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Hybrid Output-feedback Guaranteed Cost H_∞ Robust Control for Linear Systems

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***Abstract*— This paper focuses on the hybrid output-feedback guaranteed cost H_∞ robust control problem for a class of uncertain linear systems. Time-varying uncertainties are contained in both the state matrix and input matrix. Suppose there exist a finite number of candidate controllers with known controller gain matrices and none of the controllers can solve the guaranteed cost H_∞ robust control problem. Based on the single Lyapunov function method, sufficient conditions for hybrid output-feedback guaranteed cost H_∞ robust control are given and the switching laws are constructed. A sufficient condition for hybrid output-feedback guaranteed cost H_∞ robust control is also derived and the switching laws are constructed by means of multiple Lyapunov function technique. The simulation demonstrates the effectiveness of the method proposed in this paper.**

I. INTRODUCTION

In many practical engineering applications, the nature of the problem itself may often lead to the situation where a single controller can not stabilize the system[1]. On the other hand, due to high uncertainty in the model or due to sensor and/or actuator limitation, a single control law may be difficult to find. But this problem may be solvable by switching among a set of candidate controllers. For example, controller switching strategy is frequently adopted in the applications of computer disk[2], some robot control systems[3] , automobile transmission systems and so on. So it has important theoretical and practical interest to design switching law among a set of finite candidate controllers to make the system stabilizable [1-6]. In [6], the aim of designing switching law is such that the system can trace the reference model. In the case that there exist a finite number of candidate controllers and none of them can stabilize the system, a sufficient condition of hybrid H_∞ disturbance attenuation was presented in [1] .

On the other hand, in order that the closed loop system not only has the capacity of disturbance attenuation but also have satisfactory dynamic performance, the mix H_2/H_∞ control problem has be considered for controlled systems. For a system with a single continuous controller, some multi-objective control problems including the mixed H_2/H_∞ control problem have been studied widely using the algebraic approach (see, for example, [7-9]). However, when the system can not be stabilized and the above mentioned multi-objective control problems can not be solved by a continuous controller, few results have been proposed by now due to its complexity.

This paper studies the hybrid output-feedback guaranteed cost H_∞ robust control problem for a class of uncertain linear systems. Suppose that there exist finite candidate static state feedback controllers in a set of controllers, and none of the individual controller solve the problem. Based on single-Lyapunov function technique and multi-Lyapunov function technique, a hybrid state feedback controller is designed such that the closed-loop systems satisfy hybrid output-feedback guaranteed cost robust control with H_∞ disturbance attenuation γ. Finally, a simulation example illustrates the main results of this paper.

II. PRELIMINARIES

Consider a class of switched controller systems described by the state-space model of the form

$$\dot{x}(t) = (A + \triangle A)x(t) + (B + \triangle B)u_{\sigma(t)} + B_1 w, \quad (1)$$

$$z = C_1 x(t),$$

$$y = Cx(t),$$

$$x(0) = x_0,$$

where $x \in R^n$ is the state, $w(t) \in R^p$ is the disturbance input, $u(t) \in R^m$ is the control input, $z(t) \in R^q$ is the controlled output, $y(t) \in R^l$ is the measure output, A, B, C, B_1, C_1 are known real constant matrices of appropriate dimensions that describe the nominal system of (1), and $\Delta A, \Delta B$ are real-valued matrix functions representing time-varying parameter uncertainty in the system model. Suppose that uncertain matrices $\Delta A, \Delta B$ have the following structure

$$[\Delta A, \Delta B] = DF(t)[E_1, E_2], \quad (2)$$

where D, E_1, E_2 are constant matrices of appropriate dimensions, $F(t)$ is unknown function matrix and satisfies the following condition

$$F(t)^T F(t) \leq I. \quad (3)$$

$\sigma(t) : [0, \infty) \to M = \{1, 2, \cdots, m\}$ is a piecewise constant function of time t or state x, $u_{\sigma(t)} = K_{\sigma(t)} y$ is generated by the switching among the output feedback controllers

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$u_1, \cdots, u_m$, where $K_{\sigma(t)} \in [K_1, \cdots, K_m]$, $K_i, (i \in M)$ are known real constant matrices of appropriate dimensions.

