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On Controllability and Reachability of Switched Systems with Digraph-Directed Switchings

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***Abstract*— Controllability and reachability of a class of linear switched systems is investigated, where the switching signal can not be chosen freely but have to be determinated according to a switching digraph. Based on such a digraph, the systems are divided into two types: irreducible systems and reducible systems. For irreducible systems, necessary and sufficient criterion for controllability is established. It is also proved that in this case controllability is equivalent to reachability. As for reducible systems, only sufficient condition is established and it is shown that in this case controllability is not equivalent to reachability in general by two simple examples.**

I. Introduction

Many real systems in physics, biology and engineering can be modelled as hybrid systems. Linear switched systems(LSSs) are an important class of hybrid dynamic systems(HDSs) which consist of a family of linear time-invariant systems and a switching law specifying the switching between them. In recent years, there has been increasing interest in the control problems of switched systems due to their significance both in theory and applications.

Controllability and observability of LSSs have been studied by a number of papers. Ezzine and Haddad first studied controllability and observability for periodic type LSSs in [1]. Sun, Ge and Lee established necessary and sufficient geometric type criteria for controllability and observability of general LSSs in [2]. Then they expanded the results to the discrete-time case in [3]. Meanwhile, Xie and Wang proved that the controllability can be realized by a single switched sequence in [4], a direct consequence is the criteria given by Sun, Ge and Lee in [2]. For discrete-time systems, a corresponding result is also given by Xie and Wang in [5]. Different from the above work, Xu and Antsaklis investigated the reachability of a class of 2-dimensional LSSs in [6].

A common characteristic of the models in the above works is that the switching between two subsystems is arbitrary. However, in many real world systems, this is not true. For example, in an automobile power train, switching from one gear to another gear goes step by step. Other examples include autopilot systems, car driving systems, etc. In order to describe this characteristic, a switching digraph is introduced into the model. We'll investigate the controllability and reachability of this new class of LSSs.

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The remainder of this paper is organized as follows. Section II gives the system description. Sections III and IV investigate controllability and reachability of irreducible systems and reducible systems, respectively. Finally, we provide the conclusion in Section V.

II. Problem Formulation and Preliminaries

A. Problem Formulation

Consider a linear switched system given by

$$\begin{aligned} \dot{x}(t) &= A_{r(t)}x(t) + B_{r(t)}u(t) \\ y(t) &= C_{r(t)}x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the input, $y(t) \in \mathbb{R}^q$ is the output and the left continuous piecewise constant function $r(t) : \mathbb{R}^+ \to \mathcal{I} = \{1, \cdots, N\}$ is the switching signal to be designed. (A_i, B_i, C_i), $i \in \mathcal{I}$ denote $N(< \infty)$ subsystems. $r(t) = i$ implies that the subsystem (A_i, B_i, C_i) is activated at time instant t.

Moreover, a digraph D with N nodes is given to prescribe the switching between the subsystems(A brief introduction of digraph theory can be seen in Appendix A). This digraph is called as *switching digraph*. If there exists an arc from node i to node j in the switching digraph D, then one can let the switching signal change its value from i to j instantly. This means that the switching from subsystem (A_i, B_i, C_i) to subsystem (A_j, B_j, C_j) is admissible. On the other hand, if there does not exist an arc from node i to node j, such a switching is not admissible.

The adjacency matrix $\Gamma = [\gamma_{ij}]_{N\times N}$ of the switching digraph D is called as *switching matrix*. Here we assume that the diagonal elements of Γ are all ones, i.e., $\gamma_{11} = \cdots = \gamma_{NN} = 1$, which means the switching signal can keep the same value as long as one wills. According as whether the corresponding switching matrix is reducible or not, LSSs are generally divided into two categories: *reducible* LSSs and *irreducible* LSSs.

In the paper, by system Σ, we mean the LSS given by (1) without switching digraph, i.e., there is no restriction on the design of switching signal for system Σ; by system Σ_D, we mean the LSS given by (1) with a switching digraph D.

Now, we introduce the concept of switching sequence to describe the switching signal.

Definition 1 (Switching Sequence): A switching sequence is defined as

$$\pi := \{(i_1, h_1), (i_2, h_2), \cdots, (i_M, h_M)\} \tag{2}$$

where $M < \infty$ is the *length* of π , $i_m \in \{1, \cdots, N\}$ is the index of the *m*th subsystem, and $h_m > 0$ is the dwell time of the *m*th subsystem, for $m = 1, \cdots, M$. Denote $T_{[\pi]} = h_1 + h_2 + \cdots + h_M$, and we call $T_{[\pi]}$ the *period* of the switching sequence π.

