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Trajectory Tracking Control of Bimodal Piecewise Affine Systems

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Abstract— **This paper deals with a trajectory tracking problem of a class of bimodal piecewise affine systems, which have rarely been discussed so far. This would be very challenging because of the discontinuous changes of their vector fields. First, we introduce an error variable and an error system as a generalization of the tracking error and its system. As an error variable, a function switched by the mode of a piecewise affine system is adopted to overcome an inherent difficulty in trajectory tracking of piecewise affine systems. Next, we design a tracking controller which stabilizes the error system using a Lyapunov-like function, which can be applied to systems including state jumps. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

Hybrid systems, which involve the interaction of discrete and continuous dynamics, have been studied from various directions. In the control community, hybrid systems are considered as systems governed by differential equations with discrete events [1]. Piecewise affine (PWA) systems belong to a special class of hybrid systems, which are defined by partitioning the state space in a finite number of polyhedral regions and associating each region with a different affine dynamic model. PWA systems have been investigated actively because of their wide applicability.

Analysis and synthesis of PWA systems have been studied in various papers. A method to compute piecewise quadratic Lyapunov functions for analyzing stability has been proposed [2]. Synthesis based on stability with such Lyapunov functions has been discussed [3]. Furthermore, a well-posedness condition has been obtained in terms of algebraic conditions [4]. On the other hand, a trajectory tracking problem, which is important in practical applications, for PWA systems would be very challenging because the tracking output of these systems is not necessarily smooth. This problem has discussed by Solymon and Rantzer in [5], where they have estimated a tracking error, but have not guaranteed its convergence to 0. Although tracking problems of nonlinear systems have been investigated by many researchers, e. g. [6], these methods are not applicable to PWA systems because of the discontinuous changes of their vector fields.

This paper addresses a trajectory tracking problem for bimodal PWA systems. A tracking error is guaranteed to converge to 0, that is perfect tracking is achieved, under some conditions on reference trajectories. First, an error variable and an error system are introduced as a generalization of the tracking error and its system. As an error variable, a function switched by the mode of a PWA system is adopted in order to overcome an inherent difficulty in trajectory tracking control of PWA systems. Next, we design a tracking controller which stabilizes the error system using a Lyapunov-like function, which can be applied to systems including state jumps. Furthermore, the feasibility condition of tracking for single-input single-output (SISO) piecewise linear (PWL) systems is simplified. Finally, a numerical example is given to illustrate the effectiveness of the proposed method. All proofs are omitted because of the lack of space in this paper.

We will use the following notations in this paper. The triple (A, B, C) represents the linear system $\dot{x} = Ax + Bu,\ y = Cx$. The notations I_n and $\mathbb{M}_+^n$ denote the $n \times n$ identity matrix and the set of $n \times n$ lower triangular matrices whose diagonal components are positive, respectively. $|\mathbb{S}|$ gives the number of elements of a countable set $\mathbb{S}$. For a function $g : \mathbb{R} \to \mathbb{R}^n$, $g(t^\pm)$ represents $\lim_{\varepsilon \to \pm 0} g(t + \varepsilon)$. $\mathbb{PC}$ is the set of scalar functions which are piecewise continuous and right continuous on any finite interval. For a positive definite matrix $P \in \mathbb{R}^{n\times n}$, a vector $x \in \mathbb{R}^n$ and a matrix $Q \in \mathbb{R}^{m\times n}$, $||\cdot||_P$ is defined by $||x||_P := \sqrt{x^{\mathrm{T}} P x}$ and $||Q||_P := \sigma_{\max}(QP^{-1/2})$, respectively, where $\sigma_{\max}(\cdot)$ is the largest singular value of matrices. Note that $||Qx|| \le ||Q||_P ||x||_P$ holds.

II. PROBLEM FORMULATION

In this paper, we consider a bimodal PWA system given by

$$\Sigma_x \left\{ \begin{array}{l} \dot{x} = A_I x + B_I u + E_I \\ I = \left\{ \begin{array}{l} 1, \text{ if } Cx + D \le 0 \\ 2, \text{ if } Cx + D \ge 0 \end{array} \right. \\ z = Fx \end{array} \right. , \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $I(t) \in \{1, 2\}$ is the mode, $u(t) \in \mathbb{R}^m$ is the input, $z(t) \in \mathbb{R}^l$ is the tracking output, $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1, B_2 \in \mathbb{R}^{n\times m}$, $E_1, E_2 \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$ and $F \in \mathbb{R}^{l\times n}$. Σ_x is called a target system.

When $x(t)$ satisfies $Cx(t) + D = 0$ at $t = t_0$, $I(t_0)$ is determined by the behavior of $x(t)$ in the time interval $t \in [t_0, t_0 + \varepsilon]$ for a small constant $\varepsilon > 0$. From this viewpoint, a solution of Σ_x is defined as follows [4].

