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A Generic Approach to the Design of Decentralized Linear Output-Feedback Controllers

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***Abstract*—A sufficient condition for failure-tolerant performance stabilization in a desirable performance region under decentralized linear output feedback is established. To exploit the flexibility in decentralized control beyond multivariable pole assignment, and to address the subsystem design objectives along with those of the overall system, a generic problem on decentralized linear output feedback is then defined. The problem is reformulated in terms of a constrained nonlinear optimization problem. The proposed methodology results in the optimal reconciliation of failure-tolerant robust performance of the overall system, and (maximal) robustness, disturbance rejection, noninteractive performance, reliability and low actuator gains in the isolated subsystems in the face of unstructured perturbations in the controller and plant parameters. The effectiveness of the proposed approach is demonstrated by an example.**

I. Introduction

IN the broad sense, the existing results on large-scale systems appear in two main directions. On the one hand, some structural properties such as fixed modes, impulsive fixed modes, decentralized fixed modes, etc. have been explored, and on the other hand, some stabilization methods have been developed (see [1]-[5] and the references therein). With reference to stabilization, the literature is teaming with algorithms. All the existing algorithms, however, suffer from some of the following defects: a) dependence on full state information, and lack of: b) performance robustness, c) failure tolerance, d) reliability, e) disturbance rejection, f) noninteractive performance, g) low actuator gains, and h) optimality. Moreover, the complex nature of the problem has encouraged the use of nonlinear control in many of the existing results. Nonlinear controllers, nonetheless, are more expensive and more difficult to implement than the linear ones. Thus, linear controllers are considered in this paper.

In this paper, following the results of [1]-[4], [6]-[8], the abovementioned shortcomings, a-h, are addressed. This work is organized as follows. In Section II, the problem of finding suitable decentralized static output-feedback controllers for the subsystems of a large-scale system is formulated. The proposed formulation introduces some flexibility to the design procedure. In Section III, a sufficient condition for failure-tolerant performance stabilization in a desirable performance region is established. In Section IV, in order to exploit the flexibility in decentralized control beyond multivariable pole placement, and to address the subsystem design objectives in addition to those of the overall system, a generic problem on decentralized linear output feedback is defined. To solve this problem, its objectives are formulated separately. In particular, a new solution to the problem of minimal sensitivity design is presented. The above formulated objectives are then put back together in Section V where a restatement of the original problem is obtained in terms of a constrained nonlinear optimization problem. The proposed methodology results in the optimal reconciliation of failure-tolerant robust performance of the overall system, and (maximal) robustness, disturbance rejection, noninteractive performance, reliability and low actuator gains in the isolated subsystems in the face of unstructured perturbations in the controller and plant parameters. Finally, the effectiveness of the proposed approach is demonstrated by an example.

Throughout the paper it is assumed that the desirable closed-loop eigenvalues are distinct, since they possess better robustness properties than the repeated ones. Also, since the design of a dynamic controller can be reduced to that of a static one [3], [5], all the formulations are given for the static case. All the results are presented for output feedback; state feedback thus follows directly. Besides, to distinguish between a large-scale system and its subsystems, the terms failure tolerance and reliability are used for them, respectively. For the sake of notational simplicity, it is assumed that the transfer functions of the actuators and sensors are one; an actuator/a sensor failure (i.e., a loop disconnection) is thus represented by the suppression of its controller gain to zero.

It should also be noted that in the literature (see e.g. [3], [6]-[8], [10], [11]), by some misuse of the terminology, sometimes it is said that a matrix has (its eigenvalues have)

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low/minimal sensitivity to unstructured uncertainties in its elements if an upper bound (which is the condition number of its modal matrix) of the sensitivities of its eigenvalues is minimized/at its minimum. In other words, instead of minimizing the sensitivities themselves, an upper bound of the sensitivities is minimized. This terminology is used also in this paper. Moreover, since the derived condition for failure-tolerant performance stabilization has some inherent robustness to unstructured perturbations in the controller and plant parameters, the terminology failure-tolerant *performance robustness* of the overall system is used in this paper (see Sections IV.D). It should also be noted that due to lack of space some proofs are omitted.