Given a switching sequence π given by (2), or briefly, $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, an associated switching signal $r(t), t \in [0, T_{[\pi]}]$ can be determined as

$$r(t) = i_m, \quad \text{if } t \in (t_{m-1}, t_m] \tag{3}$$

where $t_0 = 0$, $t_m = \sum_{l=1}^{m} h_l$, for $m = 1, \cdots, M$.

For system Σ_D, a switching sequence $\{(i_m, h_m)\}_{m=1}^{M}$ is said to be *admissible* if the component of the switching matrix γ_{ij} satisfies that $\gamma_{i_1 i_2} = \cdots = \gamma_{i_{M-1} i_M} = 1$, i.e., there exists an arc from node i_{m-1} to node i_m, for $m = 2, \cdots, M$; otherwise, it is said to be *inadmissible*.

The set of all admissible switching sequence for system Σ_D is denoted by $\Pi_a(\Sigma_D)$[1].

As to system Σ, it is obvious that any switching sequence given by (2) is admissible since there is no restriction on switching.

Here we give a simple example to illustrate the above concepts.

Example 1: Consider an LSS with $N = 5$, i.e., $\mathcal{I} = \{1, 2, 3, 4, 5\}$, its switching digraph D is given in Fig.1.

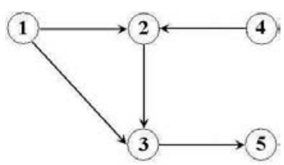

Fig. 1. Switching Digraph D of the LSS in Example 1.

Its switching matrix is given by

$$\Gamma = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the admissible switching sequence for the LSS in Example 1 must be one of the following forms:

$$\begin{aligned}
&\{(i, h_1)\},\ i \in \mathcal{I};\\
&\{(1, h_1), (2, h_2)\};\\
&\{(1, h_1), (3, h_2)\};\\
&\{(1, h_1), (2, h_2), (3, h_3)\};\\
&\{(1, h_1), (2, h_2), (3, h_3), (5, h_4)\};\\
&\{(1, h_1), (3, h_2), (5, h_3)\};\\
&\{(2, h_1), (3, h_2)\};\\
&\{(2, h_1), (3, h_2), (5, h_3)\};\\
&\{(3, h_1), (5, h_2)\};\\
&\{(4, h_1), (2, h_2)\};\\
&\{(4, h_1), (2, h_2), (3, h_3)\};\\
&\{(4, h_1), (2, h_2), (3, h_3), (5, h_4)\}.
\end{aligned}$$

where $h_1, h_2, h_3, h_4 > 0$.

[1]Here we assume that the switching sequence can start from any subsystem.

In the sequel, denote $\mathbb{U}$ the set of functions piecewise continuous. As usual, assume that all the control input $u(t) \in \mathbb{U}$. By $\prod_{i=1}^{n} A_i$ denote the matrices product $A_1 \cdots A_n$ and by $\prod_{i=n}^{1} A_i$ denote the matrices product $A_n \cdots A_1$.

The concepts of controllability and reachability for LSSs are defined as follows.

Definition 2 (Controllability): For a LSS Σ_D, a nonzero state x_0 is *controllable*, if there exist a switching sequence $\pi \in \Pi_a(\Sigma_D)$ and an input $u(t)) \in \mathbb{U}$, $t \in [0, T_{[\pi]}]$ such that the system is driven from $x(0) = x_0$ to $x(T_{[\pi]}) = 0$. The system is (completely) *controllable* if any nonzero state x is controllable.

Definition 3 (Reachability): For a LSS Σ_D, a nonzero state x_f is *reachable*, if there exist a switching sequence $\pi \in \Pi_a(\Sigma_D)$ and an input $u(t)) \in \mathbb{U}$, $t \in [0, T_{[\pi]}]$ such that the system is driven from $x(0) = 0$ to $x(T_{[\pi]}) = x_f$. The system is (completely) *reachable* if any nonzero state x is reachable.

Definition 4 (Controllable State Set): Given a switching sequence $\pi \in \Pi_a(\Sigma_D)$, the controllable state set of π is defined as

$$\begin{aligned}\mathcal{C}(\pi) :=& \{x \mid \exists\, u(t), t \in [0, T_{[\pi]}],\\ &\text{s. t. the system is driven from } x(0) = x \text{ to } x(T_{[\pi]}) = 0.\}\end{aligned} \tag{4}$$

The system controllable state set is defined as

$$\mathcal{C} = \bigcup_{\forall \pi \in \Pi_a(\Sigma_D)} \mathcal{C}(\pi) \tag{5}$$

Thus, the system is controllable if and only if $\mathcal{C} = \mathbb{R}^n$.