Definition 1: For a bounded function $u : \mathbb{R}_+ \to \mathbb{R}^m$, $(x(t), I(t))$ is said to be a solution of Σ_x on $[0, t_0)$ for the initial state $x_0 \in \mathbb{R}^n$, if $x(t)$ and $I(t)$ satisfy the second relation of (1) and $x(t) = x_0 + \int_0^t f(x(s), I(s), u(s))\mathrm{d}s$ where $f(x, I, u)$ is the right-hand side of the first equation of (1), and there is no left accumulation point in the set of the discontinuous points of $I(t)$ on $[0, t_0)$.

Note that Definition 1 does not allow trajectories with Zeno behavior nor sliding modes.

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Our purpose in this paper is to make output z track a reference trajectory r, that is to satisfy the following condition for some set $\mathbb{D} \subset \mathbb{R}^n$.

$$\lim_{t\to\infty}(z(t) - r(t)) = 0,\ x(0) = x_0,\ \forall x_0 \in \mathbb{D} \tag{2}$$

Generally, for trajectory tracking problems like this, reference trajectories need to be generated by systems which have similar structures to target systems, which is a well-known property, *internal model principle*, for linear systems [7] and nonlinear smooth systems [6]. Similarly, we suppose that the reference trajectory r is generated by a PWA system

$$\Sigma_w \begin{cases} \dot{w} = Q_J w + R_J v + U_J \\ J = \begin{cases} 1, & \text{if } Sw + T \le 0 \\ 2, & \text{if } Sw + T \ge 0 \end{cases} \\ r = Ww \end{cases}, \tag{3}$$

where $w(t) \in \mathbb{R}^p$, $J(t) \in \{1,2\}$, $v(t) \in \mathbb{R}^q$, $Q_1, Q_2 \in \mathbb{R}^{p\times p}$, $R_1, R_2 \in \mathbb{R}^{p\times q}$, $U_1, U_2 \in \mathbb{R}^p$, $S \in \mathbb{R}^{1\times p}$, $T \in \mathbb{R}$ and $W \in \mathbb{R}^{l\times p}$. Σ_w is called a reference system. A procedure to construct a reference system from a given scalar function r is presented in Section VI.

Now, we formulate the trajectory tracking problem discussed in this paper for the tracking condition (2).

Problem 1: For (1) and (3), find a condition such that (2) is achievable with some feedback input u and some set of the initial states $\mathbb{D} \subset \mathbb{R}^n$ whose volume is not 0 [1], and give the feedback input u and the set $\mathbb{D}$ achieving (2).

We give several notations and assumptions on systems Σ_x and Σ_w.

(H.1) Let d_i^x and d_i^w be the minimum values of the relative degrees of (A_i, B_i, C) and (Q_i, R_i, S), respectively. Assume that $d_x := \min\{d_1^x, d_2^x\} \ge 2$ and $d_w := \min\{d_1^w, d_2^w\} \ge 2$.

(H.2) Let $\mathbb{J}$ be the set $\{t : J(t) \neq J(t^-)\}$, that is the set of the instants when $J(t)$ changes. Assume that there exists a positive constant δ such that $\tau_a - \tau_b \ge \delta$ for any $\tau_a \in \mathbb{J}$ and $\tau_b \in \mathbb{J}$ satisfying $\tau_a > \tau_b$, which implies that $w(t)$ has no accumulations. Let η be $\sup_{t>0}(|\mathbb{J} \cap [0,t]|/t)$, which is less or equal to $1/\delta$.

III. Basic Strategy for Trajectory Tracking

This section presents a basic strategy for solving the trajectory tracking problem by generalizing tracking methods taken in various papers. In paper [6], a tracking error between states x and w of target and reference systems are represented by $\epsilon = x - \pi(w)$ for a function π, and trajectory tracking of nonlinear systems is obtained by stabilizing the system of ϵ. From this viewpoint, we define an *error variable* which generalizes the tracking error as follows.

Definition 2: For a function $\varphi : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ and a function parameter $\bar{w} : \mathbb{R}_+ \to \mathbb{R}^{p_1}$, $\epsilon = \varphi(x, w, \bar{w})$ is called an *error variable* if there exists a class $\mathcal{K}$ function [2] $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$||z(t) - r(t)|| \le \alpha(||\epsilon(t)||),\ \forall t \in \mathbb{R}_+, \tag{4}$$

where $\epsilon(t) = \varphi(x(t), w(t), \bar{w}(t))$, $z(t) = Fx(t)$, $r(t) = Ww(t)$, and $p_1 \ge 0$ and $p_2 > 0$ are integers.

See the next section to know the usage of a function parameter $\bar{w}$.

Let Σ_ϵ be the system governing ϵ, and be called an *error system*. System Σ_ϵ is derived by differentiating $\epsilon = \varphi(x, w, \bar{w})$ and substituting (1) and (3). Our strategy for achieving the tracking condition (2) is to assign an appropriate function φ satisfying (4) and to stabilize its error system Σ_ϵ. The following lemma will be available to guarantee stability of the error system using a *Lyapunov-like function*.

