2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

An Improved Non-Sequential MIMO QFT Design Method

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Abstract—**This paper presents an improved non-sequential MIMO QFT design method for uncertain systems. The paper considers the robust stability of non-sequential MIMO QFT designed closed-loop control systems. A non-sequential MIMO QFT stability theorem is derived that serves as the basis for an improvement of the design methodology, whereby it can be successfully applied to non-minimum phase systems, albeit with a degree of conservatism partially inherent in independent and decentralised design methodologies. The results improve the non-sequential MIMO QFT design methodology and provide insight into the plant cases for which the methodology can be successfully applied.**

I. INTRODUCTION

The extension of the single-input single-output (SISO) Quantitative feedback theory (QFT) design methodology to multivariable systems has provided a number of competing techniques. These can be generalised into two classes, being the non-sequential design methodologies and the sequential design methodologies [1],[2]. The fundamental issue of robust stability has been addressed for both the non-sequential and sequential multi-input multi-output (MIMO) QFT design methodologies [3]-[5]. In regards to the non-sequential design methodology, termed NS MIMO QFT herein, Kerr *et. al.* [3] confirmed the robust stability properties of a successful design when applied to a minimum phase (MP) plant family and hence confirmed the stability properties of the design methodology as presented in [1]. This work represents an extension of the results of [3] that facilitates, through a minor modification of the NS MIMO QFT design methodology, the application of NS MIMO QFT to non-minimum phase (NMP) plant families.

The stability theory developed herein is based on that of [3]. The theory employs non-negative matrix theory and the multivariable Nyquist stability theorem. Existing results from the theory of non-negative matrices and their application to dominance theory can be found in [6], and specifically applied to MIMO QFT designs in [7],[8]. The stability theory developed herein combines the effect of satisfying the robust stability specifications on each loop in the decentralised design and the MIMO performance specifications. The conservatism in the cross-coupling bounds for the satisfaction of the performance specifications is exposed as the mechanism by which NS MIMO QFT couples the robust stability of the true plant family to the robust stability of the equivalent system. This exposition provides valuable insight into the properties of the NS MIMO QFT design methodology.

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II. NS MIMO QFT DESIGN METHODOLOGY

The NS MIMO QFT design methodology provides a design procedure to synthesise a fixed diagonal controller transfer function matrix (TFM) $\mathbf{G}(s)$ and prefilter $\mathbf{F}(s)$ to satisfy specifications on the closed-loop system shown in Fig. 1. Herein this system is referred to as the true system, as it preserves the structure of the MIMO plant $\mathbf{P}(s)$. In contrast to this, the NS MIMO QFT design is performed on a decoupled system, termed the equivalent system. In this section we first describe the properties of the plant family and controller, and the associated assumptions that will be carried through the developments of the paper.

Consider the linear time invariant (LTI) structured plant family $\mathbf{P}(s,\mathbf{\Psi})$ described by an $n\times n$ TFM

$$\mathbf{P}(s,\alpha)=\left[p_{(i,j)}(s,\alpha)\right]=\mathbf{N}_P(s,\alpha)/d_P(s,\alpha), \qquad (1)$$

with uncertain parameter vector α, $\alpha\in\mathbf{\Psi}$, and $\mathbf{P}(s,\alpha)$ a minimal realization. The numerator polynomial matrix $\mathbf{N}_P(s,\alpha)$ and the denominator $d_P(s,\alpha)$ are described by

$$\mathbf{N}_P(s,\alpha)=\left[n_{P(i,j)}(s,\alpha)\right],$$

$$n_{P(i,j)}(s,\alpha)=n_{P(i,j)(m)}(\alpha)s^m+\ldots+n_{P(i,j)(0)}(\alpha)s^0,$$

$$d_P(s,\alpha)=d_{P(l)}(\alpha)s^l+\ldots+d_{P(0)}(\alpha)s^0. \qquad (2)$$

The LTI, proper, diagonal controller is described by an $n\times n$ TFM

$$\mathbf{G}(s)=\text{diag}\left[g_{(i,i)}(s)\right]. \qquad (3)$$

The design of the prefilter $\mathbf{F}(s)$ is not explicitly considered here. The closed-loop characteristic equation for the true system in Fig. 1 is

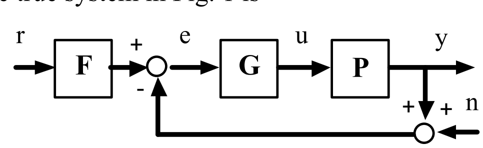

Figure 1. Two Degree of Freedom MIMO Control Structure

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$$\phi_P(s,\boldsymbol{\Psi}) = \det\left[\mathbf{I} + \mathbf{P}(s,\boldsymbol{\Psi})\mathbf{G}(s)\right]. \quad (4)$$

Assumptions on the plant and controller [3]:

(i) $\boldsymbol{\Psi}$ is a compact and connected set.

(ii) $d_{P(k)}(\alpha)$, $k = 0,\ldots,l$, $n_{P(i,j)(k)}(\alpha)$, $k = 0,\ldots,m$, are continuous with respect to α.

(iii) $0 \notin d_{P(l)}(\boldsymbol{\Psi})$.

(iv) The plant $\mathbf{P}(s,\boldsymbol{\Psi})$ is strictly proper and nonsingular.

