Robust and Gain-Scheduled $\mathcal{H}_2$ Synthesis for LFT Parameter-Dependent Systems

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***Abstract*— In this paper, we solve the robust $\mathcal{H}_2$ synthesis problem for linear fractional transformation (LFT) systems affected by white noise disturbances. Using an intermediate matrix, we develop robust and gain-scheduling control synthesis conditions in linear matrix inequality (LMI) forms. The newly proposed robust $\mathcal{H}_2$ control approach has been applied to several examples, and demonstrated less conservative performance than other control designs based on robust $\mathcal{H}_\infty$ method.**

I. INTRODUCTION

Robust control techniques, such as $\mathcal{H}_\infty$ control and μ-synthesis [1], [8], [17], are popular design tools for the control of complex and uncertain systems. Given an uncertain system $\mathbf{T}$ is robustly stable, suppose that a suitable measure of the performance of the system $\mathbf{T}$ is the rejection effect of the disturbance signal d on the regulation or tracking error signal e. Robustness performance analysis ascertaining the induced ℓ_2 performance in the worst sense, i.e., with respect to the uncertain elements in a given set. However, the robust $\mathcal{H}_\infty$ control techniques leaning heavily on robustness, and is often not very satisfactory as a disturbance rejection criterion. For example, if the disturbance is described as a white noise signal, the best measure of the effect of the disturbance signal on the error signal is the so-called $\mathcal{H}_2$ norm [17]. This is because the induced ℓ_2 norm in effect measures gain in the worst case to persistent sinusoidal signals, and is obviously conservative in most applications.

Unfortunately, the $\mathcal{H}_2$ norm is difficult to extend to the case where $\mathbf{T}$ is not linear time-invariant [15], [18]. Possible extension of $\mathcal{H}_2$ performance index for uncertain systems has been proposed in [15], which is basis dependent and very complicated to solve. Reference [4] provided analysis results of robust $\mathcal{H}_2$ performance for uncertain linear systems using multipliers. In [9], [10], a set theoretic approach with signals in ℓ_2 is adopted for extending the concept of $\mathcal{H}_2$ norm to uncertain linear systems. The definition of [9] is based on the use of sets in $\mathcal{L}_2$ for approximating white noise signals in the frequency domain, and is in spirit closely related to the classical white-noise rejection interpretation of the $\mathcal{H}_2$ norm. The attractiveness of this approach is also partly due to the fact that tight robust analysis type conditions may be derived for the computation of robust $\mathcal{H}_2$ performance bounds. Moreover, [10] demonstrates how the infinite-dimensional conditions can be converted to a state-space formula. Various robust $\mathcal{H}_2$ measures have been compared in [11], which also provides extensive review of recent progress in robust $\mathcal{H}_2$ research field.

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Consider an uncertain linear system in LFT form

$$\begin{bmatrix} x(k+1) \\ q(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 \\ C_0 & D_{00} & 0 \\ C_1 & D_{10} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \\ d(k) \end{bmatrix}, \tag{1}$$

$$p(k) = \Delta(k) q(k), \tag{2}$$

where $x(k) \in \mathbf{R}^n$, $p(k), q(k) \in \mathbf{R}^{n_p}$, $e \in \mathbf{R}^{n_e}$ and $d \in \mathbf{R}^{n_d}$. All the state-space matrices have compatible dimensions. Suppose A is a stable matrix. Δ is a time-varying uncertainty which obeys the following structure

$$\begin{aligned} \mathbf{\Delta} := \{ &\mathrm{diag}\{\delta_1 I_{m_1}, \ldots, \delta_s I_{m_s}, \Delta_{s+1}, \ldots, \Delta_{s+f}\} : \\ &\delta_i : \mathbf{R} \to \ell_2,\ \|\delta_i\| \le 1,\ i = 1, \ldots, s, \\ &\Delta_{s+j} : \mathbf{R} \to \ell_2^{r_j \times r_j},\ \|\Delta_{s+j}\| \le 1,\ j = 1, \ldots, f \Big\}, \end{aligned}$$

where the operator norm on δ_i and Δ_j is the induced ℓ_2 norm and $\sum_{i=1}^s m_i + \sum_{j=1}^f r_j = n_p$. To reduce the conservatism of robust stabilization and performance control problems, we will introduce the following scaling matrix set

$$\begin{aligned} \mathcal{D} = \Big\{ &\mathrm{diag}\left\{D_1, \cdots, D_s, d_{s+1} I_{r_1}, \cdots, d_{s+f} I_{r_f}\right\} : \\ &D_i \in \mathbf{S}_+^{m_i \times m_i},\ i = 1, \cdots, s,\ d_{s+j} > 0,\ j = 1, \cdots, f \Big\}. \end{aligned}$$

Clearly, we have $D\Delta = \Delta D$ for any $\Delta \in \mathbf{\Delta}$ and $D \in \mathcal{D}$.