Definition 5 (Reachable State Set): Given a switching sequence $\pi \in \Pi_a(\Sigma_D)$, the reachable state set of π is defined as

$$\begin{aligned}\mathcal{R}(\pi) :=& \{x_f \mid \exists\, u(t),\ t \in [0, T_{[\pi]}],\\ &\text{s. t. the system is driven from } x(0) = 0 \text{ to } x(T_{[\pi]}) = x_f.\}\end{aligned} \tag{6}$$

The system reachable state set is defined as

$$\mathcal{R} = \bigcup_{\forall \pi \in \Pi_a(\Sigma_D)} \mathcal{R}(\pi) \tag{7}$$

The system is reachable if and only if $\mathcal{R} = \mathbb{R}^n$.

B. A Brief Review of LSSs without Restriction on Switching

Here we give a brief review of LSSs without restriction on switching(for details, see [4]). Given a matrix $B \in \mathbb{R}^{n \times p}$, denote $\mathcal{I}m(B)$ as the *range* of B, i.e., $\mathcal{I}m(B) = \{y \mid y = Bx, x \in \mathbb{R}^p\}$. Given a matrix $A \in \mathbb{R}^{n \times n}$ and a linear subspace $\mathcal{W} \subseteq \mathbb{R}^n$, let $\langle A|\mathcal{W}\rangle$ be the *minimal invariant subspace*, i.e., $\langle A|\mathcal{W}\rangle = \sum_{i=1}^{n} A^{i-1}\mathcal{W}$. For notational simplicity, denote $\langle A|B\rangle = \langle A|\mathcal{I}m(B)\rangle$.

Lemma 1: [4] Given an admissible switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, we have

$$\mathcal{C}(\pi) = \langle A_{i_1}|B_{i_1}\rangle + \sum_{m=2}^{M} \prod_{j=1}^{m-1} \exp(-A_{i_j} h_j)\, \langle A_{i_m}|B_{i_m}\rangle \tag{8}$$

$$\mathcal{R}(\pi) = \sum_{m=1}^{M-1} \prod_{j=M}^{m+1} \exp(A_{i_j} h_j)\, \langle A_{i_m}|B_{i_m}\rangle + \langle A_{i_M}|B_{i_M}\rangle \tag{9}$$

Given a switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, denote

$$\exp(\pi) = \prod_{m=M}^{1} \exp(A_{i_m} h_m) \tag{10}$$

It is easy to verify that

$$\mathcal{R}(\pi) \equiv \exp(\pi)\mathcal{C}(\pi) \tag{11}$$

For system Σ, we define a subspace sequence as follows[2][4]

$$\mathcal{W}_1 = \sum_{i=1}^{N} \langle A_i | B_i \rangle, \mathcal{W}_m = \sum_{i=1}^{N} \langle A_i | \mathcal{W}_{m-1} \rangle, m = 2, \cdots, n \tag{12}$$

and

$$\mathcal{W} = \mathcal{W}_n. \tag{13}$$

It is easy to see that $\mathcal{C}, \mathcal{R} \subseteq \mathcal{W}$.

The sufficient and necessary criterion for controllability of LSSs without restriction on switching has been established as follows(for details, see [4]).

Lemma 2: [4] For system Σ, there exists a switching sequence π, such that $\mathcal{C}(\pi) = \mathcal{W}$. Furthermore, the following statements are equivalent:

(a) the system is controllable;

(b) the system is reachable;

(c) the corresponding linear subspace $\mathcal{W}$ is the full space, i.e., $\mathcal{W} = \mathbb{R}^n$.

III. IRREDUCIBLE SYSTEM

It is easy to obtain the fact that if an LSS Σ_D is controllable, then the corresponding LSS Σ is controllable; however, the converse is not true in general. If some conditions on Σ_D are imposed, we'll find that the converse is true as well.

For irreducible LSSs, we can establish necessary and sufficient criterion for controllability and reachability and the form of the condition is similar to that of LSSs without restriction on switching.

Theorem 1: If an LSS Σ_D is irreducible, then there exists a switching sequence $\pi \in \Pi_a$ such that $\mathcal{C}(\pi) = \mathcal{W}$, where $\mathcal{W}$ is defined by (12)(13) associated with the corresponding LSS Σ. Furthermore, the following statements are equivalent:

(a) the system is controllable;

(b) the system is reachable;

(c) the corresponding linear subspace $\mathcal{W}$ is the full space, i.e., $\mathcal{W} = \mathbb{R}^n$.

Corollary 1: If an LSS Σ_D is irreducible, the following statements are equivalent:

(a) the system Σ_D is controllable and reachable;

(b) the corresponding system Σ is controllable and reachable.

To prove Theorem 1 recall the following lemma.