II. Problem Statement

Consider a large-scale system G with the state-space equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are the system state, input and output matrices, respectively. Let the system be partitioned into N linear-time-invariant subsystems G_i described by

$$\begin{aligned} \dot{x}_i &= A_{ii}x_i + \sum\nolimits_{j=1}^{N} A_{ij}x_j + B_{ii}u_i \qquad j \neq i \\ y_i &= C_{ii}x_i \end{aligned} \tag{2}$$

in which $A_{ij} \in \mathbf{R}^{n_i \times n_j}$, $B_{ii} \in \mathbf{R}^{n_i \times m_i}$, $C_{ii} \in \mathbf{R}^{p_i \times n_i}$, $\sum_1^N n_i = n$, $\sum_1^N m_i = m$ and $\sum_1^N p_i = p$. The terms $\sum_{j=1}^N A_{ij}x_j$, $j \neq i$ describe the interactions with other subsystems. In this work, in contrast to the literature, the isolated subsystems, i.e., the triples (A_{ii}, B_{ii}, C_{ii}) $i = 1,\dots,N$, are not restricted to be minimal (see Remark 4.2).

The interaction between the subsystems and the rest of the system is considered via H in the following decomposition

$$A = A^d + H = diag\{A_{ii}^d\} + H \tag{3}$$

where

$$H_{ii} = A_{ii} - A_{ii}^d \quad , \quad H_{ij} = A_{ij} \quad i \neq j \tag{4}$$

by which some flexibility (degrees of freedom) is introduced into the design procedure by the freedom in choosing A_{ii}^d (see Remark 5.2). The decentralized linear output-feedback controller has the form $K = diag\{K_i\}$ ($i = 1,\dots,N$). As stated before, for notational simplicity it assumed that the transfer functions of the actuators and sensors are one. Thus, the ith local controlled input is given by $u_i = -K_i y_i$. By the application of this controller to the system (2) the closed-loop system will be

$$\dot{x} = (A_{cl} + H)x \tag{5}$$

where $A_{cl} = A^d - BKC = diag\{A_{cl_i}\}$ in which $A_{cl_i} = A_{ii}^d - B_{ii}K_iC_{ii}$ denotes the state matrix of the ith isolated closed-loop subsystem.

III. Failure-Tolerant Performance Stabilization

A sufficient condition for failure-tolerant performance stabilization in a desirable performance region is derived in this Section. The result is developed in two steps, as follows.

A. Decentralized Performance Stabilization

Performance stabilization of a system refers to assigning the poles of the system in some prescribed region which represents the requirements on the stability and performance. From among the common desirable performance regions, i.e., sector, elliptical, vertical strip and parabolic regions [6], sector region is adopted in this paper. The proceeding analysis and synthesis, nevertheless, is applicable to all of the abovementioned regions.

Ω *region*: This is part of the complex s-plane defined by

$$\Omega = \{\lambda : (\Re(\lambda) + \alpha)\cos\delta \pm \Im(\lambda)\sin\delta \leq 0\} \tag{6}$$

where $0 \leq \delta < \pi/2$ and $\alpha \geq 0$ as in the Fig. 1.

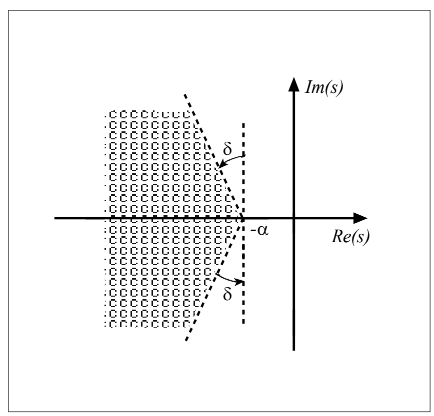

Fig. 1. Ω region as the desired performance region

In case of the sector Ω region defined above, performance stability of the system is equivalent to the stability of its associated augmented system defined below, as stated in the next Theorem.

Augmented system: The system described by

$$\dot{x} = \Theta(\delta) \otimes (A_{cl} + H + \alpha I)x \tag{7}$$

is called the augmented system associated with the system (5) in which

$$\Theta(\delta)=\begin{bmatrix}\cos\delta & -\sin\delta\\ \sin\delta & \cos\delta\end{bmatrix} \tag{8}$$

with $\alpha \ge 0$, $0 \le \delta < \pi/2$ and where $\otimes$ is the Kronecker product of matrices.

Theorem 3.1 [9]: The decentralized controller K stabilizes the overall system (5) into the Ω region iff it stabilizes its associated augmented system (7) into the open left-half complex s-plane.

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Utilizing the above Theorem, a sufficient condition for decentralized performance stabilization is presented in the following Theorem.

Theorem 3.2: If the subcontrollers K_i ($i=1,\ldots,N$) are designed such that

$$\max_{\substack{i\\ i=1,\ldots,N}} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes A_{cl_i} + \Theta(\delta)^T \otimes A_{cl_i}^T}{2}\right) < -\alpha\cos\delta - \lambda_{\max}\left(\frac{\Theta(\delta)\otimes H + \Theta(\delta)^T\otimes H^T}{2}\right) \tag{9}$$

in which $\lambda_{\max}(.)$ denotes the maximum eigenvalue of $(.)$, then performance stability of the system G into the Ω region is assured.