Now we introduce two performance indexes:

(1). quadratic cost function performance index

$$J_1 = \int_0^{+\infty} [x(t)^T Q x(t) + u_{\sigma(t)}^T R u_{\sigma(t)}] dt, \quad (4)$$

where Q and R are symmetric positive definite weighted matrix.

(2). H_∞ performance index

$$J_2 = \int_0^{+\infty} (z^T z - \gamma^2 w^T w) dt. \quad (5)$$

When $J_2 < 0$, i.e., $\|z\|_2 < \gamma \|w\|_2$ for any nonzero $w \in L_2[0, \infty)$, the system (1) is said to be stabilizable with an H_∞ disturbance attenuation γ. Let $L_2[0, \infty)$ denote the space of square integrable functions on $[0, \infty)$ and $\|\cdot\|$ stand for the usual L_2-norm.

The following gives the definition of hybrid output-feedback guaranteed cost robust control with H_∞ disturbance attenuation γ for system (1).

Definition 1 Let the constant $\gamma > 0$ be given. The problem of hybrid output-feedback guaranteed cost robust control with H_∞ disturbance attenuation γ is to find a linear output feedback control law $u^*_{\sigma(t)} = K_{\sigma(t)} y$, such that for all admissible parameter uncertainty $\Delta A, \Delta B$ the following conditions are satisfied simultaneously.

(1) The closed-loop system of the system (1) is asymptotically stable when $w = 0$.

(2) There exists a cost upper-bound J_1^* such that quadratic cost function performance index for the closed-loop system of the system (1) satisfies $J_1 < J_1^*$.

(3) Subjected to the assumption of the zero initial condition, the controlled output z satisfies $J_2 < 0$, i.e., $\|z\|_2 < \gamma \|w\|_2$.

Remark 1 If one of the controllers in the controller set $u_1, \cdots, u_m$, say, j-th controller, solves the problem of guaranteed cost robust control with H_∞ disturbance attenuation γ, then the problem becomes trivial. Therefore, we assume none of the individual controller solves the problem.

For convenience, we adopt the following notations for system (1). In particular, a switching sequence with an initial state x_0 is expressed by

$$\begin{aligned} \Sigma = \{&x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_n, t_n), \cdots, \\ &i_k \in M = \{1, 2, \cdots, m\}, \\ &k \in N = \{1, 2, \cdots\} \bigcup \{0\}\} \end{aligned} \quad (6)$$

in which (i_n, t_n) means that the system evolves according to the i_k-th subsystem for $t_k \le t < t_{k+1}$. We denote the trajectory of system (1) $x_\Sigma(\cdot)$. For any j, $1 \le j \le m$,

$$\begin{aligned} \sum j = \{&[t_{j_1}, t_{j_1+1}), [t_{j_2}, t_{j_2+1}), \cdots, [t_{j_n}, t_{j_n+1}), \cdots, \\ &\sigma(t) = j, t_{j_n} \le t < t_{j_n+1}, n \in N\} \end{aligned} \quad (7)$$

denotes the sequence of switching times of the j-th subsystem which is switched on at t_{j_n} and switched off at t_{j_n+1}.

III. MAIN RESULTS

A. Single Lyapunov function technique

Let the matrix pencil be, $\gamma_{\alpha_1, \alpha_2, \cdots, \alpha_m}(A_1, A_2, \cdots, A_m)$ i.e.,

$$\begin{aligned} &\gamma_{\alpha_1, \alpha_2, \cdots, \alpha_m}(A_1, A_2, \cdots, A_m) = \\ &\{\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_m A_m : \\ &\alpha_1, \alpha_2, \cdots, \alpha_m \in [0, 1], \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\} \end{aligned} \quad (8)$$