(v) There are no RHP pole-zero cancellations between $\mathbf{P}(s,\boldsymbol{\Psi})$ and $\mathbf{G}(s)$ or in forming $\phi_P(s,\boldsymbol{\Psi})$.

Consider now the design equations and steps involved in a NS MIMO QFT design. In all the MIMO QFT designs the plant TFM is first transformed into what is termed an equivalent plant TFM on which the design is performed. Denote the plant inverse TFM by

$$\boldsymbol{\Pi}(s,\boldsymbol{\Psi}) = \left[\pi_{(i,j)}(s,\boldsymbol{\Psi})\right] = \mathbf{P}(s,\boldsymbol{\Psi})^{-1}, \quad (5)$$

and
$$\mathbf{Q}(s,\boldsymbol{\Psi}) = \left[q_{(i,j)}(s,\boldsymbol{\Psi})\right] = \left[1/\pi_{(i,j)}(s,\boldsymbol{\Psi})\right]. \quad (6)$$

Decompose $\boldsymbol{\Pi}(s,\boldsymbol{\Psi})$ using a diagonal splitting such that

$$\boldsymbol{\Pi}(s,\boldsymbol{\Psi}) = \mathbf{B}(s,\boldsymbol{\Psi}) + \boldsymbol{\Lambda}(s,\boldsymbol{\Psi})^{-1}, \quad (7)$$

where $\mathbf{B}(s,\boldsymbol{\Psi})$ is the off-diagonal part and $\boldsymbol{\Lambda}(s,\boldsymbol{\Psi})^{-1}$ is the diagonal part of $\boldsymbol{\Pi}(s,\boldsymbol{\Psi})$. With the compensator assumed diagonal, the design is based on the diagonal elements of $\mathbf{Q}(s,\boldsymbol{\Psi})$, which is the equivalent system described by

$$\boldsymbol{\Lambda}(s,\boldsymbol{\Psi}) = \operatorname{diag}\left[q_{(i,i)}(s,\boldsymbol{\Psi})\right] = \operatorname{diag}\left[1/\pi_{(i,i)}(s,\boldsymbol{\Psi})\right]. \quad (8)$$

The elements $q_{(i,i)}(s,\boldsymbol{\Psi})$ are referred to as the equivalent plants for the NS MIMO QFT design. The closed-loop characteristic equation for the equivalent system is then

$$\phi_\Lambda(s,\boldsymbol{\Psi}) = \det\left[\mathbf{I} + \boldsymbol{\Lambda}(s,\boldsymbol{\Psi})\mathbf{G}(s)\right]. \quad (9)$$

Let $\mathbf{E}(s,\boldsymbol{\Psi}) = \boldsymbol{\Lambda}(s,\boldsymbol{\Psi})\mathbf{B}(s,\boldsymbol{\Psi})$. From Eqs. (5) and (7)

$$\mathbf{P}(s,\boldsymbol{\Psi}) = \left[\mathbf{I} + \mathbf{E}(s,\boldsymbol{\Psi})\right]^{-1}\boldsymbol{\Lambda}(s,\boldsymbol{\Psi}). \quad (10)$$

The TFM $\mathbf{E}(s,\boldsymbol{\Psi})$ represents the relative interaction in the system and the relative error in approximating the true system $\mathbf{P}(s,\boldsymbol{\Psi})$ by the diagonal system $\boldsymbol{\Lambda}(s,\boldsymbol{\Psi})$. In NS MIMO QFT, the phase information in the elements of $\mathbf{B}(s,\boldsymbol{\Psi})$ and hence $\mathbf{E}(s,\boldsymbol{\Psi})$ is generally ignored. The robust internal stability problem, under assumptions (iv) and (v) and the properness of the controller, can be stated as

RS: Synthesise a fixed diagonal controller $\mathbf{G}(s)$ such that $\phi_P(s,\boldsymbol{\Psi})$ is Hurwitz.

In the NS MIMO QFT design method of [1] the fixed diagonal controller TFM $\mathbf{G}(s)$ is chosen such that the closed-loop system with equivalent plant family $\boldsymbol{\Lambda}(s,\boldsymbol{\Psi})$ is robustly internally stable. Considering the robust stability (RS) specification, the question that is posed in this paper is under what conditions on the design of the fixed diagonal controller $\mathbf{G}(s)$ for RS of the equivalent system will the RS specification on the true system be satisfied? Here it is shown that in NS MIMO QFT this is connected to the satisfaction of the robust performance (RP) specifications. Furthermore, it is shown that for NMP plant TFMs, the existence of a successful design is contingent on the exploitation of the design freedom provided by not stabilising the equivalent closed loop system with plant TFM $\boldsymbol{\Lambda}(s,\boldsymbol{\Psi})$.