Lemma 1: Consider the systems (1) and (3). For an error variable $\epsilon = \varphi(x, w, \bar{w})$, assume that the following conditions (C.1) and (C.2) are satisfied for some input u and some set $\mathbb{E} \subset \mathbb{R}^{p_2}$ which includes the origin and the volume of the set

$$\mathbb{D} = \{x : \varphi(x, w(0), \bar{w}(0)) \in \mathbb{E}\} \subset \mathbb{R}^n \tag{5}$$

is not 0. Then, (2) holds for the pair of these u and $\mathbb{D}$, which is a solution of Problem 1.

(C.1) *The origin of Σ_ϵ is an equilibrium point, that is*

$$\epsilon(0) = 0 \Rightarrow \epsilon(t) = 0,\ \forall t \in \mathbb{R}_+. \tag{6}$$

(C.2) *For a class $\mathcal{KL}$ function [3] $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, there exists a function $V : \mathbb{R}^{p_2} \to \mathbb{R}_+$ satisfying*

$$V(0) = 0 \text{ and } V(\epsilon) > 0,\ \forall \epsilon \in \mathbb{E}\backslash\{0\} \tag{7}$$

$$V(\epsilon(t)) \le \beta(V(\epsilon(0)), t),\ \forall t \in \mathbb{R}_+ \text{ if } \epsilon(0) \in \mathbb{E}. \tag{8}$$

Remark 1: Condition (8) guarantees the decrease of $V(\epsilon)$ instead of the common condition $\dot{V}(\epsilon) < 0$. Lyapunov-like functions enable us to deal with systems including discontinuous changes of the state. Note that error system Σ_ϵ can include the discontinuous changes of the state if $\varphi(\cdot,\cdot,\cdot)$ is not differentiable. See Section IV for the detail.

IV. New Error Variable and Error System

Before proposing a new error variable, consider the variable given by the linear combination $\hat{\epsilon} = \Psi x - w$ as a candidate for error variable, which is from the result of linear systems [7]. Since the output error $z - r$ is given by $W\hat{\epsilon} + (F - W\Psi)x$, $\hat{\epsilon}$ is an error variable if the following condition holds from (4).

$$F = W\Psi \tag{9}$$

Unfortunately, $\hat{\epsilon}(t)$ can not be guaranteed to converge to the origin in general although it converges to a bounded domain under some condition, which is discussed in [5]. However,

[1] The volume of a set $\mathbb{D} \subset \mathbb{R}^n$ is not 0 iff there exist a vector $x_* \in \mathbb{R}^n$ and a constant $\varepsilon > 0$ such that $\{x : ||x - x_*|| < \varepsilon\} \subset \mathbb{D}$.

[2] A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if $\alpha(s)$ is strictly increasing and $\alpha(0) = 0$.

[3] A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if for each fixed t, $\beta(s,t)$ belongs to class $\mathcal{K}$ with respect to s, and for each fixed s, it is decreasing with respect to t, and $\lim_{t\to\infty}\beta(s,t) = 0$.

for the tracking condition (2), an error variable is necessary to converge to the origin.

In order to overcome the difficulty of the tracking problem of PWA systems, we introduce a new error variable including a function parameter $\bar{w}$. Consider

$$\bar{\epsilon} = \Psi x - w_{I,J}, \quad w_{i,j} = \begin{cases} w, \text{ if } i = j \\ \bar{w}, \text{ if } i \neq j \end{cases} \tag{10}$$

as a candidate for error variable, where $\Psi \in \mathbb{R}^{p \times n}$ is a row full rank matrix and $\bar{w} : \mathbb{R}_+ \to \mathbb{R}^p$ is a function satisfying

$$\begin{aligned} \dot{\bar{w}} &= Q_{3-J}\bar{w} + R_{3-J}\bar{v} + U_{3-J} \\ \bar{w}(t) &= w(t), \ \forall t \in \{0\} \cup \mathbb{J}. \end{aligned} \tag{11}$$

Remark 2: The differential equation of $\bar{w}(t)$ always differs from the one of $w(t)$. This error variable $\bar{\epsilon}(t)$ can be discontinuous when $I(t)$ or $J(t)$ changes, which is discussed later.

Since $w_{I,J}$ is assigned instead of w in (10), $\bar{\epsilon}$ is not necessarily an error variable. We derive a condition for $\bar{\epsilon}$ to be an error variable. The output error $z - r$ is expressed as

$$\begin{aligned} z - r = (F - W\Psi)x + W\bar{\epsilon} \\ + \begin{cases} 0, \text{ if } I(t) = J(t) \\ W(\bar{w}(t) - w(t)), \text{ if } I(t) \neq J(t) \end{cases} \end{aligned}$$

from (1), (3) and (10). Thus, $\bar{\epsilon}$ satisfies (4) if (9) holds and there exists a nonnegative real number κ such that

$$||\bar{w}(t) - w(t)||_P \leq \kappa ||\bar{\epsilon}(t)||_P \tag{12}$$

for any $t \geq 0$ satisfying $I(t) \neq J(t)$. From this viewpoint, the following lemma gives a sufficient condition for $\bar{\epsilon}$ to be an error variable.