For known control gain matrices K_i, output matrix C and constant matrix E_2, $C^T K_i^T (R + E_2^T E_2) K_i C$ is known semi-positive definite matrices. Therefore, there exists a matrix H, such that

$$H - C^T K_i^T (R + E_2^T E_2) K_i C \ge 0, i \in M \quad (9)$$

Theorem 1 If there exist a matrix $\bar{K} \in \gamma_{\alpha_1, \alpha_2, \cdots, \alpha_m}(A_1, A_2, \cdots, A_m)$ and a symmetric positive definite matrix P such that

$$\begin{aligned} &Q + H + P(A + B\bar{K}C) \\ &+(A + B\bar{K}C)^T P + 2PDD^T P + E_1^T E_1 \\ &+\gamma^{-2} P B_1 B_1^T P + C_1^T C_1 < 0, \end{aligned} \quad (10)$$

then there exists a switching law $\sigma(t) : [0, \infty) \to M = \{1, 2, \cdots, m\}$ such that $u_{\sigma(t)} = K_{\sigma(t)} y$ solves the problem of hybrid guaranteed cost robust control with H_∞ disturbance attenuation γ. In this case, the cost upper-bound of system (1) is given by $J_1^* = x_0^T P x_0$, where $\gamma > 0$ and H satisfies (9).

Proof: Since $\bar{K} \in \gamma_{\alpha_1, \alpha_2, \cdots, \alpha_m}(A_1, A_2, \cdots, A_m)$, there exist $\alpha_1, \alpha_2, \cdots, \alpha_m \in [0, 1]$, such that

$$\sum_{i=1}^m \alpha_i = 1, \quad (11)$$

and

$$\bar{K} = \sum_{i=1}^m \alpha_i K_i, \quad (12)$$

Substituting (12) into (10) gives

$$\begin{aligned} &Q + H + P(A + B \textstyle\sum_{i=1}^m \alpha_i K_i C) \\ &+(A + B \textstyle\sum_{i=1}^m \alpha_i K_i C)^T P + 2PDD^T P \\ &+E_1^T E_1 + \gamma^{-2} P B_1 B_1^T P + C_1^T C_1 < 0, \end{aligned} \quad (13)$$

i.e. $\sum_{i=1}^m \alpha_i \prod_i < 0$, where

$$\begin{aligned} \prod\nolimits_i &= Q + H + P(A + BK_i C) \\ &+(A + BK_i C)^T P + 2PDD^T P \\ &+E_1^T E_1 + \gamma^{-2} P B_1 B_1^T P + C_1^T C_1 < 0, \end{aligned} \quad (14)$$

which implies for $\forall x \in R^n \backslash \{0\}$

$$x^T \sum_{i=1}^m \alpha_i \Pi_i x < 0, \quad (15)$$

Let

$$\Omega_i = \{x | x^T \Pi_i x < 0\}, \quad (16)$$

Then $\bigcup_{i=1}^{m}\Omega_i = R^n\backslash\{0\}$. Construct $\bar{\Omega}_1 = \Omega_1, \cdots, \bar{\Omega}_i = \Omega_i - \bigcup_{j=1}^{i-1}\bar{\Omega}_j, \cdots, \bar{\Omega}_m = \Omega_m - \bigcup_{j=1}^{m-1}\bar{\Omega}_j$. Obviously $\bigcup_{i=1}^{m}\bar{\Omega}_i = R^n\backslash\{0\}$ and $\bar{\Omega}_i\bigcap\bar{\Omega}_j = \phi$, $i \neq j$. Now we choose the switching law as

$$\sigma(x(t)) = i, x(t) \in \bar{\Omega}_i, i \in M. \tag{17}$$

We first prove that system (1) is asymptotically stable.

Define the Lyapunov function $V(x) = x(t)^T Px(t)$. When $w = 0$, $u_i = K_i y$, differentiating $V(x)$ along system (1) yields

$$\begin{aligned}
\dot{V}(x) &= 2x^T P\dot{x} \\
&= 2x^T P[(A + \Delta A)x + (B + \Delta B)K_i Cx] \\
&= x^T[P(A + BK_iC) + (A + BK_iC)^T P \\
&\quad + 2PDFE_1 + 2PDFE_2K_iC]x \\
&\leq x^T[P(A + BK_iC) + (A + BK_iC)^T P \\
&\quad + 2PDD^T P + E_1^T E_1 + C^T K_i^T E_2^T E_2 K_i C]x,
\end{aligned}$$

So, by (16), (17) we have $\dot{V}(x) < 0$.

Therefore, $V(x)$ decreases along solutions of system (1), which implies asymptotic stability.