Lemma 3: Given a square matrix sequence $P_1, \cdots, P_M$ and a linear subspace sequence $\mathcal{Q}_1, \cdots, \mathcal{Q}_M$ such that the linear space $\mathcal{H} = \sum_{m=1}^{M} P_m \mathcal{Q}_m$ is the full space, moreover, given square matrices of appropriate dimensions $S_1, S_2, \cdots, S_M$ and positive scalars $h_1, h_2, \cdots, h_M$, if $\max_{1 \le i \le M} h_i$ is selected small enough, then the linear space $\widetilde{\mathcal{H}} = \sum_{m=1}^{M} P_m \exp(-S_m h_m) \mathcal{Q}_m$ is also the full space.

Proof: We choose matrix $O_1, \cdots, O_M$ such that $\mathcal{I}m(O_m) = \mathcal{Q}_m$, $m = 1, \cdots, M$, then we have the matrix $O = [P_1 O_1, \cdots, P_M O_M]$ is of full rank. Set $\widetilde{O} = [P_1 \exp(-S_1 h_1) O_1, \cdots, P_M \exp(-S_M h_M) O_M]$, it is obvious that $\mathcal{I}m(\widetilde{O}) = \widetilde{\mathcal{H}}$.

Denote $E(h_1, \cdots, h_M) = \widetilde{O} - O$. We have $\mathrm{rank}(\widetilde{O}) = \mathrm{rank}(\widetilde{O}\widetilde{O}^T) = \mathrm{rank}(OO^T + D(h_1, \cdots, h_M))$, where $D(h_1, \cdots, h_M) = E(h_1, \cdots, h_M)O^T + OE^T(h_1, \cdots, h_M) + E(h_1, \cdots, h_M)E^T(h_1, \cdots, h_M)$. Since O is of full rank, OO^T is positive definite. Therefore we can take $\max_i h_i$ small enough such that $\lambda_{\min}(D(h_1, \cdots, h_M)) \ge -0.5\lambda_{\min}(OO^T)$. It follows that $\lambda_{\min}(\widetilde{O}\widetilde{O}^T) \ge \lambda_{\min}(OO^T) + \lambda_{\min}(D(h_1, \cdots, h_M)) \ge 0.5\lambda_{\min}(OO^T)$. Thus, $\widetilde{O}$ is of full rank. Hence, $\widetilde{\mathcal{H}}$ is the full space. ■

Now we give the proof of Theorem 1.

Proof: [Proof of Theorem 1] By Lemma 2, for system Σ, there exist a switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, such that $\mathcal{C}(\pi) = \mathcal{W} = \mathbb{R}^n$.

For system Σ_D, we try to find an admissible switching sequence such that its controllable state set is equal to $\mathcal{W}$ as well. If such a switching sequence exists, the proof of the rest part of Theorem 1 is trivial. In fact, such a switching sequence can be constructed based on the above switching sequence π.

Consider the nodes i_1 and i_2, since the system is irreducible, there must exist a path of finite length from the node i_1 to the node i_2. Without loss of generality, suppose this path is $i_1, j_1, j_2, \cdots, j_{N_1}, i_2$. At first, we consider the switching sequence

$$\pi_1^1 = \{(i_1, h_1), (j_1, g_1), (i_2, h_2), \cdots, (i_M, h_M)\}$$

where g_1 needs to be chosen. We have

$$\begin{aligned} \mathcal{C}(\pi_1^1) &= \langle A_{i_1} | B_{i_1} \rangle + \exp(-A_{i_1} h_1) \langle A_{j_1} | B_{j_1} \rangle \\ &+ \exp(-A_{i_1} h_1) \exp(-A_{j_1} g_1) \\ &\left(\langle A_{i_2} | B_{i_2} \rangle + \sum_{m=3}^{M} \prod_{k=j+1}^{m-1} \exp(-A_{i_k} h_k) \langle A_{i_m} | B_{i_m} \rangle \right) \end{aligned}$$

By Lemma 3, we can choose g_1 small enough such that $\mathcal{C}(\pi_1^1) = \mathbb{R}^n$. Then, we can repeat this process to construct the follow switching sequences by choosing g_m, $m = 2, \cdots, N_1$ small enough

$$\pi_1^m = \{(i_1, h_1), (j_1, g_1), \cdots, (j_m, g_m), (i_2, h_2), \cdots, (i_M, h_M)\},$$
$$m = 2, \cdots, N_1,$$

which satisfying $\mathcal{C}(\pi_1^m) = \mathbb{R}^n, m = 2, \cdots, N_1$.

Up to now, we have constructed a switching sequence $\pi_1 = \pi_1^{N_1}$ in which the former part from i_1 to i_2 is admissible, i.e., $\{(i_1, h_1), (j_1, g_1), \cdots, (j_m, g_m), (i_2, h_2)\}$ is admissible.