Lemma 2: Assume that (9) holds and that there exists a non-negative constant κ and a positive definite matrix P such that (12) holds for $t \in \mathbb{R}_+$ satisfying $I(t) \neq J(t)$. Then the variable $\bar{\epsilon}$ given by (10) is an error variable.

A verifiable condition on Lemma 2 is given in Section V-C.

Now, the error system for $\bar{\epsilon}$ is derived by differentiating (10) and substituting (1) and (3). Assume that $I \in \mathbb{PC}$, and let $\mathbb{I}$ be the set $\{t : I(t) \neq I(t^-)\}$. Consider feedback input u in the form of a linear combination of x, $w_{I,J}$ and $v_{I,J}$ whose coefficient matrices are dependent on I as

$$\begin{aligned} u &= \tilde{\Gamma}_I x + \Lambda_I w_{I,J} + \Theta_I v_{I,J} + \Delta_I \\ v_{i,j} &= \begin{cases} v & \text{if } i = j \\ \bar{v} & \text{if } i \neq j \end{cases}, \end{aligned}$$

where $\tilde{\Gamma}_i \in \mathbb{R}^{m \times n}$, $\Lambda_i \in \mathbb{R}^{m \times p}$, $\Theta_i \in \mathbb{R}^{m \times q}$ and $\Delta_i \in \mathbb{R}^m (i = 1, 2)$. Then, the error system of $\bar{\epsilon}$ is given by

$$\Sigma_{\bar{\epsilon}} \begin{cases} \dot{\bar{\epsilon}} = (Q_I - \Psi B_I \Lambda_I)\bar{\epsilon} + \psi_{I,J} \\ \bar{\epsilon}(t) = \bar{\epsilon}(t^-) + \phi(t)(w(t) - \bar{w}(t)) \end{cases}, \tag{13}$$

where $\psi_{i,j} \in \mathbb{R}^m \ (i, j = 1, 2)$ and $\phi(t) \in \{-1, 0, 1\}$ denote

$$\begin{aligned} \psi_{i,j} = (\Psi A_i + \Psi B_i \tilde{\Gamma}_i - Q_i \Psi + \Psi B_i \Lambda_i \Psi)x \\ + (\Psi B_i \Theta_i - R_i)v_{i,j} + (\Psi E_i + \Psi B_i \Delta_i - U_i) \\ \phi(t) = |I(t) - J(t)| - |I(t^-) - J(t^-)|. \end{aligned}$$

Note that $\bar{\epsilon}(t)$ can change discontinuously according to the second equation of (13) at $t \in \mathbb{I}$.

V. Trajectory Tracking Control

In this section, we realize the trajectory tracking of the PWA system (1) by stabilizing the error system (13) using Lemma 1.

A. Equilibrium Point of Error System

First, consider condition (C.1) in Lemma 1. The origin of $\Sigma_{\bar{\epsilon}}$ is an equilibrium point, that is (6) is satisfied, if the following conditions hold in (13).

- $\bar{\epsilon} = 0 \Rightarrow \dot{\bar{\epsilon}} = 0$ holds in the first equation, that is $\psi_{i,j} = 0 (i, j = 1, 2)$.
- $\bar{\epsilon}(t^-) = 0 \Rightarrow \bar{\epsilon}(t) = 0$ holds in the second equation, that is $\bar{\epsilon}(t^-) = 0 \Rightarrow \phi(t)(w(t) - \bar{w}(t)) = 0$ holds for $t \in \mathbb{R}_+$.

These statements are related to a well-posedness condition of PWL systems [4] and a trajectory tracking condition of linear systems [7], respectively. From this viewpoint, a sufficient condition for (C.1) is given as follows.

Lemma 3: The error variable $\bar{\epsilon}$ given by (10) satisfies (6) if there exist matrices $\Psi \in \mathbb{R}^{p \times n}$, $X, \Omega_i \in \mathbb{M}_+^d$, $\Gamma_i \in \mathbb{R}^{m \times n}$, $\Theta_i \in \mathbb{R}^{m \times q}$ and $\Delta_i \in \mathbb{R}^m (i = 1, 2)$ such that

$$\mathcal{C}_1 = X\mathcal{C}_2, \ \mathcal{D}_1 = X\mathcal{D}_2 \tag{14}$$

$$\mathcal{C}_i = \Omega_i \mathcal{S}_i \Psi, \ \mathcal{D}_i = \Omega_i \mathcal{T}_i \tag{15}$$