Sketch of the Proof: Let λ_i and v_i ($i=1,\ldots,N$) denote an eigenvalue and its associated right eigenvector of the system (7). Thus, the system *G* has performance stability in the Ω region iff $\Re(\lambda_i) < 0$. Now, use the following:

$$v_i^H v_i \Re\left(\lambda_i\left(\Theta(\delta)\otimes(A_{cl}+H+\alpha I)\right)\right) = \alpha\cos\delta\, v_i^H v_i + v_i^H\left(\frac{\Theta(\delta)\otimes A_{cl} + \Theta(\delta)^T\otimes A_{cl}^T}{2}\right)v_i + v_i^H\left(\frac{\Theta(\delta)\otimes H + \Theta(\delta)^T\otimes H^T}{2}\right)v_i$$

and

$$x^H\left(\frac{M+M^T}{2}\right)x \le x^H x \lambda_{\max}\left(\frac{M+M^T}{2}\right).$$

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In order to reduce the modeling effort, in practice often an upper bound of the interaction is measured; not are all its elements known. In this case the following Corollary can be used.

Corollary 3.1: If only an upper bound of the interaction is known, (9) can be substituted by (10) where $\bar{\sigma}(H)$ and $\|H\|_F$ denote the maximum singular value and the Frobenious norm of H, respectively.

$$\max_{\substack{i\\ i=1,\ldots,N}} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes A_{cl_i} + \Theta(\delta)^T \otimes A_{cl_i}^T}{2}\right) < -\alpha\cos\delta - \bar{\sigma}(H) < -\alpha\cos\delta - \|H\|_F \tag{10}$$

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In the following subsection, tolerance to actuator and/or sensor failure is embedded to the above result.

B. Failure-Tolerant Decentralized Performance Stabilization

Failure in all the actuators and/or sensors of a subsystem results in the disconnection of that subsystem. The disconnection of a subsystem is called islanding, especially in power systems. A sufficient condition for decentralized performance stabilization with tolerance to islanding of any single subsystem is presented in the subsequent Theorem.

Theorem 3.3: If the subcontrollers K_i ($i=1,\ldots,N$) are designed such that

$$\max_{\substack{i\\ i=1,\ldots,N}} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes A_{cl_i} + \Theta(\delta)^T \otimes A_{cl_i}^T}{2}\right) < -\alpha\cos\delta - \max_{\substack{j\\ j=0,\ldots,N}} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes(H+F_j) + \Theta(\delta)^T\otimes(H^T+F_j^T)}{2}\right) \tag{11}$$

where the loop-failure matrix F_j is given by $F_j = diag\{F_{j_{kk}}\}$ in which for $j,k=1,\ldots,N$

$$F_{j_{kk}} = 0_{n_k\times n_k} \qquad k \ne j \tag{12}$$

$$F_{j_{jj}} = -B_{jj}K_j C_{jj} \tag{13}$$

$$F_0 = 0_{n\times n} \tag{14}$$

then tolerance to islanding of any single subsystem and performance stability in the Ω region is guaranteed.

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In the same manner, a sufficient condition for tolerance to islanding of any two (or more) single subsystems can be derived. For the sake of notational simplicity, this is not addressed in this paper. Moreover, for the sake of brevity, the case of the interactions known by an upper bound is not further mentioned in the rest of the paper.

IV. The Decentralized Linear Output-Feedback Problem

Many large-scale problems arise from linearization of mathematical models in them subsystems usually have no physical meanings. In many others, like urban traffic networks, digital communication networks, cooperating robotic systems, and power systems, however, subsystems have real physical entities. In such cases some subsystem design objectives like robustness, disturbance rejection and noninteractive performance, in addition to the overall system design objectives, usually do exist. Thus, to address the subsystem design objectives and to exploit the flexibility in decentralized control beyond multivariable pole placement, a generic problem on decentralized linear output feedback is defined as in the sequel.

Problem $\mathcal{P}$: Design a decentralized static output-feedback controller which achieves robust performance of the overall system with tolerance to islanding of any single subsystem, as well as robustness, disturbance rejection, noninteractive performance, reliability and low actuator gains in the isolated subsystems in the face of unstructured perturbations in the controller and plant parameters.

As is seen, the first objective of the problem $\mathcal{P}$ has already been partly addressed. All the objectives are completely handled in this Section. To this end, these objectives are separately formulated. They are put back together in the next Section where a restatement of the problem $\mathcal{P}$ is obtained as a constrained nonlinear optimization problem.

