Stability of linear neutral systems with linear fractional norm-bounded uncertainty

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***Abstract* — This paper is concerned with the stability problem of uncertain linear neutral systems using a discretized Lyapunov functional approach. The uncertainty under consideration is *linear fractional* norm-bounded uncertainty which includes the routine norm-bounded uncertainty as a special case. A delay-dependent stability criterion is derived and is formulated in the form of linear matrix inequalities (LMIs). The criterion can be used to check the stability of linear neutral systems with both *small* and *non-small* delays. For nominal systems, the analytical results can be approached with fine discretization. For uncertainty systems with *small* delay, numerical examples show significant improvement over approaches in the literature. For uncertainty systems with *non-small* delay, the effect of the uncertainty on the maximum time-delay interval for asymptotic stability is also studied.**

I. Introduction

DURING the past few years, delay-dependent stability of linear neutral systems has attracted considerable attention, see for example, [1, 4-7, 9-10] and references therein. The goal is to obtain the maximum allowed upper bound on the delay that guarantees the stability of a linear neutral system. Therefore, the admissible allowed upper bound on the delay is the main "performance index" for measuring the conservatism of the conditions obtained.

There are two kinds of linear neutral systems. One kind of system is that the discrete delay lies in a given interval $[0, r_{\max}]$, where the delay is called a *small* delay. The other kind of system is one which is stable with some *nonzero* discrete delay, but is unstable without the delay, see Example 2 in this paper; in this case the discrete delay lies in an interval $[r_{\min}, r_{\max}]$, where $r_{\min} > 0$, and it is called a *non-small* delay.

In the time-domain, the direct Lyapunov method is a powerful tool for studying the stability of linear neutral systems. In the recent years, Lyapunov-Krasovskii approach is widely employed to consider the delay-dependent stability problem for the systems. In the following some more recent results in the time-domain using Lyapunov-Krasovskii approach are mentioned.

For linear neutral systems with *small* discrete delay, there exist many stability results in the literature. For example, based on a neutral model transformation, some sufficient conditions were obtained in [4, 9]. In [5], using the neutral model transformation and the decomposition technique for the discrete delay term, a delay-dependent stability criterion was proposed. Employing a descriptor model transformation and the bounding technique for cross product terms, some sufficient criteria were given in [1]. Recently, He *et al.* [7] proposed a new method that employed some free weighting matrices (*relaxation* matrices) to express the relationship between the terms in the Leibniz-Newton formula which was used to derive a delay-dependent stability criterion. It was shown through some examples that the new method improved the results in [1, 5]. However, the results mentioned in this paragraph can only handle the *small* discrete delay case, they fail to make any conclusion for the *non-small* discrete delay case. Moreover, the results are more conservative, which can be seen by applying these results to a constant time-delay system without uncertainty and compare with analytical results.

For linear neutral systems with *non-small* discrete delay, simple Lyapunov-Krasovskii functionals are no longer to be valid to judge the stability problem since they require that the corresponding system without delay needs to be stable. Therefore, one has to find some more general Lyapunov-Krasovskii functionals to handle the problem.

For a linear system of *retarded* type with a constant delay, it has been proven that the existence of a more general quadratic form Lyapunov-Krasovskii functional is *necessary* and *sufficient* for the stability of an uncertainty-free time-delay system [8]. A discretized Lyapunov functional approach has been proposed to enable one to write the stability criterion in an LMI form [2]. The criteria have shown significant improvements over the existing results even under very coarse discretization [2, 6].

In this paper, a more general quadratic form Lyapunov-Krasovskii functional will be introduced and the discretized Lyapunov functional approach will be employed to derive a

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delay-dependent stability criterion for linear neutral systems. The criterion is effective for linear neutral systems with both *small* and *non-small* delays. Two numerical examples will be given to illustrate the effectiveness of the criterion.

Notation. For a symmetric matrix W, "$W > 0$" denotes that W is a positive definite matrix. Similarly, "$\geq$", "$<$" and "$\leq$" denote positive semi-definiteness, negative definiteness and negative semi-definiteness. Use $\dot{W}$ to denote the derivative of W with respect to time t while $\dot{W}(\alpha)$ denotes the derivative of W with respect to the argument and evaluated at α. $\lambda_i(C)$ is the ith eigenvalue of matrix C. Use I to stand for the identity matrix of appropriate dimensions.