$$\Psi A_i + \Psi B_i \Gamma_i = Q_i \Psi \tag{16}$$

$$\Psi B_i \Theta_i = R_i, \ \Psi E_i + \Psi B_i \Delta_i = U_i \tag{17}$$

where $\mathcal{S}_i = [S^{\mathrm{T}} \ (SQ_i)^{\mathrm{T}} \ \cdots \ (SQ_i^{d-1})^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{d \times p}$, $\mathcal{T}_i = [T \ SU_i \ SQ_iU_i \ \cdots \ SQ_i^{d-2}U_i]^{\mathrm{T}} \in \mathbb{R}^d$, $\mathcal{C}_i = [C^{\mathrm{T}} \ (CA_i)^{\mathrm{T}} \ \cdots \ (CA_i^{d-1})^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{d \times n}$ and $\mathcal{D}_i = [D \ CE_i \ CA_iE_i \ \cdots \ CA_i^{d-2}E_i]^{\mathrm{T}} \in \mathbb{R}^d$. The input u which achieves (C.1) is given by

$$u = (\Gamma_I - \Lambda_I \Psi)x + \Lambda_I w_{I,J} + \Theta_I v_{I,J} + \Delta_I \tag{18}$$

for any matrix $\Lambda_i \in \mathbb{R}^{m \times p} (i = 1, 2)$.

Note that equations (16)–(18) reduce the error system (13) to the following simple form.

$$\Sigma_{\bar{\epsilon}} \begin{cases} \dot{\bar{\epsilon}} = (Q_I - \Psi B_I \Lambda_I)\bar{\epsilon} \\ \bar{\epsilon}(t) = \bar{\epsilon}(t^-) + \phi(t)(w(t) - \bar{w}(t)) \end{cases} \tag{19}$$

B. Construction of Lyapunov-like Function

We consider condition (C.2) in Lemma 1 to guarantee the stability of the error system on the assumption that the conditions in Lemmas 2 and 3 are satisfied. We shall assign a quadratic function as a candidate for Lyapunov-like function and estimate its value for the error system (19) in the two cases: $I(t)$ changes or does not. First, when $I(t)$ does not change, $\bar{\epsilon}(t)$ is governed by the linear system given by the first equation of (19) with either of $I = 1$ or $I = 2$. Thus, a quadratic function $V(\bar{\epsilon}) = \bar{\epsilon}^{\mathrm{T}} P \bar{\epsilon} (P > 0)$ can be guaranteed to decrease by a well-known stability theory of linear systems: if there exist matrices $P > 0$, Λ_i and a positive constant μ satisfying

$$(Q_i - \Psi B_i \Lambda_i)^{\mathrm{T}} P + P(Q_i - \Psi B_i \Lambda_i) + 2\mu P < 0 \quad (20)$$

for $i = 1, 2$, then $V(\bar{\epsilon}(t))$ exponentially decreases with the rate of convergence 2μ. Next, at the instance when I changes, $\bar{\epsilon}(t)$ changes discontinuously according to the second equation of (19). Assume that $\bar{\epsilon}$ is an error variable, that is (12) holds, then $V(\bar{\epsilon})$ satisfies $V(\bar{\epsilon}(t^+)) \leq (1+\kappa)^2 V(\bar{\epsilon}(t^-))$.

From the above discussion, we obtain the following upper bound of the value of $V(\bar{\epsilon}(t))$.

$$\begin{aligned} V(\bar{\epsilon}(t)) &\leq (1+\kappa)^{2|\mathbb{I}\cap[0,t]|} V(\bar{\epsilon}(0)) e^{-2\mu t} \\ &= V(\bar{\epsilon}(0)) e^{2(|\mathbb{I}\cap[0,t]| \log(1+\kappa) - \mu t)} \end{aligned} \quad (21)$$

Note that $|\mathbb{I} \cap [0,t]|$ denotes the number of changes of I before t. Thus, if the relation

$$|\mathbb{I} \cap [0,t]| \log(1+\kappa) - \mu t \to -\infty (t \to \infty) \quad (22)$$

is satisfied, then $V(\bar{\epsilon}(t))$ converges to 0. To check relation (22), we have to estimate the value of $|\mathbb{I} \cap [0,t]|$. Fortunately, I behaves in a similar manner to J on some condition of $\bar{\epsilon}(0)$, and the difference between the numbers of changes of I and J before t is less than 1 for any $t \geq 0$. This is referred in the next paragraph. Then, hypothesis (H.2) gives

$$|\mathbb{I} \cap [0,t]| \leq |\mathbb{J} \cap [0,t]| + 1 \leq \eta t + 1, \quad (23)$$

which guarantees that the left-hand side of (22) is less than or equal to $\{\eta \log(1+\kappa) - \mu\} t + \log(1+\kappa)$. Then, (22) is fulfilled if the following holds.

$$\mu > \eta \log(1+\kappa) \quad (24)$$

Then, from (21), (23) and (24), $V(\bar{\epsilon}(t))$ satisfies (8) for a class $\mathcal{KL}$ function $\beta(s,t) = (1+\kappa)^2 s e^{-2\rho t}$, where ρ is the difference between the left-hand and right-hand sides of (24).