The closed-loop complementary sensitivity TFM and sensitivity TFM for the feedback configuration in Fig. 1 are respectively

$$\mathbf{T}_P = \left[t_{P(i,j)}\right] = \left[\mathbf{I} + \mathbf{PG}\right]^{-1}\mathbf{PGF}, \quad (11)$$

and
$$\mathbf{S}_P = \left[s_{P(i,j)}\right] = \left[\mathbf{I} + \mathbf{PG}\right]^{-1}. \quad (12)$$

The RP problem can be stated as

RP: Synthesise a fixed diagonal controller $\mathbf{G}(s)$ and prefilter $\mathbf{F}(s)$ such that robust tracking, disturbance rejection and sensitivity specifications of the following form are satisfied, $\forall\omega \in [0,\infty)$, $\forall i,j$:

(i) $t^l_{(i,j)}(\omega) \le \left|t_{P(i,j)}(j\omega,\boldsymbol{\Psi})\right| \le t^u_{(i,j)}(\omega)$,

(ii) $\left|s_{P(i,j)}(j\omega,\boldsymbol{\Psi})\right| \le s^u_{(i,j)}(\omega)$,

where the performance bounds are $\mathbf{T}^l = \left[t^l_{(i,j)}\right]$, $\mathbf{T}^u = \left[t^u_{(i,j)}\right]$ and $\mathbf{S}^u = \left[s^u_{(i,j)}\right]$. In NS MIMO QFT the RP problem is satisfied by satisfying the following more stringent RP problem on the equivalent plant TFM [1].

RP: Synthesise a fixed diagonal controller $\mathbf{G}(s)$ and prefilter $\mathbf{F}(s)$ such that, $\forall\omega \in [0,\infty)$, $\forall i,j$:

(i) $t^l_{(i,j)}(\omega) \le \left|t_{e(i,j)}(j\omega,\boldsymbol{\Psi})\right| \le t^u_{(i,j)}(\omega)$,

(ii) $\left|s_{e(i,j)}(j\omega,\boldsymbol{\Psi})\right| \le s^u_{(i,j)}(\omega)$,

where the inequalities are on the elements of the closed-loop TFM in the equivalent system. The closed-loop complementary sensitivity TFM and sensitivity TFM for the equivalent system are respectively

$$\mathbf{T}_e = \left[t_{e(i,j)}\right] = \left[\mathbf{I} + \boldsymbol{\Lambda}\mathbf{G}\right]^{-1}\left[\boldsymbol{\Lambda}\mathbf{GF} + \mathbf{E}\left(\mathbf{H} \circ \mathbf{T}^u\right)\right], \quad (13)$$

and
$$\mathbf{S}_e = \left[s_{e(i,j)}\right] = \left[\mathbf{I} + \boldsymbol{\Lambda}\mathbf{G}\right]^{-1}\left[\mathbf{I} + \mathbf{E} + \mathbf{E}\left(\mathbf{H} \circ \mathbf{S}^u\right)\right], \quad (14)$$

where $\mathbf{H} = \left[h_{(i,j)}\right] = \left[\Delta_{(i,j)}\right]$, $\bar{\sigma}\left(\Delta_{(i,j)}\right) \le 1$, $\forall i,j \in \{1,\ldots,n\}$. The operator $\circ$ is the Schur product where

$\mathbf{A}\circ\mathbf{B}=\left[a_{(i,j)}b_{(i,j)}\right]$. The matrix $\mathbf{H}$ and the Schur product are introduced to account for the loss of phase information in the cross-coupling. Consequently, the stability conditions in NS MIMO QFT are only sufficient. Define

$$\mathbf{T}_{e(r)}=\left[t_{e(r)(i,j)}\right]=\left[\mathbf{I}+\mathbf{\Lambda G}\right]^{-1}\left[\mathbf{\Lambda G F}\right], \quad (15)$$

$$\mathbf{T}_{e(c)}=\left[t_{e(c)(i,j)}\right]=\left[\mathbf{I}+\mathbf{\Lambda G}\right]^{-1}\left[\mathbf{E}\left(\mathbf{H}\circ\mathbf{T}^{u}\right)\right], \quad (16)$$

$$\mathbf{S}_{e(r)}=\left[s_{e(r)(i,j)}\right]=\left[\mathbf{I}+\mathbf{\Lambda G}\right]^{-1}\left[\mathbf{I}+\mathbf{E}\right], \quad (17)$$

$$\mathbf{S}_{e(c)}=\left[s_{e(c)(i,j)}\right]=\left[\mathbf{I}+\mathbf{\Lambda G}\right]^{-1}\left[\mathbf{E}\left(\mathbf{H}\circ\mathbf{S}^{u}\right)\right]. \quad (18)$$

With the phase of the cross-coupling unknown, the worst case phase relationships must be simultaneously satisfied. Using Eqs. (15)-(18) the RP specifications can be represented as

$$t_{(i,j)}^{l}(\omega)\le\left|t_{e(r)(i,j)}(j\omega,\mathbf{\Psi})\right|\pm\left|t_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|\le t_{(i,j)}^{u}(\omega), \quad (19)$$

$$\left|s_{e(r)(i,j)}(j\omega,\mathbf{\Psi})\right|+\left|s_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|\le s_{(i,j)}^{u}(\omega), \quad (20)$$

where the inequalities are to be satisfied $\forall\omega\in[0,\infty)$ and $\forall i,j$. Let $\delta_{T(i,j)}=t_{(i,j)}^{u}-t_{(i,j)}^{l}$. The inequalities in Eqs. (19) and (20) are satisfied by dividing the allowable regions $\delta_{T(i,j)}$ and $s_{(i,j)}^{u}(\omega)$ at each frequency between the two terms in the inequality. Here we assume the allowable regions for the terms in the inequality are nonzero, which is the case in any practical design. From Eqs. (19) and (20), under the above assumption, the satisfaction of the following inequalities on the cross-coupling disturbance transfer functions is a necessary condition for the satisfaction of the RP specification in a NS MIMO QFT design:

$$\left|t_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|<\delta_{T(i,j)}(\omega)\le t_{(i,j)}^{u}(\omega),\ \forall\omega\in[0,\infty), \quad (21)$$

and $$\left|s_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|<s_{(i,j)}^{u}(\omega),\ \forall\omega\in[0,\infty). \quad (22)$$