II. Problem Statement

Consider an uncertain linear neutral system

$$\dot{x}(t) - C\dot{x}(t-r) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t-r) \quad (1)$$

with initial condition

$$x(t) = \varphi(t)\,,\ \dot{x}(t) = \dot{\varphi}(t)\,, \forall t \in [-r, 0] \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $r > 0$ is a constant time delay, $\varphi(t)$ is the initial condition, $C \in \mathbb{R}^{n\times n}$ is a constant matrix, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are constant matrices, $\Delta A(t)$ and $\Delta B(t)$ are uncertain matrices representing the system's time-varying parameter uncertainties and satisfy

$$\begin{pmatrix}\Delta A(t) & \Delta B(t)\end{pmatrix} = L(I - F(t)D)^{-1}F(t)\begin{pmatrix}E_a & E_b\end{pmatrix} \quad (3)$$

where L, D, E_a, and E_b are known constant matrices, $F(t)$ is an unknown matrix satisfying

$$F^T(t)F(t) \leq I \quad (4)$$

In order to guarantee the uncertainty in (3) to be well defined, we assume that

$$I - D^T D > 0 \quad (5)$$

Remark 1: It should be noted that when $D \equiv 0$, the *linear fractional norm-bounded uncertainty* (3) reduces to the routine norm-bounded uncertainty. Therefore, the linear fractional norm-bounded uncertainty is more general than the routine norm-bounded uncertainty.

Define $\mathcal{C}$ as the set of continuous $\mathbb{R}^n$ valued function on the interval $[-r,\ 0]$, and let $x_t \in \mathcal{C}$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t+\theta)\,,\ -r \leq \theta \leq 0\,. \quad (6)$$

In this paper, we will attempt to formulate a practically computable criterion for robust stability of uncertain system described by (1) to (5).

Define the operator

$$\mathcal{D}\phi = \phi(0) - C\phi(-r)\,. \quad (7)$$

Throughout this paper, we assume that

A1: All the eigenvalues of matrix C are inside the *open* unit circle, i.e. $|\lambda_i(C)| < 1$ ($i = 1, 2, \cdots, n$).

Choose a Lyapunov-Krasovskii functional $V(t,\phi)$ of a quadratic form

$$V(t,\phi):\ \mathbb{R}\times\mathcal{C}\ \mapsto \mathbb{R}$$

$$\begin{aligned} V(t,\phi) &= \frac{1}{2}(\mathcal{D}\phi)^T P(\mathcal{D}\phi) + (\mathcal{D}\phi)^T \int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &+\frac{1}{2}\int_{-r}^{0}\left[\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta\right]d\xi \\ &+\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)S(\xi)\phi(\xi)d\xi \end{aligned} \quad (6)$$

where

$$P \in \mathbb{R}^{n\times n}\,,\ P = P^T\,;\ Q:[-r,0]\to\mathbb{R}^{n\times n}$$

$$S:[-r,0]\to\mathbb{R}^{n\times n}\,,\ S^T(\xi) = S(\xi)$$

$$R:[-r,0]\times[-r,0]\to\mathbb{R}^{n\times n}\,,\ R(\eta,\xi) = R^T(\xi,\eta)$$

and Q, R and S are Lipschitz matrix functions with piecewise continuous derivatives.

By Theorem 8.1 in Chapter 9 in [3], for asymptotic stability of system (1)-(5), we have the following result.

Theroem 1: Under A1, the system (1)-(5) is asymptotically stable if there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and a quadratic Lyapunov-Krasovskii functional V of the form (6) which staifies

$$\varepsilon_1(\mathcal{D}\phi)^T(\mathcal{D}\phi) \leq V(t,\phi) \quad (9)$$

and its derivative along the solution of (1) satisfies

$$\dot{V}(\phi) \leq -\varepsilon_2\phi^T(0)\phi(0) \quad (10)$$

for any $\phi \in \mathcal{C}$, where $\dot{V}(t,\phi) \overset{\Delta}{=} \frac{d}{dt}V(t,x_t)\Big|_{x_t=\phi}$.