A. Reliability

The possibility of the occurrence of actuator/sensor failure in a system is proportional to the number of its actuators/sensors. Thus, it is said that reliability is increased by suppressing some controller gains to zero [6], [7]. Hence, some elements of K_i are excluded from the search space (elements of K) and riveted to

$$K_{i_{rs}} = K_{i_{rs}}^{\ d} \tag{15}$$

where $K_{i_{rs}}^{\ d}$'s for some *r*,*s* are some prescribed fixed values, in particular zero. This reduces the controller complexity, alleviates its implementation, and renders a reduction in the computational burden at each iteration, however, at the expense of sacrificing some degrees of freedom.

B. Feasibility

To comply with practical implementation requirements it is necessary to check the feedback gain magnitude. A possibility is to minimize the Frobenious norm of the feedback controller of each isolated subsystem [6]. Thus

$$J_{1,i} = \left\| K_i \right\|_F \tag{16}$$

for $i = 1,\ldots,N$, is introduced as an optimization cost function.

C. Disturbance Rejection and Noninteractive Performance

Considering the effect of the exogenous disturbances u_l^{dist} through the additive term $D_{ll}u_l^{dist}$ on the state equations of the *l*th isolated subsystem, where $D_{ll} \in \mathbf{R}^{n_l \times q_l}$ and $u_l = K_l(r_l - y_l)$, the *l*th isolated closed-loop subsystem will be

$$\begin{aligned} \dot{x} &= (A - BKC)x + BKr \ + Du^{dist} \\ y &= Cx \end{aligned} \tag{17}$$

in which the index l has been omitted for notational simplicity. Based on this notation, let d_j , b_j , c_k , λ_i , v_i , w_i , v_{ij} , y_k , x_j , r_j , u_j^{dist} and $x(0)$ denote the *j*th column of D , the *j*th column of B , the *k*th row of C , the *i*th distinct eigenvalue/mode, its associated right eigenvector, its associated left eigenvector, the *j*th element of the *i*th right eigenvector, the *k*th output, the *j*th state, the *j*th reference input, the *j*th disturbance, and the initial value, respectively. The following Theorems present some conditions for different kinds of decoupling in this subsystem.

Theorem 4.1 [6], [7]: For the unexcited system, i.e., $r = 0$ and $u^{dist} = 0$, the following two propositions are true:

a) A necessary and sufficient condition to decouple the state x_j from the mode λ_i is that $v_{ij}w_i^T x(0) = 0$ which reduces to $v_{ij} = 0$ if λ_i appears in the other modes.

b) A necessary and sufficient condition to decouple the output y_k from the mode λ_i is that $c_k v_i w_i^T x(0) = 0$ which reduces to $c_k v_i = 0$ if λ_i appears in the other outputs.

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Theorem 4.2 [6], [7]: For the undisturbed system with zero initial conditions, i.e., $u^{dist} = 0$ and $x(0) = 0$, the following four propositions are true:

a) A necessary and sufficient condition to decouple the mode λ_i from the input r_j is that $w_i^T \tilde{b}_j = 0$ where $\tilde{b}_j$ is the jth column of BK.

b) If the inputs are linearly independent, a necessary and sufficient condition to decouple the state x_j from the mode λ_i is that $v_{ij} w_i^T BK = 0$ which reduces to $v_{ij} = 0$ if λ_i appears in the other states, i.e., if there exists coupling between λ_i and all of the inputs. If the inputs are linearly dependent, the above condition is a sufficient condition only.

c) If the inputs are linearly independent, a necessary and sufficient condition to decouple the output y_k from the mode λ_i is that $c_k v_i w_i^T BK = 0$ which reduces to $c_k v_i = 0$ if λ_i appears in the other outputs, i.e., if there exists coupling between λ_i and all of the inputs. If the inputs are linearly dependent, the above condition is a sufficient condition only.

d) To decouple the states and the outputs from the inputs, using the static pre-compensator E, the state equations of the lth isolated closed-loop subsystem will be $\dot{x} = (A - BKC)x + BKEr$. Let the jth column of BKE be denoted by $\bar{b}_j$, then: d1) A necessary and sufficient *condition to decouple the state* x_i *from the input* r_j *is that* $v_{xi} w_x^T \bar{b}_j = 0$ *for* $x = 1,...,n$ *. d2) A necessary and sufficient condition to decouple the output* y_k *from the input* r_j *is that* $c_k v_x w_x^T \bar{b}_j = 0$ *for* $x = 1,...,n$ *.*