Defining the family of sets

$$\mathcal{W}_\eta := \left\{ f \in \ell_2^p : \left\| \int_0^s f(\omega) f^*(\omega) \frac{d\omega}{2\pi} - \frac{s\|f\|^2}{2p\pi} I_p \right\|_\infty < \eta \right\},$$

where $\|P(s)\|_\infty$ denotes the maximum absolute value of the elements of $P(s)$ over $s \in [0, 2\pi]$, $\|f\|$ denotes the ℓ_2^p norm of f, and $f(\omega)$ denotes the Fourier transform of f. For fixed $\eta > 0$ and any not necessarily linear nor time-invariant system $\mathbf{T} : \ell_2^p \to \ell_2^q$, define

$$\|\mathbf{T}\|_{\mathcal{W}_\eta} := \sup_{f \in \mathcal{W}_\eta,\ \|f\|=1} \|\mathbf{T} f\|.$$

Then it was shown in [9] that a suitable extension of the notion of the $\mathcal{H}_2$ norm to the not necessarily linear time-invariant system $\mathbf{T}$ is

$$\|\mathbf{T}\|_2 := \sqrt{p}\ \lim_{\eta \to 0} \|\mathbf{T}\|_{\mathcal{W}_\eta}. \tag{3}$$

This definition captures the white noise response characteristics of $\mathbf{T}$, and will be used in this paper.

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For time-varying uncertainties, the robust $\mathcal{H}_2$ analysis condition for the uncertain system (1)-(2) has been derived in frequency-domain [9] using a scaling matrix $X \in \mathcal{D}$ and a frequency-dependent function $Y(e^{j\omega})$ with

$$M^\star(e^{j\omega}) \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X & 0 \\ 0 & Y(e^{j\omega}) \end{bmatrix} < 0, \quad (4)$$

$$\int_0^{2\pi} \operatorname{tr}\left[Y(e^{j\omega})\right] \frac{d\omega}{2\pi} < \gamma^2, \quad (5)$$

where

$$M(e^{j\omega}) = \begin{bmatrix} D_{00} \\ D_{10} \end{bmatrix} + \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} (e^{j\omega} I - A)^{-1} B_0.$$

Following the approach in [10], the discrete-time robust $\mathcal{H}_2$ analysis condition can be converted into its equivalent state-space form by determining positive-definite matrices P_-, P_+ and Q, and scaling matrix $X \in \mathcal{D}$, such that

$$\begin{bmatrix} P_- & & \\ & X & \\ & & I \end{bmatrix} - [\star] \begin{bmatrix} P_- & \\ & X \end{bmatrix} \begin{bmatrix} A & B_0 \\ C_0 & D_{00} \\ C_1 & D_{10} \end{bmatrix}^T > 0, \quad (6)$$

$$\begin{bmatrix} P_+ & & \\ & X & \\ & & I \end{bmatrix} - [\star] \begin{bmatrix} P_+ & \\ & X \end{bmatrix} \begin{bmatrix} A & B_0 \\ C_0 & D_{00} \\ C_1 & D_{10} \end{bmatrix}^T > 0, \quad (7)$$

$$\begin{bmatrix} Q & B_1^T \\ B_1 & P_+ - P_- \end{bmatrix} > 0. \quad (8)$$

$$\operatorname{tr}(Q) < \gamma^2 \quad (9)$$

This set of robust $\mathcal{H}_2$ analysis conditions is analogous to robust $\mathcal{H}_\infty$ results and expands the usefulness of robust control theory.

Although the robust $\mathcal{H}_2$ analysis problem has been adequately solved, there are less research work addressing the synthesis of robust controllers based on the new $\mathcal{H}_2$ performance measure. For time-varying uncertainty, we will derive finite-dimensional control synthesis conditions for robust and gain-scheduling $\mathcal{H}_2$ problems parallel to robust $\mathcal{H}_\infty$ control techniques. The proposed approach is based on an intermediate matrix variable and idea from [6], which leads to convex synthesis conditions in LMI forms. We will then apply the proposed robust $\mathcal{H}_2$ techniques to several examples, and demonstrate the advantages of the new approach.

II. ROBUST $\mathcal{H}_2$ STATE-FEEDBACK CONTROL

Given an open-loop uncertain linear system as

$$\begin{bmatrix} x(k+1) \\ q(k) \\ e(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & 0 & D_{02} \\ C_1 & D_{10} & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \\ d(k) \\ u(k) \end{bmatrix}, \quad (10)$$

$$p(k) = \Delta(k) q(k), \quad (11)$$

where $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$, and all the other variables have their dimensions as before. We also assume

(A1) the pair (A, B_2) is stabilize and (A, C_2) is detectable,

(A2) $D_{01} =$ and $D_{11} = 0$.