Choose Q, R and S to be continuous piecewise linear [2], i.e.,

$$\begin{aligned} &Q^i(\alpha) \overset{\Delta}{=} Q(\delta_{i-1}+\alpha h) = (1-\alpha)Q_{i-1} + \alpha Q_i \\ &S^i(\alpha) \overset{\Delta}{=} S(\delta_{i-1}+\alpha h) = (1-\alpha)S_{i-1} + \alpha S_i \\ &R(\delta_{i-1}+\alpha h, \delta_{j-1}+\eta h) = R^{ij}(\alpha,\eta) \\ &\quad\triangleq \begin{cases}(1-\alpha)R_{i-1,j-1} + \eta R_{ij} + (\alpha-\eta)R_{i,j-1},\ \alpha \geq \eta \\ (1-\eta)R_{i-1,j-1} + \alpha R_{ij} + (\eta-\alpha)R_{i-1,j},\ \alpha < \eta\end{cases} \end{aligned} \quad (11)$$

for $0 \leq \alpha \leq 1$, $0 \leq \eta \leq 1$, where

$$\delta_i = -r_m + ih\,,\ i = 0,\ 1,\ 2,\ \cdots,\ N\,,\ h = r/N\,.$$

i.e., N is the number of divisions of the interval $[-r,\ 0]$, and h is the length of each division. It is convenient to write

$$\phi^i(\alpha) = \phi(\delta_{i-1} + \alpha h)\,.$$

III. Main Results

With the choice of piecewise linear functions, the Lyapunov-Krasovskii functional condition (9) can be written in the form of a linear matrix inequality. Using similar argument to the proof of Proposition 3 in [2] yields the following result.

Proposition 1: For piecewise linear Q, S and R as described by (11), there exists an $\varepsilon_1 > 0$ such that the Lyapunov-Krasovskii functional satisfies (9) if

$$\tilde{S} > 0 \quad (12)$$

and

$$\begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \frac{1}{h}\hat{S} + \tilde{R} \end{bmatrix} > 0 \quad (13)$$

where

$$\tilde{S} = diag(S_0\ S_1 \cdots S_{N-1}\ S_N)$$

$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix}$$

$$\tilde{Q} = \left(Q_0,\ Q_1,\ \cdots,\ Q_N\right).$$

System described by (1), (3)-(5) can be rewritten as

$$\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r) + Lu(t) \quad (14)$$

$$y(t) = E_a x(t) + E_b x(t-r) + Du(t) \quad (15)$$

subject to uncertain feedback

$$u(t) = F(t)y(t) \quad (16)$$

or equivalently, in view of (4) and (15),

$$\frac{1}{2}u^T(t)u(t) \le \frac{1}{2}\left(E_a x(t) + E_b x(t-r) + Du(t)\right)^T \times\left(E_a x(t) + E_b x(t-r) + Du(t)\right) \quad (17)$$

In the following we employ (14) and (17) to formulate the Lyapunov-Krasovskii derivative condition in the form of a linear matrix inequality. We have the following proposition.

Proposition 2: For piecewise linear Q, S and R as described by (11), equation (10) is satisfied for some $\varepsilon_2 > 0$, and arbitrary $\phi \in \mathcal{C}$ if

$$\begin{pmatrix} G_{00} & -G_{01} & -G_{02} & H_0^s & H_0^a \\ -G_{01}^T & G_{11} & -G_{12} & H_1^s & H_1^a \\ -G_{02}^T & -G_{12}^T & G_{22} & H_2^s & H_2^a \\ H_0^{sT} & H_1^{sT} & H_2^{sT} & \frac{1}{h}S_d + R_d & 0 \\ H_0^{aT} & H_1^{aT} & H_2^{aT} & 0 & \frac{3}{h}S_d \end{pmatrix} > 0 \quad (18)$$