At a particular frequency $\omega\in[0,\infty)$ either Eq. (21) or (22) can dominate the design. Without loss of generality, consider the dominant bound from Eqs. (21) and (22), and the associated necessary condition for RP, to be

$$\left|s_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|<s_{(i,j)}^{u}(\omega),\ \forall\omega\in[0,\infty),\ \forall i,j. \quad (23)$$

The properties of a NS QFT design that satisfies this necessary condition for RP in NS MIMO QFT, in addition to stabilising $\mathbf{\Lambda}(s,\mathbf{\Psi})$, will be analysed in Section 3 and related to the RS specification.

Remark 1: A more stringent necessary condition for the satisfaction of Eq. (19), that arises from the $\pm$ used in the inequality, is

$$\left|t_{e(c)(i,j)}(j\omega,\mathbf{\Psi})\right|<\delta_{T(i,j)}(\omega)/2,\ \forall\omega\in[0,\infty). \quad (24)$$

III. Robust Stability in NS MIMO QFT

In this section a robust stability theorem is presented which confirms the stability properties of a successful NS MIMO QFT design for MP systems and extends the application of NS MIMO QFT to general NMP systems. The stability theorem combines the effect of satisfying the RP and RS specifications in a NS MIMO QFT design. Importantly, the theorem does not restrict the design to stabilise the equivalent closed-loop with plant TFM $\mathbf{\Lambda}(s,\mathbf{\Psi})$.

In the results to follow the right half plane (RHP) refers to the closed right half of the complex plane. Furthermore, let $N(k,h(s))$ denote the net number of clockwise encirclements of the point $(k,0)$ by the image of the Nyquist contour under $h(s)$. Let $\mathbf{A}\ge\mathbf{0}$ ($\mathbf{A}>\mathbf{0}$) imply that each element of the matrix $\mathbf{A}$ is greater than or equal to zero (greater than zero). $\mathbf{A}$ is then referred to as a nonnegative (positive) matrix. Subsequently, $\mathbf{A}\ge\mathbf{B}$ ($\mathbf{A}>\mathbf{B}$) implies that each element of $\mathbf{A}$ is greater than or equal to (greater than) the corresponding element of $\mathbf{B}$.
The return difference TFM can be decomposed as

$$\left[\mathbf{I}+\mathbf{PG}\right]=\left[\mathbf{P}\right]\left[\mathbf{\Lambda}^{-1}+\mathbf{G}\right]\left[\mathbf{I}+\mathbf{S}_{\Lambda}\mathbf{E}\right], \quad (25)$$

or equivalently

$$\left[\mathbf{I}+\mathbf{PG}\right]=\left[\mathbf{P}\right]\left[\mathbf{\Lambda}^{-1}\right]\left[\mathbf{I}+\mathbf{\Lambda G}\right]\left[\mathbf{I}+\mathbf{S}_{\Lambda}\mathbf{E}\right], \quad (26)$$

and $$\left[\mathbf{I}+\mathbf{PG}\right]=\left[\mathbf{P}\right]\left[\mathbf{G}\right]\left[\mathbf{I}+\mathbf{\Lambda}^{-1}\mathbf{G}^{-1}\right]\left[\mathbf{I}+\mathbf{S}_{\Lambda}\mathbf{E}\right], \quad (27)$$

where $\mathbf{S}_{\Lambda}=\left[\mathbf{I}+\mathbf{\Lambda G}\right]^{-1}$. The following theorem is an application of the multivariable Nyquist stability theorem to the decomposed return difference TFM in Eq. (25). The theorem requires the poles of $\left[\mathbf{\Lambda}^{-1}+\mathbf{G}\right]$ to be defined. When $\left[\mathbf{\Lambda}^{-1}+\mathbf{G}\right]$ is non-proper the alternate decompositions in Eqs. (26) and (27) can be employed with minor modification to the theorem.

Theorem 1. Assume there are no RHP pole-zero cancellations between $\mathbf{P}$ and $\mathbf{G}$ or $\mathbf{\Lambda}$ and $\mathbf{G}$, and no RHP pole-zero cancellations in forming ϕ_P or ϕ_Λ. Let $\mathbf{P}$, $\mathbf{G}$ and $\mathbf{\Lambda}$ respectively have $p_\mathbf{P}$, $p_\mathbf{G}$ and $p_\mathbf{\Lambda}$ poles and $z_\mathbf{P}$, $z_\mathbf{G}$ and $z_\mathbf{\Lambda}$ finite zeros in the RHP. Let $\left[\mathbf{\Lambda}^{-1}+\mathbf{G}\right]$ (equivalently ϕ_Λ) have z_ϕ finite zeros in the RHP. The closed-loop system is internally stable if and only if

$$N\left(0,\det\left[\mathbf{I}+\mathbf{S}_\mathbf{\Lambda}\mathbf{E}\right]\right)=z_\mathbf{\Lambda}-z_\mathbf{P}-z_\phi. \quad (28)$$