Secondly, we prove that system (1) have cost upper-bound $J_1^* = x_0^T Px_0$.

$$\begin{aligned}
J_1 &= \int_0^{+\infty}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)}]dt \\
&= \sum_{k=0}^{+\infty}\{\int_{t_k}^{t_{k+1}}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)} \\
&\quad + \dot{V}(x)]dt - V(x)|_{t_k}^{t_{k+1}}\} \\
&= \sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)} \\
&\quad + \dot{V}(x)]dt - \sum_{k=0}^{+\infty}V(x)|_{t_k}^{t_{k+1}} \\
&= \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)} \\
&\quad + \dot{V}(x)]dt + V(x_0) - V(x(t_1)) \\
&\quad + V(x(t_1)) - \cdots - V(x(+\infty))
\end{aligned}$$

Applying (15) and (16) gives rise to

$$\begin{aligned}
&\int_{t_{j_n}}^{t_{j_n+1}}[x(t)^T Qx(t) + u_{\sigma(t)}^T Ru_{\sigma(t)} + \dot{V}(x)]dt \\
&\leq \int_{t_{j_n}}^{t_{j_n+1}} x^T[Q + C^T K_j^T RK_j C \\
&\quad + P(A + BK_jC) + (A + BK_jC)^T P \\
&\quad + 2PDD^T P + E_1^T E_1 + C^T K_j^T E_2^T E_2 K_j C]xdt \\
&\leq \int_{t_{j_n}}^{t_{j_n+1}} x^T[Q + H + P(A + BK_jC) \\
&\quad + (A + BK_jC)^T P + 2PDD^T P + E_1^T E_1]xdt \\
&< 0
\end{aligned}$$

Therefore, we have

$$J_1 \leq V(x_0) = x_0^T Px_0$$

which means that the cost upper-bound of system (1) is $J_1^* = x_0^T Px_0$.

Thirdly, we show that system (1) have H_∞ disturbance attenuation γ.

In order to establish the upper-bound γ, for any nonzero $w \in L_2[0, \infty)$,we assume $x(0) = 0$. A simple calculation shows

$$\begin{aligned}
J_2 &= \int_0^{+\infty}(z^T z - \gamma^2 w^T w)dt \\
&= \int_0^{+\infty}[z^T z - \gamma^2 w^T w + \dot{V}(x)]dt - x(\infty)^T Px(\infty) \\
&\leq \sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}}[z^T z - \gamma^2 w^T w + \dot{V}(x)]dt \\
&= \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}}[z^T z - \gamma^2 w^T w + \dot{V}(x)]dt \\
&= \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}}\{z^T z - \gamma^2 w^T w \\
&+ x^T[P(A + BK_jC) + (A + BK_jC)^T P \\
&+ 2PDFE_1 + 2PDFE_2K_jC]x + 2x^T PB_1 w\}dt \\
&= \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}} x^T[P(A + BK_jC) \\
&+ (A + BK_jC)^T P + 2PDFE_1 + 2PDFE_2K_jC \\
&+ \gamma^{-2}PB_1B_1^T P + C_1^T C_1]xdt - \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}} \\
&(\gamma^{-1}B_1^T Px - \gamma w)^T(\gamma^{-1}B_1^T Px - \gamma w)dt \\
&\leq \sum_{j=1}^{m}\sum_{n=1}^{+\infty}\int_{t_{j_n}}^{t_{j_n+1}} x^T[P(A + BK_jC) \\
&+ (A + BK_jC)^T P + 2PDD^T P + E_1^T E_1 \\
&+ C^T K_j^T E_2^T E_2 K_j C + \gamma^{-2}PB_1B_1^T P + C_1^T C_1]xdt
\end{aligned}$$

It follows from (15) and (16) that $J_2 < 0$, i.e., $\|z\|_2 < \gamma\|w\|_2$, for any nonzero $w \in L_2[0, \infty)$.

Remark 2 There are infinite number of such a matrix H satisfying inequality (9). The choice of H may have an impact on the solutions of matrix the inequality (10). To minimize the conservativeness, the problem of choosing H can be turned into an optimal control problem with LMI's constraints, which can be realized by solving the following optimal control problem.

$min \;\; \gamma$

s.t.

1) $diag[H - C^T K_1^T(E_2^T E_2 + R)K_1C, \cdots,$
$H - C^T K_i^T(E_2^T E_2 + R)K_iC, \cdots,$
$H - C^T K_m^T(E_2^T E_2 + R)K_mC] < 0,$

$$2)\quad \begin{bmatrix} -\gamma I & H \\ H & -\gamma I \end{bmatrix}, \tag{18}$$

which provides a matrix H satisfying inequality (9) and having the smallest maximal singular value.