Obviously, we cant repeat this similar treatment M times by considering the paths between nodes i_{m-1} and i_m, $m = 2, \cdots, M$. Finally, we can get an admissible switching sequence π_M for system Σ_D which controllable state set $\mathcal{C}(\pi_M)$ is still the full space.

Thus, we have constructed a switching sequence $\pi_M \in \Pi_a$ such that $\mathcal{C}(\pi_M) = \mathcal{W}$. We have finished the first part of Theorem 1. As for the rest, the proof is trivial. ∎

IV. Reducible system

In this section, we investigate the controllability and reachability of a reducible system mainly by virtue of its corresponding switching matrix. Obviously, the switching matrix is reducible and nonnegative(i.e. all the elements of the matrix are nonnegative).

Before proceeding further, we first present a simple form for the switching sequence.

Lemma 4: [7] Given $\Gamma \in \mathbb{R}^{N\times N}$ is a nonnegative matrix, there exists an $N \times N$ permutation matrix P such that

$$\hat{\Gamma} = P\Gamma P^T = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 & \Gamma_{1,k+1} & \Gamma_{1,k+2} & \cdots & \Gamma_{1,s} \\ 0 & \Gamma_2 & \cdots & 0 & \Gamma_{2,k+1} & \Gamma_{2,k+2} & \cdots & \Gamma_{2,s} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & \Gamma_k & \Gamma_{k,k+1} & \Gamma_{k,k+2} & \cdots & \Gamma_{k,s} \\ 0 & \cdots & \cdots & 0 & \Gamma_{k+1} & \Gamma_{k+1,k+2} & \cdots & \Gamma_{k+1,s} \\ 0 & \cdots & \cdots & 0 & 0 & \Gamma_{k+2} & \cdots & \Gamma_{k+2,s} \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \Gamma_s \end{bmatrix} \tag{14}$$

where $\Gamma_i \in \mathbb{R}^{l_i \times l_i} (i = 1, \cdots, s)$ are irreducible and nonnegative matrices, $\sum_{i=1}^{s} l_i = N$, $1 \leq k \leq s \leq N$. Moreover, at least one of the matrices in the following matrix sequences:

$$\Gamma_{m,k+1}, \cdots, \Gamma_{m,s}, \tag{15}$$

is nonzero, $m = 1, \cdots, k$.

By Lemma 4, since a permutation transformation just changes the index of the subsystems, without loss of generality, we just let Γ be in the form of (14).

Set $l_0 = 0$. For $m = 1, \cdots, s$, by Σ_m denote the LSS which takes the subsystems

$$(A_i, B_i, C_i), i \in \{\sum_{j=0}^{m-1} l_j + 1, \cdots, \sum_{j=0}^{m} l_j\},$$

as its subsystems; by $\Sigma_{m,D}$ denoted the LSS which has the same subsystems as those of system Σ_m and takes digraph composed by the nodes $\sum_{j=0}^{m-1} l_j + 1, \cdots, \sum_{j=0}^{m} l_j$ and the arcs between them appeared in the digraph D as its switching digraph D_m, it is easy to verify that the adjacency matrix of D_m is just the submatrix Γ_m.

For system Σ_m, we can determinate its controllable state set by calculating the following subspace sequence

$$\begin{aligned} \mathcal{W}_{m,1} &= \sum_{i=\sum_{j=0}^{m-1} l_j+1}^{\sum_{j=0}^{m} l_j} \langle A_i | B_i \rangle, \\ \mathcal{W}_{m,\rho} &= \sum_{i=\sum_{j=0}^{m-1} l_j+1}^{\sum_{j=0}^{m} l_j} \langle A_i | \mathcal{W}_{m,\rho-1} \rangle, \rho = 2, \cdots, n. \end{aligned} \tag{16}$$

Obviously, the controllable state set of system Σ_m is

$$\mathcal{W}_{\Sigma_m} = \mathcal{W}_{m,n}. \tag{17}$$

Since Γ_m is irreducible, by Corollary 1, system $\Sigma_{m,D}$ is controllable if and only if system Σ_m is controllable.

Based on the above analysis, we can established the following theorem directly.

Theorem 2: If the reducible LSS Σ_D satisfies $k = s$, i.e., the switching matrix Γ is just a block diagonal matrix:

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_s \end{bmatrix},$$

then the following two statements are equivalent:

(a) the system is controllable and reachable;

(b) there exists $m \in \{1, \cdots, s\}$ such that system Σ_m is controllable and reachable.