To show the first inequality in (23), let $\mathbb{H}_0$ be $\{0\}$ and $\mathbb{H}_i$ be an open neighborhood of the time τ_i which is the i-th element of $\mathbb{J}$. Then, the number of changes of J before t is given as

$$|\mathbb{J} \cap [0,t]| = \begin{cases} i-1 & \text{if } t \in [\sup \mathbb{H}_{i-1}, \tau_i) \\ i & \text{if } t \in [\tau_i, \sup \mathbb{H}_i) \end{cases}.$$

If the following holds for a positive constant ζ_1, then I changes at $t \in \mathbb{H}_i$ only once for every i, and that it does not change on the rest of these intervals for some initial error as J does.

$$|Sw(t) + T| \geq \zeta_1, \ t \in \mathbb{R} \backslash \bigcup_i \mathbb{H}_i \quad (25)$$

This means that $|\mathbb{I} \cap [0,t]|$ is given by $i-1$ or i also, which implies the first inequality of (23). Note that this is fulfilled for the initial error $\bar{\epsilon}(0)$ belonging to a set $\mathbb{E} = \{\bar{\epsilon} : ||\bar{\epsilon}||_P < \iota\}$ for a positive constant ι, which means that $x(0) \in \mathbb{D}$ for

$$\mathbb{D} = \{x : ||\Psi x - w(0)||_P < \iota\}. \quad (26)$$

The volume of this set is not 0 because Ψ is a row full rank matrix. Note that (26) is derived by substituting (10) in (5) and using the fact that $\bar{\epsilon}(0) = \Psi x(0) - w(0)$ because of $w(0) = \bar{w}(0)$ from (11). From this viewpoint, a sufficient condition for (C.2) is given as follows.

Lemma 4: Assume that all assumptions in Lemma 3 hold. If there exist a positive definite matrix $P \in \mathbb{R}^{p\times p}$ and positive constants κ and μ satisfying the assumption in Lemma 2, (20) and (24), then $V(\bar{\epsilon}) = \bar{\epsilon}^{\mathrm{T}} P \bar{\epsilon}$ satisfies (7) and (8) for the initial state $x(0)$ and the reference trajectory w satisfying (26) and (25), respectively.

C. Main Result

In this subsection, a trajectory tracking method is given using Lemmas 1, 2, 3 and 4. Before giving out the main theorem, we derive a condition that the reference trajectory satisfies the assumption in Lemma 2. Inequality (12) holds on the time interval $\mathbb{H}_i$ under some condition on $w(t)$ and $\bar{w}(t)$ as follows.

Lemma 5: There exists a positive constant κ such that (12) holds at $t \in \mathbb{H}_i$, if there exists a positive constant ω such that $C = \omega S$, $D = \omega T$ and there exist a positive definite matrix P and positive constants λ_1, λ_2 and γ satisfying the following for $i = 1, 2, \ldots$.

$$\inf_{t\in\mathbb{H}_i} |S(Q_{J(t)} w(t) + U_{J(t)})| \geq \lambda_1 \quad (27)$$

$$\inf_{t\in\mathbb{H}_i} |S(Q_{3-J(t)} \bar{w}(t) + U_{3-J(t)})| \geq \lambda_2 \quad (28)$$

$$\sup_{t\in\mathbb{H}_i} ||w(t) - \bar{w}(t)||_P / |t - \tau_i| \leq \gamma \quad (29)$$

Inequality (12) holds at $t \in \mathbb{H}_i$ for at least κ satisfying

$$\kappa \geq ||S||_P \gamma / \min\{\lambda_1, \lambda_2\}. \quad (30)$$

Remark 3: For a sufficiently small interval $\mathbb{H}_i$, (27) and (28) are the conditions for $w(t)$ and $\bar{w}(t)$ to traverse the planes $Sw + T = 0$ and $S\bar{w} + T = 0$, respectively, and (29) is always satisfied from (11).

Now, we summarize the above discussion and guarantee the tracking condition (2). First, $\bar{\epsilon}$ is an error variable from Lemmas 2 and 5. Second, the origin of the error system is an equilibrium point from Lemma 3. Third, Lemma 4 guarantees the existence of a Lyapunov-like function. Finally, trajectory tracking is achieved from Lemma 1. From this viewpoint, the main result of this paper is presented as follow.

Theorem 1: Consider systems Σ_x and Σ_w. Assume that all assumptions in Lemmas 3 and 5 hold, and that there exist a positive definite matrix $P \in \mathbb{R}^{p\times p}$ and positive constants μ and κ satisfying (20), (24) and (30). Then, the trajectory tracking problem 1 is feasible with the feedback input u (18).

Remark 4: Although the unique solution of system Σ_x is not necessary to exist for every initial state, the solution exists uniquely from the initial state which is included by the set $\mathbb{D}$ we have discussed.