where

$$G_{00} = I - D^T D$$

$$G_{01} = L^T P + D^T E_a$$

$$G_{02} = -L^T PC + D^T E_b$$

$$G_{11} = -PA - A^T P - S_N - Q_N - Q_N^T - E_a^T E_a$$

$$G_{12} = PB - A^T PC - Q_N^T C - Q_0 + E_a^T E_b$$

$$G_{22} = C^T PB + B^T PC - C^T Q_0 - Q_0^T C + S_0 - E_b^T E_b$$

$$S_d = diag(S_{d1}\ S_{d2}\ \cdots\ S_{dN})$$

$$S_{di} = \frac{1}{h}(S_i - S_{i-1})$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}$$

$$R_{dij} = \frac{1}{h}(R_{ij} - R_{i-1,j-1});\ i, j = 1,\ 2,\ \cdots,\ N$$

$$H_j^s = [H_{j1}^s\ \ H_{j2}^s\ \ \cdots\ \ H_{jN}^s],\ j = 0,1,2;$$

$$H_{0i}^s = -\frac{1}{2}L^T(Q_i + Q_{i-1})$$

$$H_{1i}^s = -\frac{1}{2}A^T(Q_i + Q_{i-1}) + \frac{1}{h}(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,N}^T + R_{i-1,N}^T),\ i = 1,\ 2,\ \cdots,\ N$$

$$H_{2i}^s = -\frac{1}{2}B^T(Q_i + Q_{i-1}) - \frac{1}{h}C^T(Q_i - Q_{i-1}) + \frac{1}{2}(R_{i,0}^T + R_{i-1,0}^T),\ i = 1,\ 2,\ \cdots,\ N$$

$$H_j^a = [H_{j1}^a\ \ H_{j2}^a\ \ \cdots\ \ H_{jN}^a],\ j = 0,1,2;$$

$$H_{0i}^a = \frac{1}{2}L^T(Q_i - Q_{i-1})$$

$$H_{1i}^a = \frac{1}{2}A^T(Q_i - Q_{i-1}) + \frac{1}{2}(R_{i,N}^T - R_{i-1,N}^T),\ i = 1,\ 2,\ \cdots,\ N$$

$$H_{2i}^a = \frac{1}{2}B^T(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,0}^T + R_{i-1,0}^T),\ i = 1,\ 2,\ \cdots,\ N.$$

Proof. Using (1), one obtains

$$\begin{aligned} \dot{V}(t,\phi) &= (\mathcal{D}\phi)^T P[A\phi(0) + B\phi(-r) + Lu(t)] \\ &+ [A\phi(0) + B\phi(-r) + Lu(t)]^T \int_{-r}^0 Q(\xi)\phi(\xi)d\xi \\ &+ (\mathcal{D}\phi)^T \int_{-r}^0 Q(\xi)\dot{\phi}(\xi)d\xi \\ &+ \int_{-r}^0 d\xi \int_{-r}^0 \phi^T(\xi)R(\xi,\eta)\dot{\phi}(\eta)d\eta \\ &+ \int_{-r}^0 \phi^T(\xi)S(\xi)\dot{\phi}(\xi)d\xi \end{aligned} \quad (19)$$

Through integration by parts, we have

$$\begin{aligned} \dot{V}(t,\phi) &= u^T(t)L^T P\phi(0) - u^T(t)L^T PC\phi(-r) \\ &- \frac{1}{2}\phi^T(0)[-PA - A^T P - Q(0) - Q^T(0) - S(0)]\phi(0) \\ &+ \phi^T(0)[PB - A^T PC - Q^T(0)C - Q(-r)]\phi(-r) \\ &- \frac{1}{2}\phi^T(-r)[C^T PB + B^T PC + S(-r) \\ &\qquad - C^T Q(-r) - Q^T(-r)C]\phi(-r) \\ &+ u^T(t)\int_{-r}^0 L^T Q(\xi)\phi(\xi)d\xi \end{aligned}$$

$$+\phi^T(0)\int_{-r}^{0}[A^T Q(\xi)-\dot{Q}(\xi)+R^T(\xi,0)]\phi(\xi)d\xi$$
$$+\phi^T(-r)\int_{-r}^{0}[B^T Q(\xi)+C^T\dot{Q}(\xi)-R^T(\xi,-r)]\phi(\xi)d\xi$$
$$-\frac{1}{2}\int_{-r}^{0}d\xi\int_{-r}^{0}\phi^T(\xi)\left(\frac{\partial R(\xi,\eta)}{\partial\xi}+\frac{\partial R(\xi,\eta)}{\partial\eta}\right)\phi(\eta)d\eta$$
$$-\frac{1}{2}\int_{-r}^{0}\phi^T(\xi)\dot{S}(\xi)\phi(\xi)d\xi\,. \tag{20}$$