Proof: (b) ⇒ (a) is trivial. (a) ⇒ (b) is easy to verify since in this case, it is easy to see that the controllable state set and the reachable state set of the system Σ_D is

$$\mathcal{C} = \mathcal{R} = \bigcup_{1 \leq m \leq s} \mathcal{W}_{\Sigma_m}.$$

∎

Remark 1: In Theorem 2, if $s = N$, this means there is no switching that can happen between any pair of all these subsystems, then the conclusion of the theorem is reduced to the following trivial situation:

the system is controllable and reachable iff at least one of its subsystem is controllable and reachable and we can choose to start with the controllable and reachable subsystem.

Next, we consider another more complicated case.

Theorem 3: Suppose the reducible LSS Σ_D satisfies $k = s = 2$, i.e., the switching matrix Γ is just a block upper triangle matrix:

$$\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_{1,2} \\ 0 & \Gamma_2 \end{bmatrix}$$

where $\Gamma_{1,2}$ is nonzero, if

$$\mathcal{W}_{\Sigma_1} + \mathcal{W}_{\Sigma_2} = \mathbb{R}^n, \tag{18}$$

then there exist an admissible switching sequence π such that $\mathcal{C}(\pi) = \mathcal{W}_{\Sigma_1} + \mathcal{W}_{\Sigma_2}$, and hence, the system is controllable and reachable.

Proof: For systems $\Sigma_{1,D}$, $\Sigma_{2,D}$, by Theorem 1, there exist two admissible switching sequences $\pi_1 = \{(i_\rho, h_\rho)\}_{\rho=1}^{M_1}$, $\pi_2 = \{(j_\rho, g_\rho)\}_{\rho=1}^{M_2}$ such that

$$\mathcal{C}(\pi_1) = \mathcal{W}_{\Sigma_1}, \mathcal{C}(\pi_2) = \mathcal{W}_{\Sigma_2}.$$

Since the matrix $\Sigma_{1,2}$ is nonzero, there must exist nodes k_1, k_2 satisfying $1 \leq k_1 \leq l_1$,$l_1 + 1 \leq k_2 \leq N$ and there is an arc from node k_1 to k_2 in D.

First, we can choose positive scalars f_1, f_2 small enough to construct the follow two switching sequences(may be not admissible):

$$\widehat{\pi}_1 = \{(i_1, h_1), \cdots, (i_{M_1}, h_{M_1}), (k_1, f_1)\}, \tag{19}$$

$$\widehat{\pi}_2 = \{(k_2, f_2), (j_1, g_1), \cdots, (j_{M_2}, g_{M_2})\} \tag{20}$$

such that

$$\mathcal{C}(\widehat{\pi}_1) = \mathcal{W}_{\Sigma_1}, \mathcal{C}(\widehat{\pi}_2) = \mathcal{W}_{\Sigma_2}.$$

Secondly, similar to the proof process of Theorem 1, we can construct two admissible switching sequences base on $\widehat{\pi}_1, \widehat{\pi}_2$, denoted as $\widetilde{\pi}_1 = \{(i'_\rho, h'_\rho)\}_{\rho=1}^{M'_1}, \widetilde{\pi}_2 = \{(j'_\rho, g'_\rho)\}_{\rho=1}^{M'_2}$ satisfying

$$\mathcal{C}(\widetilde{\pi}_1) = \mathcal{W}_{\Sigma_1}, \mathcal{C}(\widetilde{\pi}_2) = \mathcal{W}_{\Sigma_2}.$$

It is obvious that the index part of the last item of $\widetilde{\pi}_1$ is k_1 and the index part of the first item of $\widetilde{\pi}_2$ is k_2.

Finally, we can choose positive scalar η small enough to construct the following admissible switching sequence for system Σ_D:

$$\widehat{\pi}=\{(i'_1, h'_1), \cdots, (i'_{M'_1}, h'_{M'_1}), (k_2, \eta), (j'_1, g'_1), \cdots, (j'_{M'_1}, g'_{M'_2})\} \tag{21}$$

it is easy to verify that

$$\mathcal{C}(\widehat{\pi}) = \mathcal{W}_{\Sigma_1} + \mathcal{W}_{\Sigma_2}. \tag{22}$$

Up to now, we have constructed such an admissible switching sequence. The proof of the rest is trivial. ■

Remark 2: Theorem 3 can be extended to a more general case: suppose that there exist $1 \leq k_1 < k_2 \leq s$ such that Γ_{k_1,k_2} is nonzero, if

$$\mathcal{W}_{\Sigma_{k_1}} + \mathcal{W}_{\Sigma_{k_2}} = \mathbb{R}^n, \tag{23}$$

the conclusions still hold.