VI. Tracking Control of SISO PWL Systems

The feasibility condition of trajectory tracking given in Theorem 1 is sometimes difficult to check, since (25) and (27)–(29) require the design of a set $\mathbb{H}_i$ and the advance information on the reference system, e. g., its structure, $w(t)$ and $\bar{w}(t)$. In this section, we focus on SISO PWL systems whose tracking output is given by the same to the value which determines the mode, that is $E_1 = E_2 = 0$, $C = F$, $D = 0$ and $l = 1$, and shall give a feasible condition of tracking which requires only the information of reference trajectories and the coefficient matrices of target systems. Let Σ_x^{PWL} denote this system, and assume that the reference trajectory is given by a periodic continuous function $r : \mathbb{R}_+ \to \mathbb{R}$, and that it is known in advance. In the following, we present a procedure to construct a reference system Σ_w^{PWL} for r, and achieve trajectory tracking using Theorem 1. Note that the feasibility condition derived does not require any information on Σ_w^{PWL} and $\mathbb{H}_i$.

We start by giving a pair of variables w and J from r as a candidate for solution of some reference system Σ_w^{PWL}. Note that the assumption $C = F$ implies that the sign of $z(t)$ determines the mode $I(t)$ of target system Σ_x^{PWL}. Thus, the mode $J(t)$ of Σ_w^{PWL} seems to be assigned from the sign of $r(t)$. To define J uniquely from this viewpoint, let $\mathbb{J}$ be $\{t : r(t) = 0\}$, and assume that $\mathbb{J} \cap [0, t]$ is a finite set for any $t \geq 0$, which is corresponding to (H.2). Moreover, we assume that r traverses the plane $r = 0$, that is $|\dot{r}(t)| > 0$ for any t such that $r(t) = 0$, which is relevant to Remark 3. Then, the right continuous function J can be uniquely defined by

$$\begin{cases} J(t) = 1 \text{ if } r(t) < 0 \\ J(t) = 2 \text{ if } r(t) > 0 \end{cases}, \tag{31}$$

which is included by $\mathbb{PC}$. Next, a variable w is offered as a solution of some reference system Σ_w^{PWL} together with J. Assume that r is p-th time differentiable on $\mathbb{R}_+ \backslash \mathbb{J}$ for a positive integer p, and let $\hat{w}(t) \in \mathbb{R}^p$ be the following.

$$\hat{w}(t) = \begin{cases} [r(t),\ \dot{r}(t),\ \cdots,\ r^{(p-1)}(t)]^{\mathrm{T}},\ t \in \mathbb{R}_+ \backslash \mathbb{J} \\ \hat{w}(t^+),\ t \in \mathbb{J} \end{cases}$$

Note that $\hat{w}(t)$ can change discontinuously at $t \in \mathbb{J}$. In addition, assume that the discontinuous changes can be described by a mapping which depends on $\hat{w}(t^-)$ and $J(t^\pm)$. Then, using a mapping $\sigma : \mathbb{R}^p \times \{1, 2\}^2 \to \mathbb{R}^p$, r is regarded as output of the system

$$\begin{cases} \dot{\hat{w}}(t) = K\hat{w}(t) + Lr^{(p)}(t),\ t \in \mathbb{R}_+ \backslash \mathbb{J} \\ \hat{w}(t) = \sigma(\hat{w}(t^-), J(t), J(t^-)),\ t \in \mathbb{J} \\ r = N\hat{w} \end{cases}, \tag{32}$$

where $N = [1 \quad 0_{1\times(p-1)}]$, $0_{n,m} \in \mathbb{R}^{n\times m}$ represents the null matrix, and K and L are the following matrices.

$$K = \begin{bmatrix} 0_{(p-1)\times 1} & I_{p-1} \\ 0 & 0_{1\times(p-1)} \end{bmatrix},\ L = \begin{bmatrix} 0_{(p-1)\times 1} \\ 1 \end{bmatrix}$$

To use Theorem 1 for (32), the following lemma gives a piecewise linear transformation which converts the mapping $\sigma(\cdot, i, j)$ in (32) into the identity mapping for any i, j. Then, the transformed variable is continuous and is governed by an SISO PWL system, which is a reference system for r.

Lemma 6: Assume that a given reference trajectory r satisfies the following.

- (H'.1) *r is p-th time differentiable for a positive integer $p \geq 2$ at $t \in \mathbb{R}_+ \backslash \mathbb{J}$.*
- (H'.2) *$\mathbb{J} \cap [0, t]$ is a finite set for any $t \geq 0$ for $\mathbb{J} := \{t : r(t) = 0\}$.*
- (H'.3) *$w(t^+) = M^{J(t)-J(t^-)}\hat{w}(t^-)$ holds for any $t \in \mathbb{J}$ and some matrix $M \in \mathbb{R}^{p\times p}$, where J is given by (31) and $\hat{w}(t) = [r(t),\ \dot{r}(t),\ \cdots,\ r^{(p-1)}(t)]^{\mathrm{T}}$.*
- (H'.4) *$|\dot{r}(t^\pm)| > 0,\ t \in \mathbb{J}$.*

Then, the transformation $w = M^{1-J}\hat{w}$ gives a reference system for r as (3) with the matrices $Q_i = M^{1-i}KM^{i-1}$, $R_i = M^{1-i}L$, $U_i = 0_{p\times 1}$, $S = N$, $T = 0$. Moreover, this system satisfies (H.1) and (H.2), and there exist a positive constant λ and an open neighborhood $\mathbb{H}_i$ of τ_i satisfying (25), (27), (28) and (29) for every $\tau_i \in \mathbb{J}$.