With the piecewise linear Q, S, and R chosen, (20) can be written as

$$\begin{aligned}\dot{V}(t,\phi)&=u^T(t)L^T P\phi(0)-u^T(t)L^T PC\phi(-r)\\&-\frac{1}{2}\phi^T(0)G_{11}^0\phi(0)+\phi^T(0)G_{12}^0\phi(-r)\\&-\frac{1}{2}\phi^T(-r)G_{22}^0\phi(-r)\\&-\frac{h}{2}\sum_{i=1}^{N}\int_0^1\phi^{iT}(\alpha)S_{di}\phi^i(\alpha)d\alpha\\&-\frac{h^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\int_0^1\phi^i(\alpha)d\alpha\right)^T R_{dij}\left(\int_0^1\phi^j(\alpha)d\alpha\right)\\&-hu^T(t)\sum_{i=1}^{N}\int_0^1[H_{0i}^s+(1-2\alpha)H_{0i}^a]\phi^i(\alpha)d\alpha\\&-h\phi^T(0)\sum_{i=1}^{N}\int_0^1[H_{1i}^s+(1-2\alpha)H_{1i}^a]\phi^i(\alpha)d\alpha\\&-h\phi^T(-r)\sum_{i=1}^{N}\int_0^1[H_{2i}^s+(1-2\alpha)H_{2i}^a]\phi^i(\alpha)d\alpha\,.\end{aligned}\tag{21}$$

where

$$G_{11}^0=G_{11}+E_a^T E_a\,,\ G_{12}^0=G_{12}-E_a^T E_b\,,\ G_{22}^0=G_{22}+E_b^T E_b$$

Rewrite (21) as

$$\begin{aligned}\dot{V}(t,\phi)&=-\frac{1}{2}\int_0^1\left(u^T(t)\ \phi^T(0)\ \phi^T(-r)\ h\tilde{\phi}^T(\alpha)\right)\\&\times\begin{pmatrix}0&-L^T P&L^T PC&H_0^s+(1-2\alpha)H_0^a\\ *&G_{11}^0&-G_{12}^0&H_1^s+(1-2\alpha)H_1^a\\ *&*&G_{22}^0&H_2^s+(1-2\alpha)H_2^a\\ *&*&*&\frac{1}{h}S_d\end{pmatrix}\begin{pmatrix}u(t)\\ \phi(0)\\ \phi(-r)\\ h\tilde{\phi}(\alpha)\end{pmatrix}d\alpha\\&-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right).\end{aligned}\tag{22}$$

where * stands for symmetric part of the matrix and

$$\tilde{\phi}(\alpha)=[\tilde{\phi}^{1T}(\alpha)\ \ \tilde{\phi}^{2T}(\alpha)\ \ \cdots\ \ \tilde{\phi}^{NT}(\alpha)]^T\,.$$

Noting that (17), we have

$$\begin{aligned}\dot{V}(t,\phi)&\le-\frac{1}{2}\int_0^1\left(u^T(t)\ \phi^T(0)\ \phi^T(-r)\ h\tilde{\phi}^T(\alpha)\right)\\&\times\begin{pmatrix}G_{00}&-G_{01}&-G_{02}&H_0^s+(1-2\alpha)H_0^a\\ *&G_{11}&-G_{12}&H_1^s+(1-2\alpha)H_1^a\\ *&*&G_{22}&H_2^s+(1-2\alpha)H_2^a\\ *&*&*&\frac{1}{h}S_d\end{pmatrix}\begin{pmatrix}u(t)\\ \phi(0)\\ \phi(-r)\\ h\tilde{\phi}(\alpha)\end{pmatrix}d\alpha\\&-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right)\end{aligned}\tag{23}$$