Remark 3: Theorem 3 can be extended to another more general case: suppose that there exist $1 \leq k_1 < k_2 < \cdots < k_\rho \leq s$ such that $\Gamma_{k_1,k_2}, \cdots, \Gamma_{k_{\rho-1},k_\rho}$ are all nonzero, if

$$\mathcal{W}_{\Sigma_{k_1}} + \cdots + \mathcal{W}_{\Sigma_{k_\rho}} = \mathbb{R}^n, \tag{24}$$

the conclusions still hold.

As for more general reducible systems, controllability is not equivalent to reachability in general. Two numerical examples are given below to show this fact.

Example 2: Consider the 2-dimensional LSS, with

$$A_1 = 0_{2\times 2}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{25}$$

and the switching digraph is given in Fig.2.

Fig. 2. Switching Digraph of Example 2.

The admissible switching sequence has the following forms:

$$\pi_1 = \{(1, h_1), (2, h_2)\}, \pi_2 = \{(1, h_1)\}, \pi_3 = \{(2, h_1)\}$$

where $h_1, h_2 > 0$. Consider their controllable state set is

$$\begin{aligned} \mathcal{C}(\pi_1) &= \langle A_1|B_1\rangle + \exp(-A_1 h_1)\langle A_2|B_2\rangle \\ &\equiv span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \\ \mathcal{C}(\pi_2) &= \langle A_1|B_1\rangle \equiv span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \\ \mathcal{C}(\pi_3) &= \langle A_2|B_2\rangle \equiv \{0\} \end{aligned}$$

Then, we have $\mathcal{C} \equiv span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. Thus, the system is not controllable.

On the other hand, consider the reachable state set of π_1

$$\begin{aligned} &\mathcal{R}(\pi_1) = \exp(A_2 h_2)\langle A_1|B_1\rangle + \langle A_2|B_2\rangle \\ &= span\{\begin{bmatrix} \cos(h_2) \\ \sin(h_2) \end{bmatrix}\} \end{aligned}$$

Then, we have $\mathcal{R} \supseteq \bigcup_{h_2>0} span\{\begin{bmatrix} \cos(h_2) \\ \sin(h_2) \end{bmatrix}\} = \mathbb{R}^2$. Thus, the system is reachable.

In fact, given any nonzero state $x_f = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}, r > 0, \theta \in [0, 2\pi)$, there exist a switching sequence $\pi = \{(1, 1), (2, \theta)\}$ and an input $u(t) = r$, $t \in [0, 1 + \theta]$, such that the system state

$$x(t) = \begin{cases} \begin{bmatrix} r \\ 0 \end{bmatrix}, & t \in [0, 1] \\ e^{A_2(t-1)} x(1), & t \in (1, 1 + \theta] \end{cases}$$

Then we have

$$\begin{aligned} &x(T_{[\pi]}) = e^{A_2\theta} x(1) = \exp(\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}) \begin{bmatrix} r \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} \end{aligned}$$

Thus, we get $x(1 + \theta) = x_f$. This means that x_f is reachable. Since x_f is arbitrarily selected, the system is reachable indeed.

Moreover, given a state $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it is easy to prove that any switching sequence and any input can not drive the second component of the state to zero. Hence, the system is not controllable indeed.

Example 3: Consider the 2-dimensional LSS, with

$$A_1 = 0_{2\times 2}, B_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}; A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{26}$$

Fig. 3. Switching Digraph of Example 3.

and the switching digraph is given in Fig.3.

The admissible switching sequence has the following forms:

$$\pi_1 = \{(2,h_1),(1,h_2)\}, \pi_2 = \{(2,h_1)\}, \pi_3 = \{(1,h_1)\}$$

where $h_1, h_2 > 0$. Just consider the admissible switching sequence π_1, its controllable state set is

$$\begin{aligned}&\mathcal{C}(\pi_1) = \langle A_2|B_2\rangle + \exp(-A_1h_2)\langle A_1|B_1\rangle \\ &= span\{\begin{bmatrix} \cos(h_2) \\ \sin(h_2)\end{bmatrix}\}\end{aligned}$$

Then, we have $\mathcal{C} \supseteq \bigcup_{h_2>0} span\{\begin{bmatrix} \cos(h_2) \\ \sin(h_2)\end{bmatrix}\} = \mathbb{R}^2$. Thus, the system is controllable.

On the other hand, consider its reachable state set

$$\begin{aligned}\mathcal{R}(\pi_1) &= \exp(A_1h_2)\langle A_2|B_2\rangle + \langle A_1|B_1\rangle \\ &\equiv span\{\begin{bmatrix} 1 \\ 0\end{bmatrix}\} \\ \mathcal{R}(\pi_2) &= \langle A_2|B_2\rangle \equiv \{0\} \\ \mathcal{R}(\pi_3) &= \langle A_1|B_1\rangle \equiv span\{\begin{bmatrix} 1 \\ 0\end{bmatrix}\}\end{aligned}$$

Then, we have $\mathcal{R} \equiv span\{\begin{bmatrix} 1 \\ 0\end{bmatrix}\}$. Thus, the system is not reachable.