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Theorem 4.3 [6], [7]: For the unexcited system with zero initial conditions, i.e., $r = 0$ *and* $x(0) = 0$ *, the following four propositions are true:*

a) A necessary and sufficient condition to decouple the mode λ_i *from the disturbance* u_j^{dist} *is that* $w_i^T d_j = 0$ *.*

b) If the disturbances are linearly independent, a necessary and sufficient condition to decouple the state x_j *from the mode* λ_i *is that* $v_{ij} w_i^T D = 0$ *which reduces to* $v_{ij} = 0$ *if* λ_i *appears in the other states, i.e., if there exists coupling between* λ_i *and all of the disturbances. If the disturbances are linearly dependent, the above condition is a sufficient condition only.*

c) If the disturbances are linearly independent, a necessary and sufficient condition to decouple the output y_k *from the mode* λ_i *is that* $c_k v_i w_i^T D = 0$ *which reduces to* $c_k v_i = 0$ *if* λ_i *appears in the other outputs, i.e., if there exists coupling between* λ_i *and all of the disturbances. If the disturbances are linearly dependent, the above condition is a sufficient condition only.*

d) To decouple the states and the outputs from the disturbances, note that a pre-compensator for the disturbances cannot be designed and the state equations of the lth isolated closed-loop subsystem are $\dot{x} = (A - BKC)x + Du^{dist}$ *. Then, d1) A necessary and sufficient condition to decouple the state* x_i *from the disturbance* u_j^{dist} *is that* $v_{xi} w_x^T d_j = 0$ *for* $x = 1,...,n$ *. d2) A necessary and sufficient condition to decouple the output* y_k *from the disturbance* u_j^{dist} *is that* $c_k v_x w_x^T d_j = 0$ *for* $x = 1,...,n$ *.*

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The superposition theorem should then be used to obtain the conditions for the decoupling of states from modes and outputs from modes. A system described as in the above is said to have high noninteractive performance if the above decoupling conditions can be satisfied. However, in most practical situations it is neither possible nor expected to have perfect decoupling [6], [7]. In fact, the criterion for each of them is the minimization of the following cost functions, respectively:

$$J_{2,1} = | v_{ij} w_i^T x(0) | \tag{18}$$

$$J_{2,2} = | c_k v_i w_i^T x(0) | \tag{19}$$

$$J_{2,3} = | v_{ij} | \tag{20}$$

$$J_{2,4} = | c_k v_i | \tag{21}$$

$$J_{2,5} = | w_i^T \tilde{b}_j | \tag{22}$$

$$J_{2,6} = \Sigma_{x=1}^{m} \; | v_{ij} w_i^T \tilde{b}_x | + \Sigma_{z=1}^{q} \; | v_{ij} w_i^T d_z | \tag{23}$$

$$J_{2,7} = \Sigma_{x=1}^{m} \; | c_k v_i w_i^T \tilde{b}_x | + \Sigma_{z=1}^{q} \; | c_k v_i w_i^T d_z | \tag{24}$$

$$J_{2,8} = \Sigma_{x=1}^{n} \; | v_{xi} w_x^T \bar{b}_j | \tag{25}$$

$$J_{2,9} = \Sigma_{x=1}^{n} \; | c_k v_x w_x^T \bar{b}_j | \tag{26}$$

$$J_{2,10} = | w_i^T d_j | \tag{27}$$

$$J_{2,11} = \Sigma_{x=1}^{n} \; | v_{xi} w_x^T d_j | \tag{28}$$

$$J_{2,12} = \Sigma_{x=1}^{n} \; | c_k v_x w_x^T d_j | \; . \tag{29}$$

In the overall system, the decoupling performance of the lth isolated subsystem (17) is inevitably degraded due to the interaction with other subsystems through the terms $\sum_{h=1}^{N} A_{lh} x_h$, $h \neq l$. However, this degradation can be to some extent compensated for by taking into account the interaction terms as some additive disturbances, using the Theorem 4.3, and modifying the above decoupling indices accordingly. For the sake of brevity this is not further followed; the procedure is straightforward.

The right-hand side of each of the above cost functions $J_{2,s}$, $s = 1,...,12,$ is defined as a decoupling index, appropriately, e.g., $w_i^T b_j$ is the ith mode-from-jth input decoupling index. Besides, $w_i^T B$ is defined as the ith mode-from-inputs decoupling vector, Wd_j as the modes-from-jth disturbance decoupling vector, Cv_i as the outputs-from-ith mode decoupling vector, and so forth [6], [7].

D. Minimal Sensitivity

Let the condition number of a matrix be defined by the ratio of its largest singular value to its smallest one. For a linear-time-invariant system described by (1), the problem of minimal sensitivity (maximal robustness) of eigenvalues to unstructured perturbations in the system and controller parameters in linear output feedback design is defined in the sequel.