Introducing the following notation

$$G=\begin{pmatrix}G_{00}&-G_{01}&-G_{02}\\ -G_{01}^T&G_{11}&-G_{12}\\ -G_{02}^T&-G_{12}^T&G_{22}\end{pmatrix},\ \phi_{u0r}=\begin{pmatrix}u(t)\\ \phi(0)\\ \phi(-r)\end{pmatrix},$$

$$H^s=\begin{pmatrix}H_0^s\\ H_1^s\\ H_2^s\end{pmatrix},\ H^a=\begin{pmatrix}H_0^a\\ H_1^a\\ H_2^a\end{pmatrix}.$$

(23) can be written as

$$\begin{aligned}\dot{V}(t,\phi)&\le-\frac{1}{2}\phi_{u0r}^T G\phi_{0r}-h\phi_{u0r}^T\int_0^1[H^s+(1-2\alpha)H^a]\tilde{\phi}(\alpha)d\alpha\\&-\frac{1}{2}h^2\int_0^1\tilde{\phi}^T(\alpha)\frac{1}{h}S_d\tilde{\phi}(\alpha)d\alpha\\&-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right).\end{aligned}\tag{24}$$

Rewrite (24) as

$$\begin{aligned}\dot{V}(t,\phi)&\le-\frac{1}{2}\int_0^1\left(\phi_{u0r}^T[H^s+(1-2\alpha)H^a]\quad h\tilde{\phi}^T(\alpha)\right)\\&\times\begin{pmatrix}W&I\\ I&\frac{1}{h}S_d\end{pmatrix}\begin{pmatrix}[H^s+(1-2\alpha)H^a]^T\phi_{u0r}\\ h\tilde{\phi}(\alpha)\end{pmatrix}d\alpha\\&+\frac{1}{2}\phi_{u0r}^T[H^sWH^{sT}+\frac{1}{3}H^aWH^{aT}]\phi_{u0r}\\&-\frac{1}{2}\phi_{u0r}^T G\phi_{u0r}\\&-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right)\end{aligned}\tag{25}$$

for arbitrary W. If W is chosen to satisfy

$$\begin{bmatrix}W&I\\ I&\frac{1}{h}S_d\end{bmatrix}>0\tag{26}$$

then, use Lemma 1 in [2] in the first term on the right hand in (25) to obtain

$$\begin{aligned}\dot{V}(t,\phi)&\le-\frac{1}{2}\left(\int_0^1\phi_{u0r}^T[H^s+(1-2\alpha)H^a]d\alpha\quad\int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\right)\\&\times\begin{pmatrix}W&I\\ I&\frac{1}{h}S_d\end{pmatrix}\begin{pmatrix}\int_0^1[H^s+(1-2\alpha)H^a]^T\phi_{u0r}d\alpha\\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix}\\&+\frac{1}{2}\phi_{u0r}^T[H^sWH^{sT}+\frac{1}{3}H^aWH^{aT}]\phi_{u0r}-\frac{1}{2}\phi_{u0r}^T G\phi_{u0r}\end{aligned}$$

$$-\frac{1}{2}h^2\left(\int_0^1\tilde{\phi}^T(\alpha)d\alpha\right)R_d\left(\int_0^1\tilde{\phi}(\alpha)d\alpha\right)$$

$$=-\frac{1}{2}\begin{pmatrix}\phi_{u0r}^T & \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\end{pmatrix}\begin{pmatrix}H^sWH^{sT} & H^s \\ H^{sT} & \frac{1}{h}S_d\end{pmatrix}\begin{pmatrix}\phi_{u0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix}\phi_{u0r}^T & \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\end{pmatrix}\begin{pmatrix}G-H^sWH^{sT}-\frac{1}{3}H^aWH^{aT} & 0 \\ 0 & 0\end{pmatrix}$$

$$\times\begin{pmatrix}\phi_{u0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix}\phi_{u0r}^T & \int_0^1 h\tilde{\phi}^T(\alpha)d\alpha\end{pmatrix}\begin{pmatrix}0 & 0 \\ 0 & R_d\end{pmatrix}\begin{pmatrix}\phi_{u0r} \\ \int_0^1 h\tilde{\phi}(\alpha)d\alpha\end{pmatrix}$$

$$=-\frac{1}{2}\begin{pmatrix}\phi_{u0r}^T & h\int_0^1\tilde{\phi}^T(\alpha)d\alpha\end{pmatrix}\begin{pmatrix}G-\frac{1}{3}H^aWH^{aT} & H^s \\ H^{sT} & \frac{1}{h}S_d+R_d\end{pmatrix}$$

$$\times\begin{pmatrix}\phi_{u0r} \\ h\int_0^1\tilde{\phi}(\alpha)d\alpha\end{pmatrix}. \qquad (27)$$