B. Multiple Lyapunov function technique

In this section, we will employ multiple Lyapunov function method to design the switching law σ. For simplicity, we only consider the case of two candidate controllers, i.e., $M = \{1, 2\}$.

Theorem 2 Let the constant $\gamma > 0$ be given. If there exist two numbers β_1 and β_2 (either both nonnegative or both nonpositive), such that the inequalities

$$\begin{aligned} &Q + C^T K_1^T R K_1 C + P_1(A + BK_1C) \\ &+(A + BK_1C)^T P_1 + 2P_1DD^TP_1 + E_1^TE_1 \\ &+C^TK_1^TE_2^TE_2K_1C + \gamma^{-2}P_1B_1B_1^TP_1 \\ &+C_1^TC_1 + \beta_1(P_1 - P_2) < 0, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &Q + C^T K_2^T R K_2 C + P_2(A + BK_2C) \\ &+(A + BK_2C)^T P_2 + 2P_2DD^TP_2 + E_1^TE_1 \\ &+C^TK_2^TE_2^TE_2K_2C + \gamma^{-2}P_2B_1B_1^TP_2 \\ &+C_1^TC_1 + \beta_2(P_2 - P_1) < 0, \end{aligned} \tag{20}$$

have two symmetric positive definite solutions P_1 and P_2, then there exists a switching law $\sigma : [0, \infty) \to M = \{1, 2\}$ that solves the hybrid guaranteed cost robust control with H_∞ disturbance attenuation γ. The cost upper-bound is $J_1^* = max\{x_0^TP_1x_0, x_0^TP_2x_0\}$ when are β_i both nonnegative or $J_1^* = min\{x_0^TP_1x_0, x_0^TP_2x_0\}$ when are β_i both nonpositive.

Proof: Without loss of generality, suppose $\beta_1, \beta_2 \geq 0$. In view of S-procedure, it follows that:

If $x^T(P_1 - P_2)x \geq 0$, and $\forall x \in R^n \backslash \{0\}$, then

$$\begin{aligned} &x^T[Q + C^T K_1^T R K_1 C + P_1(A + BK_1C) \\ &+(A + BK_1C)^T P_1 + 2P_1DD^TP_1 + E_1^TE_1 \\ &+C^TK_1^TE_2^TE_2K_1C + \gamma^{-2}P_1B_1B_1^TP_1 \\ &+C_1^TC_1]x < 0, \end{aligned} \tag{21}$$

If $x^T(P_2 - P_1)x \geq 0$, and $\forall x \in R^n \backslash \{0\}$, then

$$\begin{aligned} &x^T[Q + C^T K_2^T R K_2 C + P_2(A + BK_2C) \\ &+(A + BK_2C)^T P_2 + 2P_2DD^TP_2 + E_1^TE_1 \\ &+C^TK_2^TE_2^TE_2K_2C + \gamma^{-2}P_2B_1B_1^TP_2 \\ &+C_1^TC_1]x < 0, \end{aligned} \tag{22}$$

Let

$$\begin{aligned} \Omega_1 &= \{x | x^T(P_1 - P_2)x \geq 0, x \neq 0\}, \\ \Omega_2 &= \{x | x^T(P_2 - P_1)x \geq 0, x \neq 0\}. \end{aligned} \tag{23}$$

then $\Omega_1 \bigcup \Omega_2 = R^n \backslash \{0\}$. Assume $x_0^TP_1x_0 \geq x_0^TP_2x_0$, we design the switching law as follows.

$$\begin{aligned} &\sigma(t) = 1, \quad x(t) \in \bar{\Omega}_1 = \Omega_1, \\ &\sigma(t) = 2, \quad x(t) \in \bar{\Omega}_2 = \Omega_2 - \bar{\Omega}_1. \end{aligned} \tag{24}$$

Now, let $w = 0$ and consider two Lyapunov functions $V_1(x(t)) = x(t)^TP_1x(t)$, $V_2(x(t)) = x(t)^TP_2x(t)$, where P_1 and P_2 are two symmetric positive definite matrices satisfying (19) and (20).