In this part, the state information is assumed to be available for measurement ($y = x$). We would like to design a static state-feedback controller in the form of $u(k) = Fx(k)$, such that the robust $\mathcal{H}_2$ norm of the closed-loop system is minimized. Correspondingly, the state-space data of the closed-loop system is

$$\begin{bmatrix} x(k+1) \\ q(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + B_2 F & B_0 & B_1 \\ C_0 + D_{02} F & D_{00} & 0 \\ C_1 + D_{12} F & D_{10} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \\ d(k) \end{bmatrix},$$

$$p(k) = \Delta(k) q(k).$$

In order to derive a solvable synthesis condition, we will introduce an intermediate matrix in place of P_- and P_+ [6]. It then provides common variables in conditions (6) and (7), and separates matrix A from P_-, P_+. Through the matrix variable replacement, we will have the following condition for robust $\mathcal{H}_2$ state-feedback control synthesis.

Theorem 1: The uncertain system (10)-(11 is robustly stabilized using a state-feedback control and has its robust $\mathcal{H}_2$ norm less than γ for all $\Delta \in \boldsymbol{\Delta}$ if there exist positive-definite matrices $P_-, P_+ \in \mathcal{S}_+^{n \times n}$ and $Q \in \mathcal{S}_+^{n_d \times n_d}$, scaling matrix $X \in \mathcal{D}$, rectangular matrices $M \in \mathbf{R}^{n_u \times n}$ and $V \in \mathbf{R}^{n \times n}$ such that

$$\left[\begin{array}{cc|ccc} V^T + V - P_+ & & & \star & \\ & X & & & \\ \hline AV + B_2 M & B_0 X & P_+ & & \\ C_0 V + D_{02} M & D_{00} X & & X & \\ C_1 V + D_{12} M & D_{10} X & & & I \end{array}\right] > 0, \quad (12)$$

$$\left[\begin{array}{cc|ccc} V^T + V - P_- & & & \star & \\ & X & & & \\ \hline AV + B_2 M & B_0 X & P_- & & \\ C_0 V + D_{02} M & D_{00} X & & X & \\ C_1 V + D_{12} M & D_{10} X & & & I \end{array}\right] > 0, \quad (13)$$

$$\begin{bmatrix} Q & B_1^T \\ B_1 & P_+ - P_- \end{bmatrix} > 0, \quad (14)$$

$$\operatorname{tr}(Q) < \gamma^2. \quad (15)$$

Furthermore, the robust state-feedback controller is given by $u(k) = MV^{-1}x(k)$.

Proof: We define $F := MV^{-1}$. Since $P_+, P_- > 0$, we have

$$P_+^{-1} \geq V^{-T}(V^T + V - P_+)V^{-1},$$
$$P_-^{-1} \geq V^{-T}(V^T + V - P_-)V^{-1}$$

for any non-singular matrix V. Therefore, a sufficient condition that guarantees the robust $\mathcal{H}_2$ analysis condition (6)-(9) is

$$\left[\begin{array}{cc|ccc} V^{-T}(V^T + V - P_+)V^{-1} & & & \star & \\ & X^{-1} & & & \\ \hline A + B_2 F & B_0 & P_+ & & \\ C_0 + D_{02} F & D_{00} & & X & \\ C_1 + D_{12} F & D_{10} & & & I \end{array}\right] > 0, \quad (16)$$

$$\left[\begin{array}{cc|ccc} V^{-T}(V^T+V-P_-)V^{-1} & & & \star & \\ & X^{-1} & & & \\ \hline A+B_2F & B_0 & P_- & & \\ C_0+D_{02}F & D_{00} & & X & \\ C_1+D_{12}F & D_{10} & & & I \end{array}\right] > 0, \tag{17}$$

$$\begin{bmatrix} Q & B_1^T \\ B_1 & P_+ - P_- \end{bmatrix} > 0, \tag{18}$$

$$\mathrm{tr}(Q) < \gamma^2. \tag{19}$$

Multiply $\mathrm{diag}\{V^T, X, I, I\}$ from the left side, and its transpose from the right to eqns. (16) and (17), and use the transformation $M = FV$, we obtain (12)-(14). ■

It is also possible to consider output-feedback controllers for robust $\mathcal{H}_2$ synthesis. However, similar to the robust $\mathcal{H}_\infty$ control problem, the resulting control synthesis condition for robust $\mathcal{H}_2$ output-feedback will be in non-convex form. The details will not be provided here.

III. GAIN-SCHEDULED $\mathcal{H}_2$ OUTPUT-FEEDBACK CONTROL

If we assume the parameter $\Delta(k)$ is measurable in real-time, then it is possible to consider a parameter-dependent output-feedback control as

$$\begin{bmatrix} \dot{x}_k(k) \\ u(k) \\ q_k(k) \end{bmatrix} = \begin{bmatrix} A_k & B_{k1} & B_{k0} \\ C_{k1} & 0 & D_{k10} \\ C_{k0} & 0 & D_{k00} \end{bmatrix} \begin{bmatrix} x_k(k) \\ y(k) \\ p_k(k) \end{bmatrix}, \tag{20}$$

$$p_k(k) = \Delta(k) q_k(k), \tag{21}$$

which has the same parameter dependency as the LFT plant. This provides a gain-scheduling control of original parameter-dependent system.