Problem Q [8]: The problem of minimal sensitivity (maximal robustness) of eigenvalues in static output feedback is to find an analytic solution for the static output feedback gain such that: a) condition number of the modal matrix of the closed-loop state matrix is at its minimum, i.e. one, and b) pole assignment is accomplished in some admissible region Ψ *which represents the requirements on the stability and performance. The region* Ψ *is restricted to produce nondefective (completely diagonalizable) closed-loop state matrices, since such matrices exhibit* *better sensitivity properties than the defective ones [10], [11].*

An analytic solution (as a compact-form sufficient condition) to the problem Q was first given in [8], see e.g. [3], [8], [10], [11]. An alternative solution (as a compact-form sufficient condition) to the problem Q is presented in the next Theorem.

Theorem 4.4: Let a linear-time-invariant multivariable plant be described by (1) with B and C of full column and row ranks, respectively, and the linear output-feedback controller $u = -Ky$. *Let* D^v *be the vector representation of a matrix D obtained by concatenating its rows into a long (column) vector. If a symmetric real matrix Z can be found such that*

$$rank\left(B \otimes C^T \quad (A^T + Z)^v\right) = mp \tag{30}$$

and pole assignment is accomplished in the admissible region Ψ, *then the problem* Q *is solved and the solution in the matrix-vector representation is given by*

$$K^v = -(B \otimes C^T)^+ (A^T + Z)^v . \tag{31}$$

Proof: If the controller K can be designed such that the closed-loop state matrix is symmetric, then part a) of the problem Q is solved [10]. This is possible iff a real symmetric matrix Z can be found such that $-BKC = A^T + Z$ or in the matrix-vector representation $-(B \otimes C^T)K^v = (A^T + Z)^v$ [10]. The last equation is solvable iff $rank\left(B \otimes C^T \quad (A^T + Z)^v\right) = rank\left(B \otimes C^T\right) = rank(B)rank(C^T) = mp$ in which case the solution is given by $K^v = -(B \otimes C^T)^+ (A^T + Z)^v$.

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A natural method to find Z is to use a random-number generator. However, a better and faster approach is to invoke a genetic algorithm [6]-[8], [13]. In order to have maximal robustness of the isolated subsystems either the above method or the method of [8] can be applied to them. However, it is well known that the minimum achievable condition number has a lower bound [11]. Consequently, condition (30) may be impossible to satisfy. In other words, maximal robustness of the isolated subsystems may not be achievable. To compensate for this problem a possibility is to adopt a condition number minimization approach. Thus, in order to have robustness in the isolated subsystems

$$J_{3,k} = cond(V_k) = \frac{\overline{\sigma}(V_k)}{\underline{\sigma}(V_k)} \tag{32}$$

is introduced as a minimization objective in which $\overline{\sigma}(V_k)$ and $\underline{\sigma}(V_k)$ denote the largest and smallest singular values of the modal matrix of A_{cl_k} ($k = 1,\dots,N$), respectively.

E. Failure-Tolerant Decentralized Performance Robustness

The following Theorem states that, independently of the robustness of the isolated subsystems, failure-tolerance *performance robustness* of the overall system is always assured provided failure-tolerant performance stabilization is already guaranteed.

Theorem 4.5: If the subcontrollers K_i ($i = 1,\dots,N$) *are designed such that*

$$\max_{i=1,\dots,N} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes A_{cl_i} + \Theta(\delta)^T \otimes A_{cl_i}^T}{2}\right) < -\alpha\cos\delta - \max_{j=0,\dots,N} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes (H+F_j) + \Theta(\delta)^T \otimes (H^T + F_j^T)}{2}\right) \tag{33}$$

where the loop-failure matrix F_j *is given by (12)-(14), then tolerance to islanding of any single subsystem and robust performance in the* Ω *region in the face of unstructured uncertainties in the controller and plant parameters is guaranteed.*

Sketch of the Proof: Failure-tolerant performance stabilization is guaranteed due to Theorem 3.3. As for the robustness, note that the matrices (.) on both sides, which are the arguments of $\lambda_{\max}(.)$, are symmetric.

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Note that since the state matrix of the overall closed-loop system is not symmetric, it does not have minimal sensitivity (in the sense used in this paper) to unstructured uncertainties in the system and controller parameters. However, since the condition (33) has some inherent robustness, it is said that the overall system has robust performance.