For $x \in \Omega_i, i = 1, 2$, the time-derivative of $V_i(x(t))$ along the trajectory of the closed-loop system (1) is given by

$$\begin{aligned} &\dot{V}_i(x) = 2x^TP_i\dot{x} \\ &= 2x^TP_i[(A + \triangle A) + (B + \triangle B)K_iC]x \\ &= x^T[P_i(A + BK_iC) + (A + BK_iC)^TP_i \\ &+2P_iDFE_1 + 2P_iDFE_2K_iC]x \\ &\leq x^T[P_i(A + BK_iC) + (A + BK_iC)^TP_i \\ &+2P_iDD^TP_i + E_1^TE_1 + C^TK_i^TE_2^TE_2K_iC]x, \end{aligned}$$

Using (21)-(24), we obtain $\dot{V}_i(x(t)) < 0$, $(i = 1, 2)$. When $x^T(P_1 - P_2)x \geq 0$ and $x \neq 0$, (21) holds. When $x^T(P_2 - P_1)x \geq 0$ and $x \neq 0$, (22) holds. Therefore, in the switching instant t_j , $V_{\sigma(t_j)}(x_{t_j}) = \lim_{t \to t_j^-} V_{\sigma(t)}(x(t))$. In view of multiple Lyapunov function technology, the asymptotic stability of the system (1) follows immediately.

Without loss of generality, suppose that $x^T(P_1 - P_2)x \geq 0$, we reach the following conclusion.

$$\begin{aligned} J_1 &= \int_0^{+\infty} [x(t)^TQx(t) + u_{\sigma(t)}^TRu_{\sigma(t)}]dt \\ &= \sum_{k=0}^{+\infty} \{ \int_{t_k}^{t_{k+1}} [x(t)^TQx(t) + u_{\sigma(t)}^TRu_{\sigma(t)} \\ &\quad + \dot{V}_{i_k}(x)]dt - V_{i_k}(x)|_{t_k}^{t_{k+1}} \} \\ &= \sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} [x(t)^TQx(t) + u_{\sigma(t)}^TRu_{\sigma(t)} \\ &\quad + \dot{V}_{i_k}(x)]dt - \sum_{k=0}^{+\infty} V_{i_k}(x)|_{t_k}^{t_{k+1}} \\ &= \sum_{j=1}^{2} \sum_{n=1}^{+\infty} \int_{t_{j_n}}^{t_{j_n+1}} [x(t)^TQx(t) + u_{\sigma(t)}^TRu_{\sigma(t)} \\ &\quad + \dot{V}_j(x)]dt - \sum_{k=0}^{+\infty} V_{i_k}(x)|_{t_k}^{t_{k+1}} \end{aligned} \tag{25}$$

similarly to the proof above, we can obtain

$$\begin{aligned} &\int_{t_{j_n}}^{t_{j_n+1}} [x(t)^TQx(t) + u_{\sigma(t)}^TRu_{\sigma(t)} + \dot{V}_j(x)]dt \\ &\leq \int_{t_{j_n}}^{t_{j_n+1}} x^T[Q + C^TK_j^TRK_jC \\ &\quad + P_j(A + BK_jC) + (A + BK_jC)^TP_j \\ &\quad + 2P_jDD^TP_j + E_1^TE_1 + C^TK_j^TE_2^TE_2K_jC]xdt \\ &< 0 \end{aligned} \tag{26}$$

By the design of the switching law and the continuity of $V_i (i = 1, 2)$, we know $V_1(x(t_n)) = V_2(x(t_n))$ at switching instant $t_n (n \in N)$. It is not difficult to see that

$$
\begin{aligned}
&-\sum_{k=0}^{+\infty} V_{i_k}(x)|_{t_k}^{t_{k+1}} \\
&= \begin{cases} V_1(x(t_0)) - V_1(x(t_1)) & \text{the first subsystem} \\ +V_2(x(t_1)) - \cdots & \text{is activated at} \\ \leq V_1(x(t_0)), & \text{the initial time.} \\ \\ V_2(x(t_0)) - V_2(x(t_1)) & \text{the second subsystem} \\ +V_1(x(t_1)) - \cdots & \text{is activated at} \\ \leq V_2(x(t_0)). & \text{the initial time.} \end{cases}
\end{aligned} \tag{27}
$$

Note that (25)-(26), we have

$$
J_1 \leq max\{V_1(x(t_0)), V_2(x(t_0))\} = max\{x_0^T P_1 x_0, x_0^T P_2 x_0\}
$$

i.e., the cost upper-bound of system (1) is $J_1^* = max\{x_0^T P_1 x_0, x_0^T P_2 x_0\}$. Similarly, When $\beta_1, \beta_1 \leq 0$, the cost upper-bound of system (1) is $J_1^* = min\{x_0^T P_1 x_0, x_0^T P_2 x_0\}$.