For the given output feedback control, the closed-loop system becomes

$$\left[\begin{array}{c} x(k+1) \\ \hline q(k) \\ q_k(k) \\ \hline e(k) \end{array}\right] = \begin{bmatrix} A_{cl} & B_{0,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{01,cl} \\ C_{1,cl} & D_{10,cl} & D_{11,cl} \end{bmatrix} \left[\begin{array}{c} x(k) \\ \hline p(k) \\ p_k(k) \\ \hline d(k) \end{array}\right], \tag{22}$$

$$\begin{bmatrix} p(k) \\ p_k(k) \end{bmatrix} = \begin{bmatrix} \Delta(k) & \\ & \Delta(k) \end{bmatrix} \begin{bmatrix} q(k) \\ q_k(k) \end{bmatrix}, \tag{23}$$

where the closed-loop data are defined as

$$\begin{aligned} &\begin{bmatrix} A_{cl} & B_{0,cl} & B_{1,cl} \\ C_{0,cl} & D_{00,cl} & D_{01,cl} \\ C_{1,cl} & D_{10,cl} & D_{11,cl} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc|c} A & 0 & B_0 & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_0 & 0 & D_{00} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{10} & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & D_{02} & 0 \\ 0 & 0 & I \\ \hline 0 & D_{12} & 0 \end{array}\right] \\ &\times \begin{bmatrix} A_k & B_{k1} & B_{k0} \\ C_{k1} & 0 & D_{k10} \\ C_{k0} & 0 & D_{k00} \end{bmatrix} \left[\begin{array}{cc|cc|c} 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{02} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{array}\right] \end{aligned}$$

with $D_{01,cl} = 0$ and $D_{00,cl} = 0$ as desired.

Similar to the state-feedback case, we will provide a convex synthesis condition for gain-scheduling $\mathcal{H}_2$ control synthesis. Note that the intermediate matrix V need not to be symmetric, thus leads to less conservative results.

Theorem 2: The LFT system (10)-(11) is exponentially stabilized by an LFT output-feedback controller and has its robust $\mathcal{H}_2$ norm less than γ for all $\Delta \in \mathbf{\Delta}$, if there exist positive-definite matrices $T_+, T_- \in \mathcal{S}_+^{2n\times 2n}$ and $Q \in \mathcal{S}_+^{n_d\times n_d}$, scaling matrices $L, J \in \mathcal{D}$, square matrices $R, S, U \in \mathbf{R}^{n\times n}$, such that

$$\left[\begin{array}{cc|ccc} \hat{T} - T_+ & & G_{31}^T & G_{41}^T & G_{51}^T \\ & \begin{pmatrix} L & I \\ I & J \end{pmatrix} & G_{32}^T & G_{42}^T & G_{52}^T \\ \hline G_{31} & G_{32} & T_+ & & \\ G_{41} & G_{42} & & \begin{pmatrix} L & I \\ I & J \end{pmatrix} & \\ G_{51} & G_{52} & & & I \end{array}\right] > 0, \tag{24}$$

$$\left[\begin{array}{cc|ccc} \hat{T} - T_- & & G_{31}^T & G_{41}^T & G_{51}^T \\ & \begin{pmatrix} L & I \\ I & J \end{pmatrix} & G_{32}^T & G_{42}^T & G_{52}^T \\ \hline G_{31} & G_{32} & T_- & & \\ G_{41} & G_{42} & & \begin{pmatrix} L & I \\ I & J \end{pmatrix} & \\ G_{51} & G_{52} & & & I \end{array}\right] > 0, \tag{25}$$

$$\begin{bmatrix} Q & H_{21}^T \\ H_{21} & T_+ - T_- \end{bmatrix} > 0, \tag{26}$$

$$\mathrm{tr}(Q) < \gamma^2, \tag{27}$$

where

$$\hat{T} = \begin{pmatrix} S + S^T & I + U^T \\ U + I & R^T + R \end{pmatrix},$$

$$\begin{aligned} \begin{bmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \\ G_{51} & G_{52} \end{bmatrix} &= \left[\begin{array}{cc|cc} AS & A & B_0 & B_0 J \\ 0 & RA & RB_0 & 0 \\ \hline 0 & LC_0 & LD_{00} & 0 \\ C_0 S & C_0 & D_{00} & D_{00} J \\ \hline C_1 S & C_1 & D_{10} & D_{10} J \end{array}\right] \\ &+ \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & I \\ 0 & D_{02} & 0 \\ \hline 0 & D_{12} & 0 \end{array}\right] \begin{bmatrix} \hat{A}_k & \hat{B}_{k1} & \hat{B}_{k0} \\ \hat{C}_{k1} & 0 & \hat{D}_{k10} \\ \hat{C}_{k0} & 0 & \hat{D}_{k00} \end{bmatrix} \\ &\times \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & C_2 & D_{20} & 0 \\ 0 & 0 & 0 & I \end{array}\right], \end{aligned}$$

$$H_{21} = \begin{bmatrix} B_1 \\ RB_1 \end{bmatrix} + \begin{bmatrix} 0 & B_2 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A}_k & \hat{B}_{k1} & \hat{B}_{k0} \\ \hat{C}_{k1} & 0 & \hat{D}_{k10} \\ \hat{C}_{k0} & 0 & \hat{D}_{k00} \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \\ 0 \end{bmatrix}.$$