Proof: From the multivariable Nyquist stability theorem, with no RHP pole-zero cancellations between $\mathbf{P}$ and $\mathbf{G}$ or in forming ϕ_P, the closed-loop system is internally stable if and only if

$$N\left(0,\det\left[\mathbf{I}+\mathbf{PG}\right]\right)=-p_{\mathbf{P}}-p_{\mathbf{G}}. \quad (29)$$

From Eq. (25), with no RHP pole-zero cancellations between $\mathbf{\Lambda}$ and $\mathbf{G}$ or in forming ϕ_{Λ},

$$\begin{aligned}&N\left(0,\det\left[\mathbf{I}+\mathbf{PG}\right]\right)=-p_{\mathbf{P}}-p_{\mathbf{G}}\\&=N\left(0,\mathbf{P}\right)+N\left(0,\det\left[\mathbf{\Lambda}^{-1}+\mathbf{G}\right]\right)+N\left(0,\det\left[\mathbf{I}+\mathbf{S}_{\mathbf{\Lambda}}\mathbf{E}\right]\right)\\&=\left(-p_{\mathbf{P}}+z_{\mathbf{P}}\right)+\left(-z_{\mathbf{\Lambda}}-p_{\mathbf{G}}+z_{\phi}\right)+N\left(0,\det\left[\mathbf{I}+\mathbf{S}_{\mathbf{\Lambda}}\mathbf{E}\right]\right).\end{aligned} \quad (30)$$

Therefore, Eq. (29) is satisfied if and only if

$$N\left(0,\det\left[\mathbf{I}+\mathbf{S}_{\mathbf{\Lambda}}\mathbf{E}\right]\right)=z_{\mathbf{\Lambda}}-z_{\mathbf{P}}-z_{\phi}. \quad (31)\square$$

Remark 2: Theorem 1 was motivated by the analogous results for designs employing the diagonal elements of $\mathbf{P}$ in [9]. Using the design equations in [10], Theorem 1 can also be extended to control systems employing a fully populated controller.

The following corollary to Theorem 1 provides a condition for the stability of the closed-loop system. The corollary is later shown to be a necessary and sufficient condition for a stable closed-loop system in a successful NS MIMO QFT design.

Corollary 1. Under the assumptions of Theorem 1, with

$$\rho\left(\mathbf{S}_{\mathbf{\Lambda}}\mathbf{E}(j\omega)\right)<1,\ \forall\omega, \quad (32)$$

the closed-loop system is stable if and only if

$$z_{\mathbf{P}}+z_{\phi}=z_{\mathbf{\Lambda}}. \quad (33)$$

Proof: Follows that for Corollary 1 in [3]. □

Based on Theorem 1 and Corollary 1, the following theorem governs the stability of a structured plant family in a NS MIMO QFT design.

Theorem 2 (NS MIMO QFT Stability Theorem). Under the assumptions of Theorem 1, the closed-loop system with diagonal controller $\mathbf{G}(s)$ and MIMO plant family $\mathbf{P}(s,\mathbf{\Psi})$ will be robustly internally stable if:

(i) The NS MIMO QFT cross-coupling specifications (Eqs. (21) and (22)) are satisfied, $\forall\alpha\in\mathbf{\Psi}$.

(ii) $z_{\mathbf{P}}+z_{\phi}=z_{\mathbf{\Lambda}}$, $\forall\alpha\in\mathbf{\Psi}$.

Proof: From Theorem 1, under the assumptions of Theorem 1 and the satisfaction of condition (ii), the closed-loop system is internally stable if and only if

$$N\left(0,\det\left[\mathbf{I}+\mathbf{S}_{\mathbf{\Lambda}}(s,\alpha)\mathbf{E}(s,\alpha)\right]\right)=0,\ \forall\alpha\in\mathbf{\Psi}. \quad (34)$$

From the proof of Corollary 1 in [3], a sufficient condition for the satisfaction of Eq. (34) is

$$\rho\left(\mathbf{S}_{\mathbf{\Lambda}}(\alpha,j\omega)\mathbf{E}(\alpha,j\omega)\right)<1,\ \forall\omega,\ \forall\alpha\in\mathbf{\Psi}. \quad (35)$$

It remains to show that condition (i) ensures Eq. (35) is satisfied. This was proven in Theorem 2 in [3]. Hence Eq. (35) is satisfied and the closed-loop system is internally stable. □

Condition (i) of Theorem 2 is already required to be satisfied in a design. Subsequently, one can conclude that a properly executed NS MIMO QFT design ensures the uncertain closed-loop system is robustly stable provided condition (ii) is satisfied. The following corollary serves as an existence condition for a stable NS MIMO QFT design.

Corollary 2. Under the assumptions of Theorem 1, the closed-loop system can be stabilised if and only if,

$$z_{\mathbf{\Lambda}}\geq z_{\mathbf{P}}. \quad (36)$$

Proof: Follows from $z_{\mathbf{P}},z_{\phi},z_{\mathbf{\Lambda}}\geq 0$ and $z_{\mathbf{P}}+z_{\phi}=z_{\mathbf{\Lambda}}$ being necessary and sufficient for closed-loop stability. □

Remark 3: Note that the above condition only ensures that each plant case in the plant family can be individually stabilised using NS MIMO QFT and not that the entire plant family can be stabilised. The condition will also apply to Theorem 4 of [10].