Remark 4.1: Consider the decomposition P=M+N of a matrix P. In general, there is no relation between the sensitivities of P, M and N [10]. Here, the closed-loop state matrix of the overall system is $A_{cl} + H = diag\{A_{ii}^d - B_{ii}K_iC_{ii}\} + H$ *and thus, in general, the low sensitivity of the overall system does not result in the low sensitivity of the isolated subsystems and vice versa. However, as mentioned, the above issues are both addressed in the present paper.*

Remark 4.2: It is known that for a dynamical system with n *states,* m *inputs and* p *outputs in a minimal representation,* $mp>n$ *is a generic (i.e., for almost all systems) sufficient condition for pole assignment by static output feedback [14]. Thus, the order (and number) of the subsystems is dictated by the conditions* $m_i p_i > m_i + p_i + n_i$ ($i = 1,\dots,N$). *Also, it is known that pole placement and entire eigenstructure assignment in linear multivariable output feedback are in effect identical; for a given spectra, different feedback matrices assign different eigenvectors [6]-[8]. Hence, no assumption is made on the controllability and* *observability of the subsystems, since, in other words, condition (9), (10), (11), or (33) is satisfied by eigenvector assignment which is possible for all poles. Consequently, the admissible eigenspectra regions must include the uncontrollable and unobservable modes.*

V. Design Procedure Using Optimization Techniqyes

So far, the design objectives of the problem $\mathcal{P}$ have been separately formulated. A restatement of the original problem can thus be obtained in terms of a constrained nonlinear optimization problem as follows.

Problem $\mathcal{P}$: Minimize the objective function

$$J = \sum_{i=1}^{N} \rho_{1,i} J_{1,i}^{\varphi_{1,i}} + \sum_{l=1}^{N} \sum_{j=1}^{12} \rho_{l,2,j} J_{l,2,j}^{\varphi_{l,2,j}} + \sum_{k=1}^{N} \rho_{3,k} J_{3,k}^{\varphi_{3,k}} \tag{34}$$

subject to:

c1) the admissible eigenspectra regions of the isolated subsystems Ψ_i 's ($i = 1,\ldots,N$), and

c2) the constraint (33),

in the search space $LL_{i_{rs}} \le K_{i_{rs}} \le UL_{i_{rs}}$, $K_{i_{rs}} = K_{i_{rs}}^{d}$ with respect to the unfixed elements of K.

The Ω region (embedded in the constraint c2), the Ψ_i regions, and the controller gain boundaries $LL_{i_{rs}}$'s and $UL_{i_{rs}}$'s depend on the specific application. The coefficients and exponents ρ and φ are positive scalar weightings which must be determined properly to achieve overall satisfactory performance.

It is clear that $\mathcal{P}$ is nonconvex. Convex optimization methods will thus result in a local solution [12], [13]. It is therefore necessary to employ global/random optimization techniques if a global solution is sought. They are the only so far available tools to globally solve nonconvex optimization problems. The superiority of such tools over the conventional ones, in dealing with sophisticated problems, has already been reported, see e.g. [6]-[8], [13] and the references therein.

Remark 5.1: Application of the proposed methodology results in the optimal solution, if any. Otherwise, the admissible eigenspectra regions and/or the controller gain boundaries must be enlarged and/or some weighting factors altered. It should be noted that addressing the constraints also requires some weighting factors. Depending on the complexity of the process and the minimization cost function, this may require some trial and error.

Remark 5.2: The objective function and thus its minimum depend on the choice of the subsystems. In case the subsystems have no physical entities, the flexibility in the decomposition (3) may help solve the problem $\mathcal{P}$ or find a smaller solution for a given cost function.

Remark 5.3: If maximal robustness of the isolated subsystems is a strict design objective, Theorem 4.5 can be substituted by the following Theorem (whose proof is clear) along with the new cost function which follows.

Theorem 5.1: If the subcontrollers K_i ($i = 1,\ldots,N$) are designed such that the isolated closed-loop subsystems have symmetric state matrices and

$$\max_{\substack{i \\ i=1,\ldots,N}} \cos\delta \; \lambda_{\max}\left(A_{cl_i}\right) < -\alpha\cos\delta - \max_{\substack{j \\ j=0,\ldots,N}} \lambda_{\max}\left(\frac{\Theta(\delta)\otimes(H+F_j)+\Theta(\delta)^T\otimes(H^T+F_j^{\,T})}{2}\right) \tag{35}$$

where the loop-failure matrix F_j is given by (12)-(14), then failure tolerance to islanding of any single subsystem and robust performance of the overall system in the Ω region as well as maximal robustness of all the isolated subsystems in the face of unstructured uncertainties in the controller and plant parameters is guaranteed.

Δ.

Another restatement of the problem $\mathcal{P}$ with maximal robustness of the isolated subsystems:

Minimize the objective function

$$J = \sum_{i=1}^{N} \rho_{1,i} J_{1,i}^{\varphi_{1,i}} + \sum_{l=1}^{N} \sum_{j=1}^{12} \rho_{l,2,j} J_{l,2,j}^{\varphi_{l,2,j}} \tag{36}$$

subject to:

c1) the admissible eigenspectra regions of the isolated subsystems Ψ_i 's ($i = 1,\ldots,N$),

c2) the constraint (35), and

c3) the controller gain boundaries $LL_{i_{rs}} \le K_{i_{rs}} \le UL_{i_{rs}}$,

in the space of the real symmetric matrices Z_i 's ($i = 1,\ldots,N$) satisfying the constraint (30).