Therefore, to satisfy (10), it is sufficient for (26) and

$$\begin{pmatrix}G-\frac{1}{3}H^aWH^{aT} & H^s \\ H^{sT} & \frac{1}{h}S_d+R_d\end{pmatrix}>0 \qquad (28)$$

to be satisfied. Use Proposition 2 in [2] to eliminate matrix W in (26) and (28) to obtain (18). Q.E.D.

From the above discussion, we now state and establish the following stability criterion.

Theorem 2: Under A1, the system (1)-(5) is asymptotically stable if there exist real matrices $P=P^T$, Q_i, S_i, ($i=0,\ 1,\ 2,\ \cdots,\ N$), and R_{ij} ($i,j=0,\ 1,\ 2,\ \cdots,\ N$) such that $S_0>0$, (13), and (18) hold.

Proof: It is easy to see that (12) is implied by $S_0>0$ and (18). Q.E.D.

Remark 2: When $C\equiv 0$ and $D\equiv 0$, sytem (1) reduces to the following system

$$\dot{x}(t)=[A+LF(t)E_a]x(t)+[B+LF(t)E_b]x(t-r)$$

It is easy to see that the main result in [6] is recovered from Theorem 2, which means that the result in [6] is extended to a more general class of systems.

IV. Numerical Examples

To illustrate the effectiveness of the approach, two numerical examples are presented.

Example 1: Consider system (1)-(5) with

$$A=\begin{pmatrix}-2 & 0 \\ 0 & -0.9\end{pmatrix},\ B=\begin{pmatrix}-1 & 0 \\ -1 & -1\end{pmatrix},\ C=\begin{pmatrix}0.1 & 0 \\ 0 & 0.1\end{pmatrix},$$

$$L=\begin{pmatrix}\alpha & 0 \\ 0 & \alpha\end{pmatrix},\ \alpha\ge 0,\ E_a=E_b=\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix},$$

$$D=\begin{pmatrix}d & 0 \\ 0 & d\end{pmatrix},\ 0\le d<1$$

For $\alpha=0$, the maximum time-delay for stability as judged by the criteria in [1, 5, 10], and the discretized Lyapunov functional approach, are estimated in Table I, along with the analytical limit $r_{\max}^{analytical}$. It is to show that the stability limit obtained by the discretized Lyapunov functional approach is much less conservative than the results in [1, 5, 10] and it converges to analytical solution as N increases.

TABLE I. BOUND $r_{\max}$ CALCULATED USING THE METHODS IN [1, 5, 10] AND THIS PAPER FOR $c=0.1$

$r_{\max}^{analytical}$	$r_{\max}^{Wu}$	$r_{\max}^{Han}$	$r_{\max}^{Fridman}$
6.0372	4.35	4.33	3.49
$r_{\max}^{analytical}$	$r_{\max}^{N=1}$	$r_{\max}^{N=2}$	$r_{\max}^{N=3}$
6.0372	5.9337	6.0308	6.0359

For $\alpha>0$, $d=0$, applying the criteria in [5, 10], and this paper for $N=3$. Table II illustrates the numerical results for different α. We can see that $r_{\max}$ decreases as α increases and the stability limit obtained by the discretized Lyapunov functional approach is much less conservative than the results in [5, 10].