To establish the upper-bound γ, we assume $x(0) = 0$. For any nonzero $w \in L_2[0, \infty)$, we have

$$
\begin{aligned}
J_2 &= \int_0^{+\infty} (z^T z - \gamma^2 w^T w) dt \\
&= \sum_{j=1}^{2} \sum_{n=1}^{+\infty} \int_{t_{j_n}}^{t_{j_n+1}} [z^T z - \gamma^2 w^T w + \dot{V}_j(x)] dt \\
&\quad - \sum_{k=0}^{+\infty} V_{i_k}(x)|_{t_k}^{t_{k+1}},
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
&\int_{t_{j_n}}^{t_{j_n+1}} [z^T z - \gamma^2 w^T w + \dot{V}_j(x)] dt \\
&\leq \int_{t_{j_n}}^{t_{j_n+1}} x^T [P_j(A + BK_jC) + (A + BK_jC)^T P_j \\
&+2P_j DD^T P_j + E_1^T E_1 + C^T K_j^T E_2^T E_2 K_j C \\
&+\gamma^{-2} P_j B_1 B_1^T P_j + C_1^T C_1] x dt \\
&- \int_{t_{j_n}}^{t_{j_n+1}} (\gamma^{-1} B_1^T P_j x - \gamma w)^T (\gamma^{-1} B_1^T P_j x - \gamma w) dt \\
&\leq \int_{t_{j_n}}^{t_{j_n+1}} x^T [P_j(A + BK_jC) + (A + BK_jC)^T P_j \\
&+2P_j DD^T P_j + E_1^T E_1 + C^T K_j^T E_2^T E_2 K_j C \\
&+\gamma^{-2} P_j B_1 B_1^T P_j + C_1^T C_1] x dt \\
&< 0
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&-\sum_{k=0}^{+\infty} V_{i_k}(x)|_{t_k}^{t_{k+1}} \\
&\leq \begin{cases} V_1(x(t_0)) & \text{the first subsystem} \\ = x_0^T P_1 x_0 & \text{is activated at} \\ = 0, & \text{the initial time.} \\ \\ V_2(x(t_0)) & \text{the second subsystem} \\ = x_0^T P_2 x_0 & \text{is activated at} \\ = 0. & \text{the initial time.} \end{cases}
\end{aligned} \tag{30}
$$

It follows from (28)-(30) that $J_2 < 0$, i.e., $\|z\|_2 < \gamma \|w\|_2$, for any nonzero $w \in L_2[0, \infty)$.

IV. EXAMPLE

Consider the following switched controller system

$$
\begin{aligned}
\dot{x}(t) &= (A + \triangle A)x(t) + (B + \triangle B)u_{\sigma(t)} + B_1 w, \quad (31) \\
z &= C_1 x(t), \\
y &= Cx(t), \\
x(0) &= [-2, 2]^T,
\end{aligned}
$$

where $A = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the uncertain term $[\Delta A, \Delta B] = DF(t)[E_1, E_2]$ with $D = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.5 \end{bmatrix}, F(t) = \sin t, E_1 = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0.1 \end{bmatrix}, E_2 = \begin{bmatrix} 0.5 & 1 \\ 0.8 & 0 \end{bmatrix}, Q = R = I$. Suppose that system (31) has two candidate output feedback controllers, i.e., $\sigma(t) : [0, \infty) \rightarrow M = \{1, 2\}, u_1 = K_1 y = \begin{bmatrix} -2.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} y, u_2 = K_2 y = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 2.5 \end{bmatrix} y$, It is easy to see that none of two controllers u_1 and u_2 makes system (31) stabilizable (see Fig.1 and Fig.2). Let $\gamma = 1$, $\alpha_1 = \alpha_2 = 0.5$, then $\bar{K} = 0.5K_1 + 0.5K_2 = \begin{bmatrix} -1 & -0.5 \\ -0.5 & -1 \end{bmatrix}$. Solving the optimization problem (18) and inequality (10), we obtain

$$
H = \begin{bmatrix} 7.0447 & 0.0052 \\ 0.0052 & 7.0776 \end{bmatrix}, P = \begin{bmatrix} 5.5595 & -2.2235 \\ -2.2235 & 4.9810 \end{bmatrix},
$$