Moreover, one of nth-order LFT output-feedback controller gains are

$$
\begin{aligned}
\begin{bmatrix} A_k & B_{k1} & B_{k0} \\ C_{k1} & 0 & D_{k10} \\ C_{k0} & 0 & D_{k00} \end{bmatrix} &= \begin{bmatrix} M & RB_2 & 0 \\ 0 & I & 0 \\ 0 & LD_{02} & L_2 \end{bmatrix}^{-1} \\
&\times \left(\begin{bmatrix} \hat{A}_k & \hat{B}_{k1} & \hat{B}_{k0} \\ \hat{C}_{k1} & 0 & \hat{D}_{k10} \\ \hat{C}_{k0} & 0 & \hat{D}_{k00} \end{bmatrix} - \begin{bmatrix} RAS & 0 & RB_0 J \\ 0 & 0 & 0 \\ LC_0 S & 0 & LD_{00} J \end{bmatrix} \right) \\
&\times \begin{bmatrix} N & 0 & 0 \\ C_2 S & I & D_{20} J \\ 0 & 0 & J_2^T \end{bmatrix}^{-1}, \qquad (28)
\end{aligned}
$$

where $MN = U - RS$, L_2, J_2 are commutable with Δ and $L_2 J_2^T = I - LJ$.

Proof: Since

$$
\begin{aligned}
P_{+,cl}^{-1} &\geq V^{-T}(V + V^T - P_{+,cl})V^{-1} = \hat{V}_+, \\
P_{-,cl}^{-1} &\geq V^{-T}(V + V^T - P_{-,cl})V^{-1} = \hat{V}_-
\end{aligned}
$$

for any non-singular matrix V, we have a sufficient condition that guarantees closed-loop stability and bounds its robust $\mathcal{H}_2$ norm for closed-loop LFT systems as

$$
\begin{bmatrix} \hat{V}_+ & & A_{cl}^T & C_{0,cl}^T & C_{1,cl}^T \\ & X^{-1} & B_{0,cl}^T & D_{00,cl}^T & D_{10,cl}^T \\ A_{cl} & B_{0,cl} & P_{+,cl} & & \\ C_{0,cl} & D_{00,cl} & & X & \\ C_{1,cl} & D_{10,cl} & & & I \end{bmatrix} > 0, \qquad (29)
$$

$$
\begin{bmatrix} \hat{V}_- & & A_{cl}^T & C_{0,cl}^T & C_{1,cl}^T \\ & X^{-1} & B_{0,cl}^T & D_{00,cl}^T & D_{10,cl}^T \\ A_{cl} & B_{0,cl} & P_{-,cl} & & \\ C_{0,cl} & D_{00,cl} & & X & \\ C_{1,cl} & D_{10,cl} & & & I \end{bmatrix} > 0, \qquad (30)
$$

$$
\begin{bmatrix} Q & B_{1,cl}^T \\ B_{1,cl} & P_{+,cl} - P_{-,cl} \end{bmatrix} > 0, \qquad (31)
$$

$$
\operatorname{tr}(Q) < \gamma^2. \qquad (32)
$$

We will partition the matrices V, V^{-1} compatibly to plant and controller state dimensions n, n_k

$$
V = \begin{bmatrix} S & ? \\ N & ? \end{bmatrix}, \qquad V^{-1} = \begin{bmatrix} R^T & ? \\ M^T & ? \end{bmatrix}
$$

with $U := RS + MN$, and also partition matrices X, X^{-1} as

$$
X = \begin{bmatrix} J & J_2 \\ J_2^T & J_3 \end{bmatrix}, \qquad X^{-1} = \begin{bmatrix} L & L_2 \\ L_2^T & L_3 \end{bmatrix},
$$

where $L_2 J_2^T = I - LJ$.