Corollary 2 provides a useful existence condition for a stabilising controller in NS MIMO QFT. If $z_{\mathbf{\Lambda}}=z_{\mathbf{P}}$, Theorem 2 demands that $z_{\phi}=0$, which is the usual approach employed in NS MIMO QFT where the equivalent closed-loop system with plant $\mathbf{\Lambda}$ is stable. This situation arises in MP plants and plants where each NMP transmission zero of the plant is pinned [11] to only one input and/or output, with a diagonal plant TFM serving as the most obvious example. However, the freedom provided by Theorem 2 is that for plant cases where the NMP transmission zeros are not pinned, robust stability of the closed-loop system with plant $\mathbf{P}$ can still be achieved by allowing $z_{\phi}>0$. Of course other standard conditions in NS MIMO QFT must still be satisfied.

An important feature of the condition $z_{\mathbf{P}}+z_{\phi}=z_{\mathbf{\Lambda}}$ in Theorem 2 is that it gives performance limitations that, whilst conservative due to the independent design method employed, are now much closer to those inherent in the MIMO design. In NMP MIMO plants, only m loops need suffer the performance limitation arising from a NMP zero of algebraic and geometric multiplicity equal to m, as shown by several investigations [12]. In NS MIMO QFT, when the NMP transmission zeros are not pinned, they appear in all the diagonal elements of the equivalent plant $\mathbf{\Lambda}$, giving the geometric and algebraic multiplicity of the zeros to be n in $\mathbf{\Lambda}$ for an $n\times n$ plant TFM. In this case the performance limitations would appear to be exacerbated in the NS MIMO QFT design. However, the increase in the multiplicity, being $n-m$, is exactly $z_{\mathbf{\Lambda}}-z_{\mathbf{P}}$. Hence closed loop stability requires $z_{\phi}=z_{\mathbf{\Lambda}}-z_{\mathbf{P}}=n-m$ and the performance limitations are no longer exacerbated. However, due to the independent design of each loop, the performance limitations arising from a NMP zero can not be shared between the loops which results in design conservatism.

Note that whilst Theorem 2 and the results presented herein permit the successful employment of NS MIMO QFT for controller design with NMP plants, the conservatism in the design due to the plant-controller alignment still exists, as highlighted in [13]. The formulation presented in [14] helps for this problem class. Additional conservatism also arises from the loss of phase information in the cross-coupling. This is captured in the following corollary to Theorem 2 which shows the spectral radius of $\mathbf{S_\Lambda E}$ is unnecessarily constrained to be less than $1/2$ by the satisfaction of the cross-coupling performance specifications in a NS MIMO QFT design.

Corollary 3. The satisfaction of the cross-coupling component of the tracking specifications in a NS MIMO QFT design, $\forall\alpha\in\mathbf{\Psi}$, $\forall\omega\in[0,\infty)$, guarantees,

$$\rho\left(\mathbf{S_\Lambda}(\alpha,j\omega)\mathbf{E}(\alpha,j\omega)\right)<1/2\,,\ \forall\omega\,,\ \forall\alpha\in\mathbf{\Psi}\,. \tag{37}$$

Proof: Follows from the proof of Theorem 2 in [3] using the necessary condition for RP of Eq. (24) in place of Eq. (23). □

Considering Theorems 1 and 2 and Corollaries 1 and 2, and satisfying the assumptions of Theorem 1, the robust internal stability problem for a structured MIMO plant family in a NS MIMO QFT design can be stated as

RS: For a plant family satisfying the existence condition $z_{\mathbf{\Lambda}}\ge z_{\mathbf{P}}$, synthesise a fixed diagonal controller $\mathbf{G}$ such that $z_{\mathbf{P}}+z_\phi=z_{\mathbf{\Lambda}}$ and the RP cross-coupling specifications are satisfied for all plants in the plant family.

The employment of the above conditions for RS facilitates the application of the NS MIMO QFT design methodology to general MP/NMP and stable/unstable MIMO plant families. Subsequently, the NS MIMO QFT design methodology that employs the stability theory presented herein can be considered to be an improved NS MIMO QFT design methodology.

IV. NS MIMO QFT Design Examples

A. Example 1

This section presents a concocted example where the existence condition $z_{\mathbf{\Lambda}}\ge z_{\mathbf{P}}$ is violated by all plants in the plant family. Hence the successful application of the NS MIMO QFT design methodology will result in an unstable closed-loop system. This example serves to demonstrate the potential failure of the NS MIMO QFT design methodology when applied to NMP systems where the existence condition is violated. The example also serves as a counter example to the stability theorem presented in [15]. Consider the plant family

$$\mathbf{P}(s,\mathbf{\Psi})=\begin{bmatrix}-k_{11}(s-2) & -k_{12}(s-2)\\ 6k_{21} & -k_{22}(s-2)\end{bmatrix}\Big/(s+1)(s+2)\,, \tag{38}$$

where $k_{11},k_{12}\in[1,2]$, $k_{21},k_{22}\in[4,5]$. The multivariable plant is stable with one NMP zero. The equivalent plant TFM for the NS MIMO QFT design is

$$\mathbf{\Lambda}=\operatorname{diag}\left[\frac{\varphi}{-k_{22}(s+1)(s+2)},\frac{\varphi}{-k_{11}(s+1)(s+2)}\right]. \tag{39}$$

where $\varphi=(k_{11}k_{22}s-2k_{11}k_{22}+6k_{12}k_{21})$. In order to apply NS MIMO QFT the existence condition $z_{\mathbf{\Lambda}}\ge z_{\mathbf{P}}$ must hold for all plants in the plant family. For this plant family

$$z_{\mathbf{\Lambda}}=0\,,\ z_{\mathbf{P}}=1\ \Rightarrow\ z_{\mathbf{\Lambda}}<z_{\mathbf{P}}\,. \tag{40}$$