In fact, given any nonzero state $x_f = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta)\end{bmatrix}, r > 0, \theta \in [0, 2\pi)$, there exist a switching sequence $\pi = \{(2,\theta),(1,1)\}$ and an input $u(t) = r$, $t \in [0, 1+\theta]$, such that the system state

$$x(t) = \begin{cases} e^{A_2 t}\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta)\end{bmatrix}, & t \in [0, h_1] \\ \begin{bmatrix} x_1(\theta) - r \\ x_2(\theta)\end{bmatrix}, & t \in (\theta, 1+\theta] \end{cases}$$

Then we have

$$\begin{aligned}&x(\theta) = e^{A_2\theta}x(0) = \exp(\begin{bmatrix} 0 & \theta \\ -\theta & 0\end{bmatrix})\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta)\end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta)\end{bmatrix}\end{aligned}$$

Thus, we get $x(\theta) = [r\ 0]^T$, $x(1+\theta) = [0\ 0]^T$. This implies that x_f is controllable. Since x_f is arbitrarily selected, the system is controllable indeed.

Moreover, given a state $x = [0,\ 1]^T$, it is easy to prove that any switching sequence and any input can not drive the second component of the state from zero to 1. Hence, the system is not reachable indeed.

V. Conclusion

This paper has studied the controllability and reachability of a general class of LSS, where the switching signal can not be chosen freely but have to be determinated according to a switching digraph. For irreducible systems, necessary and sufficient criterion for controllability has been established. It is also proved that in this case controllability is equivalent to reachability. As for reducible systems, only sufficient condition has been established and it is shown that in this case controllability is not equivalent to reachability in general. Some numerical examples have been given to illustrate our results.

Appendix A

In this Appendix, we give a brief introduction of digraph theory.

A *digraph* $D(V,E)$ consists of a set of nodes V and a set of ordered pairs of nodes E called *arcs*. The arc (u,v) points from u to v. In a digraph, a *walk* is a sequence of nodes $v_o, v_1, \cdots, v_N$ in which each pair of nodes v_i, v_{i+1} is linked by an arc (v_i, v_{i+1}) ; a *path* is a walk in which all nodes are distinct. The length of a path is defined as the number of arcs it contains.

A digraph is *strongly connected* if there exists a path of finite length from every node to every other.

The *adjacency matrix* $\Lambda = [\lambda_{ij}]_{N\times N}$ of a digraph is an $N \times N$ matrix in which $\lambda_{ij} = 1$ if $(v_i, v_j) \in E$ and $\lambda_{ij} = 0$, otherwise. Given a digraph, we can easily give its corresponding adjacency matrix; whereas, given an adjacency matrix, we can also easily give its corresponding digraph.

An $N \times N$ matrix Λ is *reducible* if there exists a permutation matrix H such that $H\Lambda H^T = \begin{bmatrix} \Lambda_1 & \Lambda_{12} \\ 0 & \Lambda_2\end{bmatrix}$, where Λ_1 is an $r\times r$ submatrix, Λ_2 is an $(N-r)\times(N-r)$ submatrix, $1 \le r < N$. Otherwise, it is *irreducible*.

Lemma 5: [8] A digraph is strongly connected if and only if its corresponding adjacency matrix is irreducible.

References

[1] Ezzine, J., Haddad, A. H.: Controllability and observability of hybrid systems. International Journal of Control. 49 (1989) 2045–2055.

[2] Sun, Z., Ge, S.S., Lee, T.H.: Controllability and reachability criteria for switched linear systems. Automatica, 38 (2002) 775–786.

[3] Ge, S. S., Sun, Z., Lee, T. H.: Reachability and controllability of switched linear discrete-time systems. IEEE Transactions on Automatic Control. 46 (2001) 1437–1441

[4] Xie, G., Wang, L.: Controllability and stabilizability of switched linear-systems. Systems and Control Letters. 48 (2003) 135–155.

[5] Xie, G., Wang, L.: Reachability realization and stabilizability of switched linear discrete-time systems. J. Math. Anal. Appl. 280 (2003) 209–220

[6] Xu, X., Antsaklis, P. J.: On the reachability of a class of second-order switched systems. Tech. Rep., ISIS-99-003, Univ. of Notre Dame.

[7] Gantmacher, F. R.: The Theory of Matrices, vol. 2, Chelsea, New York, 1959.

[8] Chen, J. L., Chen, X. H.: Special Matrices, Tsinghua University Press, Beijing, 2001.