We then apply the congruent transformation on the closed-loop LFT system as suggested in [6] and [7]. Let $Z := \begin{bmatrix} I & R^T \\ 0 & M^T \end{bmatrix}$, then

$$
VZ = \begin{bmatrix} S & I \\ N & 0 \end{bmatrix}, \qquad Z^T V Z = \begin{bmatrix} S & I \\ U & R \end{bmatrix}.
$$

We also define transformation variables W_1 and W_2 as

$$
W_1 = \begin{bmatrix} L & I \\ L_2^T & 0 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} I & J \\ 0 & J_2^T \end{bmatrix},
$$

then $XW_1 = W_2$ and

$$
W_1^T X W_1 = \begin{bmatrix} L & I \\ I & J \end{bmatrix}.
$$

Also we have

$$
\begin{aligned}
&\begin{bmatrix} Z^T (A_{cl} V) Z & Z^T (B_{0,cl} X) W_1 & Z^T B_{1,cl} \\ W_1^T (C_{0,cl} V) Z & W_1^T (D_{00,cl} X) W_1 & W_1^T D_{01,cl} \\ (C_{1,cl} V) Z & (D_{10,cl} X) W_1 & D_{11,cl} \end{bmatrix} \\
&= \left[\begin{array}{cc|cc|c} AS & A & B_0 & B_0 J & B_1 \\ 0 & RA & RB_0 & 0 & RB_1 \\ \hline 0 & LC_0 & LD_{00} & 0 & 0 \\ C_0 S & C_0 & D_{00} & D_{00} J & 0 \\ \hline C_1 S & C_1 & D_{10} & D_{10} J & 0 \end{array} \right] \\
&+ \left[\begin{array}{ccc} 0 & B_2 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & I \\ 0 & D_{02} & 0 \\ \hline 0 & D_{12} & 0 \end{array} \right] \begin{bmatrix} \hat{A}_k & \hat{B}_{k1} & \hat{B}_{k0} \\ \hat{C}_{k1} & 0 & \hat{D}_{k10} \\ \hat{C}_{k0} & 0 & \hat{D}_{k00} \end{bmatrix} \\
&\times \left[\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & C_2 & D_{20} & 0 & D_{21} \\ 0 & 0 & 0 & I & 0 \end{array} \right],
\end{aligned}
$$

where the transformed controller data relates to the original $(A_k, B_{k1}, B_{k0}, C_{k1}, C_{k0}, D_{k10}, D_{k00})$ in the following way

$$
\begin{aligned}
\begin{bmatrix} \hat{A}_k & \hat{B}_{k1} & \hat{B}_{k0} \\ \hat{C}_{k1} & 0 & \hat{D}_{k10} \\ \hat{C}_{k0} & 0 & \hat{D}_{k00} \end{bmatrix} &= \begin{bmatrix} RAS & 0 & RB_0 J \\ 0 & 0 & 0 \\ LC_0 S & 0 & LD_{00} J \end{bmatrix} \\
&+ \begin{bmatrix} M & RB_2 & 0 \\ 0 & I & 0 \\ 0 & LD_{02} & L_2 \end{bmatrix} \begin{bmatrix} A_k & B_{k1} & B_{k0} \\ C_{k1} & 0 & D_{k10} \\ C_{k0} & 0 & D_{k00} \end{bmatrix} \\
&\times \begin{bmatrix} N & 0 & 0 \\ C_2 S & I & D_{20} J \\ 0 & 0 & J_2^T \end{bmatrix}. \qquad (33)
\end{aligned}
$$

Multiply $\operatorname{diag}\left\{I, I, Z^T, W_1^T, I\right\}$ from the left side, and its transpose from the right to eqn. (29), and define $T_+ = Z^T P_{+,cl} Z, T_- = Z^T P_{-,cl} Z$, we get the condition (24). Condition (25) can be proved similarly by multiplying $\operatorname{diag}\left\{I, Z^T\right\}$ and its transpose on the eqn. (30).

Finally, since both matrices

$$
\begin{bmatrix} M & RB_2 & 0 \\ 0 & I & 0 \\ 0 & LD_{02} & L_2 \end{bmatrix} \qquad \begin{bmatrix} N & 0 & 0 \\ C_2 S & I & D_{20} J \\ 0 & 0 & J_2^T \end{bmatrix}
$$

are non-singular, the gain-scheduling output-feedback controller gain can be determined by inverting eqn. (33). ■

IV. EXAMPLES

In this section, two examples will be used to demonstrate the proposed approach. One is the two-disk problem, the other is the active magnetic bearing (AMB) problem.

For the state-feedback approach in Section 2, a two-disk problem will be used. This problem has been analyzed in [16] and its dynamics is represented by

$$M_1\left[\ddot{r}_1(t)-\Omega_1^2(t)r_1(t)\right] \\ =-b\dot{r}_1(t)-k(r_1(t)+r_2(t))+f(t), \quad (34)$$

$$M_2\left[\ddot{r}_2(t)-\Omega_2^2(t)r_2(t)\right] \\ =-b\dot{r}_2(t)-k(r_1(t)+r_2(t)), \quad (35)$$

where r_1, r_2 are positions of the first and the second slider relative to the center. Ω_1, Ω_2 are angular velocities of the first and the second rod varying between $[0,3]rad/sec$ and $[0,5]rad/sec$, respectively. f is the control force on the first slider along the slot. The masses of both sliders are $M_1 = 1.0kg$ and $M_2 = 0.5kg$, the damping coefficient is $b = 1.0kg/sec$ and the spring constant is $k = 200N/m$.