Evidently, the existence condition is not satisfied by any plant in the plant family and therefore the NS MIMO QFT design will give an unstable closed-loop system. Along with RS, the following concocted RP specifications are to be satisfied for all plants in the plant family and $\forall i,j\in\{1,2\}$, $\forall\omega\in[0,\infty)$:

RP:
$$\left|1/1+q_{(i,i)}(j\omega)g_{(i,i)}(j\omega)\right|\le 5\,\mathrm{dB}\,, \tag{41}$$

$$\left|\alpha_{(i,j)}(j\omega)\right|\le\left|t_{e(i,j)}(j\omega)\right|\le\left|\beta_{(i,j)}(j\omega)\right|, \tag{42}$$

with $\beta_{(1,1)}=\beta_{(2,2)}=1.2$, $\beta_{(1,2)}=\beta_{(2,1)}=3/5$, $\alpha_{(1,2)}=\alpha_{(2,1)}=0$, $\alpha_{(1,1)}=\alpha_{(2,2)}=1/4(s+1)$. The following controller and prefilter satisfy the RS and RP specifications:

$$\mathbf{G}=\operatorname{diag}(-1000,-1000)\,,\ \mathbf{F}=\operatorname{diag}\left(4/(s+4),4/(s+4)\right). \tag{43}$$

Analysis shows that $\phi_{\mathbf{\Lambda}}$ is Hurwitz. However, $\phi_P(s,\alpha)$ is not Hurwitz for any $\alpha\in\mathbf{\Psi}$ and therefore the uncertain closed-loop system is unstable. This is not surprising, as the true plant family $\mathbf{P}(s,\alpha)$ has a NMP zero at $s=2$, $\forall\alpha\in\mathbf{\Psi}$. With the interpolation constraints satisfied and large feedback gains achieved in the NS MIMO QFT design, it is clear that the true closed-loop system will be unstable despite the stability of the equivalent plant family and the satisfaction of the RP specifications. Hence the existence condition serves as a useful test for a stabilising NS MIMO QFT designed control system prior to controller design.

B. Example 2

This section presents an example where the existence condition is satisfied and the plant TFM is NMP and unstable. The successful application of the NS MIMO QFT design methodology should therefore result in a stable closed-loop system. Consider the following plant family with uncertainty limited to the input gain for simplicity:

$$\mathbf{P}=\begin{bmatrix}k_1(s+12) & k_2(5s+9)\\ k_1(2s-11) & k_2(s^2-5s-2)\end{bmatrix}\Big/(s+5)(s-3)\,, \tag{44}$$

where $k_1,k_2\in[1,2]$. The pole and zero polynomials are respectively $p=(s-3)(s+5)$ and $z=(s-5)$. The multivariable system is therefore unstable and NMP with neither the unstable pole or the NMP zero pinned. The equivalent plant TFM for the NS MIMO QFT design is

$$\mathbf{\Lambda} = \mathrm{diag}\left[-k_1(s-5)\big/\left(s^2-5s-2\right),\ k_2(s-5)/(s+12)\right]. \quad (45)$$

In order to apply NS MIMO QFT the existence condition $z_{\mathbf{\Lambda}} \geq z_{\mathbf{P}}$ must hold. For this plant TFM

$$z_{\mathbf{\Lambda}} = 2\,,\ z_{\mathbf{P}} = 1 \ \Rightarrow\ z_{\mathbf{\Lambda}} > z_{\mathbf{P}}\,. \quad (46)$$

The condition is satisfied and the design will give a stable closed-loop system provided $z_{\phi} = z_{\mathbf{\Lambda}} - z_{\mathbf{P}} = 1$. Hence, this design requires that the equivalent plant family is not stabilised and that $\phi_{\mathbf{\Lambda}}$ has one RHP zero. Along with RS, the following concocted RP specifications are to be satisfied, $\forall i, j \in \{1,2\}$:

$$\text{RP:}\quad \left|1/1+q_{(i,i)}(j\omega)g_{(i,i)}(j\omega)\right| \leq 5\,\mathrm{dB}\,,\ \forall\omega\in[0,\infty)\,, \quad (47)$$

$$\left|\alpha_{(i,j)}(j\omega)\right| \leq \left|t_{e(i,j)}(j\omega)\right| \leq \left|\beta_{(i,j)}(j\omega)\right|\,,\ \forall\omega\in[0,2], \quad (48)$$

where $\beta_{(1,1)} = (s/10+1)/(s+1)$, $\beta_{(2,2)} = 1.2(s/3+1)/(s+1)$, $\beta_{(1,2)} = 0.5(s/10+1)/(s+1)$, $\beta_{(2,1)} = 0.8(s/10+1)/(s/3+1)$, $\alpha_{(1,1)} = \alpha_{(2,2)} = 0.4/((2s+1)(10s+1))$, $\alpha_{(1,2)} = \alpha_{(2,1)} = 0$. The following design satisfies the RS and RP specifications:

$$\mathbf{G} = \mathrm{diag}\left(3750(s+10)\big/(s+50)^2, -3/(s+1)\right),$$

$$\mathbf{F} = \mathrm{diag}\left(1/(2s+1), 1/(2s+1)\right). \quad (49)$$

Analysis shows that $\phi_{\mathbf{\Lambda}}$ is not Hurwitz and possesses one RHP zero due to the RHP zero in $1+g_{(1,1)}q_{(1,1)}$. The choice to make loop 1 unstable rather than loop 2 is natural in the design problem, as the equivalent plant $q_{(1,1)}$ in loop 1 has a RHP dipole (RHP zero at $s=5$ and unstable pole at $s=5.37$). In the design the cross-coupling specifications are satisfied, but the tracking variation specification is slightly violated in the loop 2 design. This however does not affect stability. With the satisfaction of the cross-coupling specifications and the condition $z_{\mathbf{P}} + z_{\phi} = z_{\mathbf{\Lambda}}$, the conditions of Theorem 2 are satisfied and it can be verified that true closed-loop system is stable. This is despite the plant TFM possessing a NMP transmission zero and unstable pole. Hence this example demonstrates that the NS MIMO QFT design methodology can be successfully employed for NMP systems. Note that, consistent with Corollary 3, the condition $\rho(\mathbf{S_\Lambda E}) < 1$ is over satisfied in the design, with $\rho(\mathbf{S_\Lambda E}) < 1/2$ over the frequency range that the tracking specifications are enforced ($\omega \in [0,2]$ rad/s).

V. Conclusion

This paper re-examined the robust stability of NS MIMO QFT designed control systems and the applicability of NS MIMO QFT to design problems with NMP plant TFMs. Using a minor modification to the design method, a stability theorem was derived that permitted the application of the methodology to general NMP MIMO systems. This gives rise to an improved NS MIMO QFT design method. The existence conditions associated with the stability theorem provide *a priori* information on the likely success of the application of the NS MIMO QFT design methodology. Whilst the methodology is extended to NMP systems herein, it is clear that for systems possessing significant interaction the sequential MIMO QFT design methodology and/or fully populated controller design methodologies will be less conservative.

References

[1] I. Horowitz, "Quantitative synthesis of uncertain multi input-output feedback systems," *Int. J. Control*, vol. 30. no. 1, pp. 81-106, 1979.

[2] O. Yaniv, and I. Horowitz , "A quantitative design method for MIMO linear feedback systems having uncertain plants," *Int. J. Control*, vol. 43, pp. 401-421, 1986.

[3] M. L. Kerr, S. Jayasuriya and S. F. Asokanthan , "On stability in non-sequential MIMO QFT designs," *ASME J. Dyn. Sys., Meas., Cont.*, 2005, to be published.

[4] O. Yaniv, *Quantitative feedback design of linear and nonlinear control systems*, Norwell, MA: Kluwer Academic Publishers, 1999.

[5] M. L. Kerr, S. Jayasuriya and S. F. Asokanthan, "Robust stability of sequential MIMO QFT designs," *ASME J. Dyn. Sys., Meas., Cont.*, 2005, to be published.

[6] D. J. N. Limebeer, "The application of generalised diagonal dominance to linear system stability theory," *Int. J. Control*, vol. 36, no. 2, pp.185-212, 1982.

[7] O.D.I. Nwokah, and D. F. Thompson, "Algebraic and topological aspects of quantitative feedback theory," *Int. J. Control*, vol. 50, no. 4, pp.1057-1069, 1989.

[8] E. Boje, "Non-diagonal controllers in MIMO quantitative feedback design," *Int. J. Robust and Nonlinear Control*, vol. 12, no. 4, pp. 303-320, 2002.

[9] P. Grosdidier, and M. Morari , "Interaction measures for systems under decentralised control," *Automatica*, vol. 32, no. 3, pp. 309-319, 1986.

[10] E. Boje, and O. D. I. Nwokah, "Quantitative feedback design using forward path decoupling," *ASME J. Dyn. Sys., Meas., Cont.*, vol. 123, pp. 129-132, 2001.

[11] B.R. Holt, and M. Morari, "Design of resilient processing plants – VI. The effect of right-half-plane zeros on dynamic resilience," *Chemical Engineering Science*, vol. 40, pp. 59-74, 1985.

[12] O. Yaniv, and P.F. Gutmann, "Crossover frequency limitations in MIMO NMP feedback systems," *IEEE Trans. Autom. Control*, vol. 47, no. 9, pp. 1560-1564, 2002.

[13] J. C. Doyle, "Quantitative feedback theory (QFT) and robust control," in *Proceedings of the American Control Conference*, Seattle, WA, 1986, pp. 1691-1698.

[14] C.-Y. Lan, M.L. Kerr, and S. Jayasuriya, "Synthesis of controllers for non-minimum phase and unstable systems using non-sequential MIMO QFT," in *Proceedings of the American Control Conference*, Boston, MA, 2004, pp. 4139-4144.

[15] Y. Zhao, and S. Jayasuriya, "Robust stability of closed-loop systems resulting from nonsequential MIMO-QFT design," *ASME J. Dyn. Sys., Meas., Cont.*, vol. 118, pp. 753-756, 1996.