Construct the switching region as follows:

$$
\Omega_1 = \{x : x^T \begin{bmatrix} -52.8218 & 7.4400 \\ 7.4400 & 30.1468 \end{bmatrix} x < 0\},
$$

$$
\Omega_2 = \{x : x^T \begin{bmatrix} 33.9076 & -10.9740 \\ -10.9740 & -32.8281 \end{bmatrix} x < 0\},
$$

and

$$
\bar{\Omega}_1 = \Omega_1, \quad \bar{\Omega}_2 = \Omega_2 - \bar{\Omega}_1,
$$

Design the switching law by

$$
\sigma(t) = i, \quad \text{when} \quad x(t) \in \bar{\Omega}_i, \quad i \in \{1, 2\}.
$$

System (31) is stabilizable via switching among u_1 and u_2 (see Fig.3), the cost upper-bound is $J_1^* = x_0^T P x_0 = 59.95$, and when $x(0) = 0$, for any nonzero $w \in L_2[0, \infty)$, H_∞ performance $J_2 < 0$, i.e., $\|z\|_2 < \gamma \|w\|_2$.

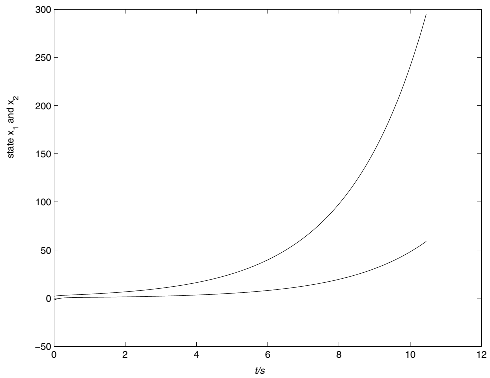

Fig. 1. The state response of system under controller u_1

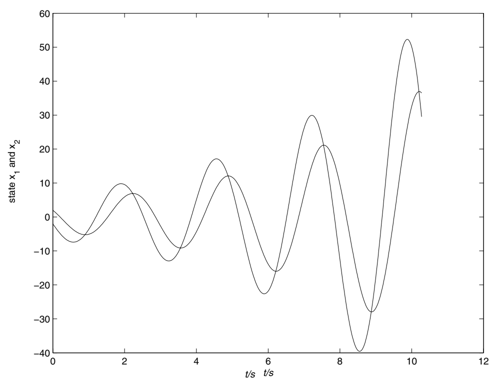

Fig. 2. The state response of system under controller u_2

V. CONCLUSIONS

This paper has addressed the problem of guaranteed cost H_∞ robust hybrid output-feedback control for a class of uncertain linear systems. In the case that no single continuous controller in a set of controllers is effective, we have used single Lyapunov function technique and multiple Lyapunov function technique to construct hybrid output feedback controllers to realize guaranteed cost robust control with H_∞ disturbance attenuation γ. The key idea is to switch among a set of candidate controllers in a given set of admissible output feedback controllers.

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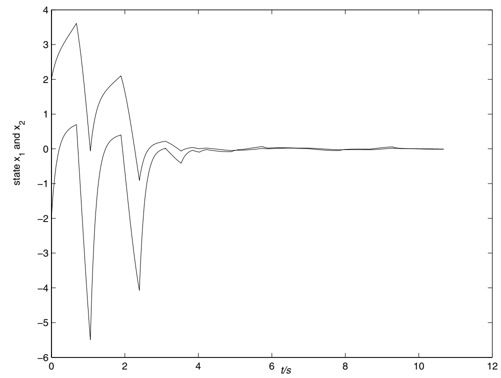

Fig. 3. The state response of system under hybrid output-feedback controller

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