Define $\delta_1 = \frac{\Omega_1^2}{4.5}-1,\ \delta_2 = \frac{\Omega_2^2}{12.5}-1$, then $\delta_1, \delta_2 \in [-1,1]$. Let $x_1 = r_1, x_2 = r_2, x_3 = \dot{r}_1, x_4 = \dot{r}_2, u = f$ and $y = r_2$, then the system $T_{\mathcal{P}}$ of plant model (34)-(35) can be written as an uncertain system in LFT form

$$\begin{bmatrix}\dot{x}\\ q\\ y\end{bmatrix}=\begin{bmatrix}A & B_0 & B_1 & B_2\\ C_0 & D_{00} & 0 & D_{02}\\ C_1 & D_{10} & 0 & D_{12}\end{bmatrix}\begin{bmatrix}x\\ p\\ d\\ u\end{bmatrix}, \quad (36)$$

$$p=\begin{bmatrix}\delta_1 & \\ & \delta_2\end{bmatrix}q, \quad (37)$$

with state space data as

$$A=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 4.5-\frac{k}{M_1} & -\frac{k}{M_1} & -\frac{b}{M_1} & 0\\ -\frac{k}{M_2} & 12.5-\frac{k}{M_2} & 0 & -\frac{b}{M_2}\end{bmatrix},$$

$$C_0=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}, C_1=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix},$$

$$B_0=\begin{bmatrix}0 & 0\\ 0 & 0\\ 4.5 & 0\\ 0 & 12.5\end{bmatrix}, B_1=\begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{0.1}{M_1} & 0\\ 0 & \frac{0.1}{M_2}\end{bmatrix}, B_2=\begin{bmatrix}0\\ 0\\ \frac{1}{M_1}\\ 0\end{bmatrix},$$

$$D_{00}=D_{02}=D_{10}=D_{12}=0.$$

We assume all the states including positions r_1, r_2 of the sliders and their velocities are measurable for state-feedback control use. The design objectives are quantified by rational weighting functions

$$W_e(s)=\frac{0.3s+1.2}{s+0.04}, \quad W_u(s)=\frac{s+0.1}{0.01s+125},$$

$$W_n(s)=\frac{s+0.4}{0.01s+400}, \quad W_a(s)=0.00001,$$

$$Act(s)=\frac{1}{0.01s+1}.$$

The continuous time plant will be discretized through zero order hold with sample time of $0.01sec$. Then the performance of robust $\mathcal{H}_2$ state-feedback control and $\mathcal{H}_\infty$ control will be compared, as shown in Table I.The $\mathcal{H}_2$ performance shown in table is divided by the square root of number of inputs, as defined in (3), which makes two performances comparable.

TABLE I

PERFORMANCE COMPARISON OF ROBUST $\mathcal{H}_2$ CONTROL.

Method	induced ℓ_2 norm
H_∞ state feedback control	0.898
H_2 state feedback control	0.478

It can be seen clearly that the proposed robust $\mathcal{H}_2$ synthesis approach provides less conservative results than $\mathcal{H}_\infty$ control method. This is due to smaller disturbance set in the definition of robust $\mathcal{H}_2$ approach.

As another example, an active magnetic bearing (AMB) system will be used for the gain-scheduling output-feedback $\mathcal{H}_2$ control. Owing to the linear dependence of the plant dynamics on the rotor speed, the nonlinear gyroscopic equations of AMB can be simplified to a set of linear time-varying differential equations as

$$\ell\ddot{\theta}=-\frac{\rho J_a}{J_r}\ell\dot{\psi}+\frac{1}{m}\left(-4c_2\ell\theta+2c_1\phi_\theta+f_{d\theta}\right), \quad (38)$$

$$\ell\ddot{\psi}=\frac{\rho J_a}{J_r}\ell\dot{\theta}+\frac{1}{m}\left(-4c_2\ell\psi+2c_1\phi_\psi+f_{d\psi}\right), \quad (39)$$

$$\dot{\phi}_\theta=\frac{1}{N}\left(e_\theta+2d_2\ell\theta-d_1\phi_\theta\right), \quad (40)$$

$$\dot{\phi}_\psi=\frac{1}{N}\left(e_\psi+2d_2\ell\psi-d_1\phi_\psi\right), \quad (41)$$

where ρ denotes the rotor speed. θ, ψ are the Euler angles denoting the orientation of rotor centerline. J_a, J_r are the moment of inertia of the rotor in axial and radial directions, respectively. ϕ_θ, ϕ_ψ are the differential magnetic flux from electromagnetic pairs, e_θ, e_ψ are the corresponding differences of electric voltage. $f_{d\theta}$, $f_{d\psi}$ are disturbance forces caused by gravity, modeling errors, imbalances, etc.

