + \tilde{P} + \tilde{M}_1 A^T + N^T B^T \\
\bar{\Phi}_{17} &= \begin{bmatrix} N^T D^T & \tilde{M}_1 C^T \end{bmatrix} \\
\bar{\Phi}_{22} &= -\tilde{Q} + h\tilde{X}_{22} + A_\tau \tilde{M}_1^T + \tilde{M}_1 A_\tau^T - \tilde{T} - \tilde{T}^T \\
\bar{\Phi}_{23} &= -\tilde{M}_1^T + \tilde{M}_1 A_\tau^T \\
\bar{\Phi}_{33} &= -\tilde{M}_1 - \tilde{M}_1^T + h\tilde{R}
\end{aligned}$$

Moreover, the controller gain can be given by

$$K = N\tilde{M}_1^{-T}$$

V. NUMERICAL EXAMPLES

In this section, some examples are used to demonstrate that the methods presented in this paper are effective and are an improvement over the existing methods.

Example 1: Consider the nominal free jump system with $w(t) = u(t) = 0$, $\Delta A_i = \Delta A_{\tau i} = 0$, and the following parameters as used in [12], [14]:

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.5 & -1 \\ 0 & -3 \end{bmatrix}, \quad A_{\tau 1} = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 0.3 \end{bmatrix} \\
A_2 &= \begin{bmatrix} -5 & 1 \\ 1 & 0.2 \end{bmatrix}, \quad A_{\tau 2} = \begin{bmatrix} -0.3 & 0.5 \\ 0.4 & -0.5 \end{bmatrix}
\end{aligned}$$

The initial condition is assumed to be $x(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $r_0 = 1$ for $-\tau \leq t \leq 0$. The generator matrix of the

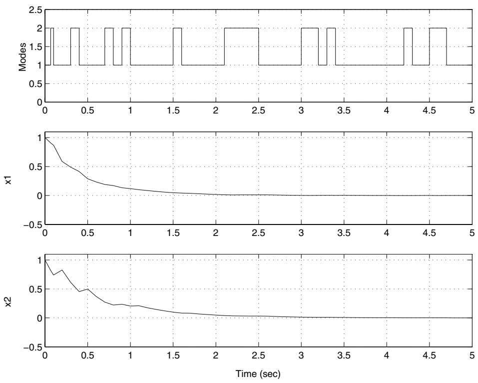

Fig. 1. The state trajectories of the free jumping system (Example 1)

TABLE I
THE UPPER BOUND OF THE TIME DELAY FOR ROBUST STABILITY

Delay bound	[15]	[16]	[17]	[18]	Our result
h	0.1958	0.2010	0.3199	0.4437	2.3970

stochastic process η_t is

$$\lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$$

When $\lambda_1 = 7$ and $\lambda_2 \leq 6$, the result of [12] cannot be applied for stability. When $\lambda_1 = 7$ and $\lambda_2 = 6$, based on the result of [13], the system is found to be delay-independent stable. If we decrease λ_2 more, e.g., $\lambda_1 = 7$ and $\lambda_2 = 3$, the result of [13] cannot guarantee the system stability. But Theorem 1 in [14] can be used to obtain a feasible solution with $\tau \leq h = 0.404$. Moreover, by Theorem 1 of our results, we can obtain a feasible solution with $\tau \leq h = 0.7316$, which is much larger than that of [14]. The state trajectories are shown in Fig. 1 when $h = 0.7316$. By this example we can see that our stability criterion gives a less conservative result than those obtained by the methods of [12], [13] and [14].

Example 2: Consider the free uncertain time-delay system, namely $w(t) = u(t) = 0$, $\eta_t = i$, $i \in S = \{1\}$, where

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \\ H &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

By comparing the robust stability criterion of Corollary 1 with those of [15]-[18] for the above system, we have Table 1. Hence, for this example, the robust stability criterion we derived for linear time-delay systems is less conservative than those reported in [15]-[18].

VI. CONCLUSIONS

In this paper, new delay-dependent conditions for robust stochastic stability and stabilization of jump linear time-delay systems are derived, where none of model transformation, bounding for cross terms and substitution is needed. The H_∞ controller guarantees the robust stability of the delayed jump linear system, while providing a certain level of disturbance attenuation. Moreover, an algorithm for calculating the delay upper bound for system stability is given. All the results are presented in terms of standard LMIs, which are very easy to solve in Matlab. Numerical examples illustrate the effectiveness of our methods. Finally, extension to time-varying systems and discrete-time case are possible.

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