Remark 5.4: In some previous works like [3], as in the Remark 5.3, elements of K_i ($i = 1,\ldots,N$) were found by optimization over the space of real symmetric $n_i \times n_i$ matrices which has $n_i(n_i+1)/2$ elements. However, to solve the problem $\mathcal{P}$ (the cost function (34)) optimization is performed over K_i ($i = 1,\ldots,N$) which has $m_i p_i$ entries. Since for most practical systems $m_i p_i < n_i(n_i+1)/2$, the proposed approach seems faster and less computationally cumbersome.

To illustrate the design procedure, without loss of generality, a small-scale system is considered in the following example.

Example: Let an unstable dynamical system with strong interactions, $\|H\| \approx \|A\|$*, be described by (1) where*

$$A = \begin{pmatrix} -8.2 & 2.3 & 0.7 & -0.1 & -2.13 & -12.3 & -0.86 & 5.15 \\ -3.2 & -4.6 & 0.43 & 0.1 & -9.81 & -0.12 & 0.15 & -2.2 \\ 0.93 & 0.21 & -7.86 & 0.38 & -11.2 & -12.59 & -0.1 & 1.4 \\ -1.2 & 0.91 & -1.32 & -5.12 & 10.56 & 0.8 & 0.2 & -1.81 \\ -0.2 & -2.3 & 0.65 & 0.23 & -9.3 & 1.2 & 4.5 & 0.66 \\ 0.1 & -0.1 & 0.15 & 0.15 & 2.1 & 0.1 & 0.01 & 0.21 \\ 2.2 & 7.29 & -1.2 & -5.1 & 0.9 & 0.5 & -2.25 & 2.1 \\ -12.32 & -0.23 & 0.12 & 4.31 & 4.2 & 5.11 & -1.4 & -8.3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C = B^T .$$

For this system solve the problem $\mathcal{P}$ *with* $J = \sum_{i=1}^{4}\left(J_{1,i} + J_{3,i}\right)$ *subject to* $|K_i| \le 50$, $i = 1,...,4$, $\Psi_i = \left\{\lambda : \Re(\lambda_j) < -2 \; ; \; j = 1,2\right\}$ $(i = 1,...,4)$ *and* $\Omega = \left\{\lambda : \left(\Re(\lambda_i) + 0.5\right)\cos(\pi/12) \pm \Im(\lambda_i)\sin(\pi/12) \le 0 \; ; \; i = 1,...,8\right\}$.

It is seen that due to the structure of the *B* and *C*, any symmetric choice of the 2×2 subsystem A_{ii}^d , $i = 1,...,4$, results in a symmetric state matrix of its closed-loop isolated subsystem. Thus, the following is tried: $J_{1,i} = 1$, $i = 1,...,4$, are excluded from the objective function and it is reduced to $J = \sum_{i=1}^{4} J_{3,i}$. The unknowns $K_i \in R$, $i = 1,...,4$, are sought to minimize the reduced objective function subject to the given constraints. For the open-loop state matrices of the isolated subsystems as $A_{11}^d = diag\{-8.2,-4.6\}$, $A_{22}^d = diag\{-7.86,-5.12\}$, $A_{33}^d = diag\{-9.3,0.1\}$ and $A_{44}^d = diag\{-2.25,-8.3\}$, a decentralized static output-feedback controller $K^0 = diag\left\{K_1, K_2, K_3, K_4\right\}$ is obtained as

$$K^0 = \begin{pmatrix} 5.69 & 0 & 0 & 0 \\ 0 & 6.65 & 0 & 0 \\ 0 & 0 & 5.81 & 0 \\ 0 & 0 & 0 & 15.65 \end{pmatrix}$$

by which all the constraints are satisfied. As stated before, it may be possible to find a smaller value for the cost function by choosing different isolated subsystems.

VI. Conclusion

To exploit the flexibility in decentralized control beyond multivariable pole assignment, and to address the subsystem design objectives in addition to those of the overall system, a generic problem on decentralized linear output feedback has been defined. The problem has been recast as a constrained nonlinear optimization problem. The proposed methodology results in the optimal reconciliation of failure-tolerant performance robustness of the overall system in a desirable performance region, and low/minimal sensitivity, disturbance rejection, noninteractive performance, reliability and low actuator gains in the isolated subsystems in the face of unstructured uncertainties in the controller and plant parameters. The effectiveness of the proposed approach has been evinced by an example.

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