Let $x^T = \begin{bmatrix}\ell\theta & \ell\psi & \ell\dot{\theta} & \ell\dot{\psi} & \phi_\theta & \phi_\psi\end{bmatrix}$, $d^T = \begin{bmatrix}f_{d\theta} & f_{d\psi}\end{bmatrix}$, and $u^T = \begin{bmatrix}e_\theta & e_\psi\end{bmatrix}$. In automatic balancing design, $f_{d\theta}$ and $f_{d\psi}$ are typically modeled as sensor noise on the measured rotor displacement. Under this assumption, the linearized equations (38)-(41) can then be written as the following LPV system

$$\begin{bmatrix}\dot{x}\\ e\\ y\end{bmatrix}=\begin{bmatrix}A(\rho) & B_1 & B_2\\ C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}x\\ d\\ u\end{bmatrix},$$

where the state-space data are

$$A(\rho)=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ -\frac{4c_2}{m} & 0 & 0 & -\frac{\rho J_a}{J_r} & \frac{2c_1}{m} & 0\\ 0 & -\frac{4c_2}{m} & \frac{\rho J_a}{J_r} & 0 & 0 & \frac{2c_1}{m}\\ \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N} & 0\\ 0 & \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N}\end{bmatrix},$$

$$B_1=0_{6\times 2}, \qquad B_2=\frac{1}{N}\begin{bmatrix}0_{4\times 2}\\ I_2\end{bmatrix}$$

$$, C_1 = \begin{bmatrix} I_2 & 0_{2\times4} \\ & 0_{2\times6} \end{bmatrix}, D_{11} = 0_{4\times2}, D_{12} = \begin{bmatrix} 0_{2\times2} \\ I_2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} I_2 & 0_{2\times4} \end{bmatrix}, \qquad D_{21} = I_2, \qquad D_{22} = 0_{2\times2}.$$

For gain-scheduled control, the rotor speed ρ is assumed to be available in real-time for control use.

For gain-scheduled control, the rotor speed ρ is assumed to be available in real-time for control use. The rotor speed is assumed to vary from $315 rad/s$ to $1100 rad/s$. The rotor dynamics exhibits strong gyroscopic effects in this speed range. The affine LPV system will then be converted into the LFT parameter-dependent system form.

The design objective of gain-scheduling control is to stabilize the system over large range of rotor speeds and minimize the disturbance effect on gap displacement at bearing locations. These design requirements are quantified by weighting functions from frozen parameter design.The weighting functions are chosen as

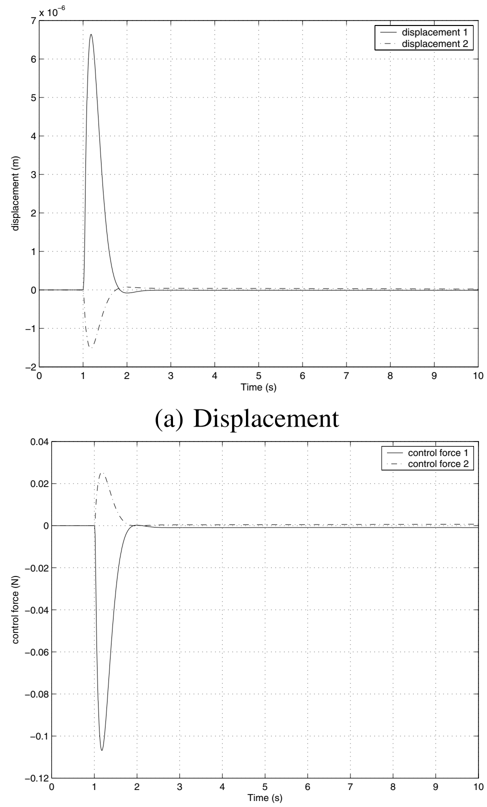
$$W_e(s) = \frac{10(s+8)}{s+0.001} I_2, \qquad W_u(s) = \frac{0.01(s+100)}{s+100000} I_2,$$
$$W_n(s) = 0.001 I_2.$$

TABLE II

PERFORMANCE COMPARISON OF GAIN-SCHEDULING CONTROL.

Method	Induced ℓ_2 norm
$\mathcal{H}_\infty$ output feedback control	8.458
$\mathcal{H}_2$ output feedback control	2.640

The time-domain simulation of proposed $\mathcal{H}_2$ gain-scheduling control is shown in Figure 1, where the disturbances $f_{d\theta}$ and $f_{d\psi}$ are chosen as step inputs with same magnitude $0.001m$ but opposite sign. As can be seen, the gain-scheduling $\mathcal{H}_2$ controller renders the displacements converging to zero quickly.



(a) Displacement

(b) Control force

Fig. 1. Time-domain simulation of gain-scheduling $\mathcal{H}_2$ control

V. CONCLUSIONS

In this paper, we have developed robust and gain-scheduling control techniques based on a robust $\mathcal{H}_2$ measure. The proposed robust control techniques are most suitable for white noise rejection and response to impulse inputs. Through several examples, it has been shown that the robust $\mathcal{H}_2$ control method could provide less conservative performance than the popular $\mathcal{H}_\infty$ control theory.

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