TABLE II. BOUND $r_{\max}$ CALCULATED USING THE METHODS IN [5, 10] AND THIS PAPER FOR DIFFERENT α AND $d\equiv 0$

α	0.00	0.05	0.10
[5]	4.33	3.61	2.90
[10]	4.35	3.64	3.06
This paper	6.03	4.93	4.05
α	0.15	0.20	0.25
[5]	2.19	1.48	0.77
[10]	2.60	2.24	1.94
This paper	3.36	2.83	2.40

For $\alpha=0.2$, $d\ge 0$, the effect of linear fractional factor d on the maximum time-delay for stability $r_{\max}$ is studied. The numerical results for $N=3$ and different d are estimated in Table III. It is easy to see that as $d\to 0$, the stability limit approaches the routine norm-bounded uncertainty case. As d increases, $r_{\max}$ decreases.

As mentioned in the introduction, the most interesting and significant contribution of this paper is that the method in this paper can be used to detect the robust stability of an uncertain linear neutral system with *non-small* delay. The existing methods on the *small* time-delay fail to check the stability of such kinds of systems. The following example illustrates the fact.

TABLE III. BOUND $r_{\max}$ CALCULATED FOR $\alpha = 0.2$ AND DIFFERENT d

d	0.0	0.1	0.2
$r_{\max}^{N=3}$	2.83	2.81	2.74
d	0.3	0.4	0.5
$r_{\max}^{N=3}$	2.64	2.47	2.35

Example 2: Consider system (1)-(5) with

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0.3 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$L = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},\ \alpha \ge 0,\ E_a = E_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix},$$

$$0 \le d < 1.$$

For $\alpha = 0$, the nominal system is asymptotically stable for $r \in (0.4302,\ 1.4163)$. The maximum time-delay interval $(r_{\min},\ r_{\max})$ for asymptotic stability is listed in Table IV for different *N*. It is clear that as *N* increases, the results converge to the analytical solutions.

For $\alpha > 0$, $d = 0$, now we are in a position to study the effect of the uncertainty bound α on the maximum time-delay interval $(r_{\min},\ r_{\max})$ for asymptotic stability. The numerical results for $N = 6$ and different α are estimated in Table V. It is easy to see that as $\alpha \to 0$, the stability interval for delay approaches the uncertain-free case. As α increases, $r_{\min}$ increases, while $r_{\max}$ decreases.

TABLE IV. BOUNDS $r_{\min}$ AND $r_{\max}$ CALCULATED FOR VARIOUS *N*

N	1	2	3
$r_{\min}$	0.4738	0.440	0.435
$r_{\max}$	0.7140	1.111	1.302
N	4	5	6
$r_{\min}$	0.433	0.432	0.432
$r_{\max}$	1.389	1.411	1.415

TABLE V. BOUNDS $r_{\min}$ AND $r_{\max}$ CALCULATED FOR VARIOUS α

α	0.00	0.01	0.02	0.03	0.04
$r_{\min}$	0.432	0.499	0.577	0.675	0.851
$r_{\max}$	1.415	1.358	1.287	1.190	1.010

For $\alpha = 0.01$, $d \ge 0$, the effect of linear fractional factor d on the maximum time-delay interval $(r_{\min},\ r_{\max})$ for asymptotic stability is also studied. Table VI lists the numerical results for $N = 6$ and different d. It is easy to see that as $d \to 0$, the stability limit approaches the routine norm-bounded uncertainty case. As d increases, $r_{\min}$ increases, while $r_{\max}$ decreases.

TABLE VI. BOUNDS $r_{\min}$ AND $r_{\max}$ CALCULATED FOR $\alpha = 0.01$ AND DIFFERENT d

d	0.0	0.1	0.2
$r_{\min}$	0.499	0.507	0.516
$r_{\max}$	1.358	1.355	1.351
d	0.3	0.4	0.5
$r_{\min}$	0.529	0.547	0.572
$r_{\max}$	1.345	1.336	1.321

V. CONCLUSION

The stability problem of linear neutral systems has been investigated using the discretized Lyapunov functional approach. A delay-dependent stability criterion has been derived and has been applicable to linear neutral systems with both *small* and *non-small* delays. For systems with *small* delay, a numerical example has shown that the results derived by the new criterion significantly improve the estimate of stability limit over the existing results in the literature. For systems with *non-small* delay, there has been an example to show the effectiveness of the new criterion while the existing methods on the *small* time-delay has failed to make any conclusion on the stability of such kinds of systems.

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