Next, we simplify the tracking condition in Theorem 1 for Σ_x^{PWL} with the following lemma.

Lemma 7: Assume that Σ_x^{PWL} satisfies (14) for some $X \in \mathbb{R}^{p\times p}$ and that r satisfies (H'.1)–(H'.4) with $M = X$. Then, Σ_w^{PWL} satisfies (9) and (15)–(17) for the matrices $\Psi = \mathcal{C}_1$, $\Gamma_i = -CA_i^p/(CA_i^{p-1}B_i)$, $\Theta_i = 1/(CA_i^{p-1}B_i)$, $\Omega_i = M^{i-1}, \Delta_i = 0$.

After rewriting other conditions in Theorem 1, we obtain the following corollary. Note that we can check the following condition only with r and the coefficient matrices of Σ_x^{PWL}.

Corollary 1: Consider Σ_x^{PWL} satisfying (14) for some matrix $X \in \mathbb{M}_+^p$ whose (2,1)-th component is 0 and $(X^{1-i}KX^{i-1} - X^{1-i}L\Lambda_i')^{\mathrm{T}}P + P(X^{1-i}KX^{i-1} - X^{1-i}L\Lambda_i') + 2\mu P < 0$ for a positive number μ, a positive matrix $P \in \mathbb{R}^{p\times p}$ and a matrix $\Lambda_i' \in \mathbb{R}^{m\times p}$ for $i = 1, 2$. Then, the tracking condition (2) is achievable with some $\mathbb{D} \subset \mathbb{R}^{n\times n}$ for a periodic scalar function r satisfying (H'.1)–(H'.4) for $M = X$ and $\mu \geq \eta \log(1 + ||S||_P\gamma'/\lambda)$, where $\eta = \max_{t\in[0,T]}(|\mathbb{J} \cap [0, t]|/t)$, $\gamma' = \max_{t\in\mathbb{J}\cap[0,T]} ||\dot{w}(t^+) - \dot{w}(t^-)||_P$, $\lambda = \min\{\lambda^-, \lambda^+\}$, $\lambda^\pm = \min_{t\in\mathbb{J}\cap[0,T]} r(t^\pm)$ and T is the period of r.

VII. Numerical Example

Consider the cart system depicted in Figure 1. Carts 1 and 2 are connected by spring k_1 and damper d_1, and physical force u can be applied to Cart 2. Cart 1 collides with a wall through spring k_2 and damper d_2. Let x_1 and x_2 be the position and velocity of Cart 1 with pointing to the right, respectively. Similarly, let x_3 and x_4 be those of Cart 2. Let $x = [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$, and $x = 0$ at the point where both carts stop, both springs are natural, and the right end of spring k_2 touches the wall. Then, force from the wall is applied to Cart 1 through spring k_2 if $x_1 \geq 0$, and is not if $x_1 \leq 0$. Let I be 1 for the former situation, and 2 for the latter one. We choose the position of Cart 1 x_1 as the

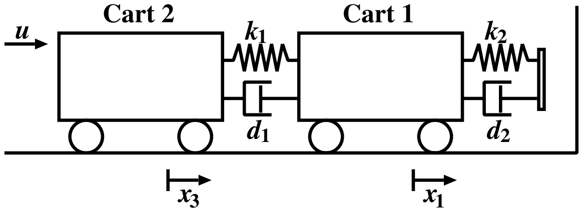

Fig. 1. 2-cart system with elastic collision

tracking output z, and this system is described by (1) with

$$A_1=\begin{bmatrix}0&1&0&0\\-k_1&-d_1&k_1&d_1\\0&0&0&1\\k_1&d_1&-k_1&-d_1\end{bmatrix},\ B_1=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$A_2=\begin{bmatrix}0&1&0&0\\-k_1-k_2&-d_1-d_2&k_1&d_1\\0&0&0&1\\k_1&d_1&-k_1&-d_1\end{bmatrix},\ B_2=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$C=F=[1\ \ 0\ \ 0\ \ 0],\ D=0,\ E_1=E_2=0,$$

where k_i and d_i represent the coefficients of the springs and dampers. Let $k_1=d_1=1$ and $k_2=d_2=0.1$, and the reference trajectory r be the periodic function

$$r(t)=\begin{cases} t^2/4-1/2,\ t\in[0,T/4)\cup[Ti-T/4,Ti+T/4)\\ t^5/320+(2-\sqrt{2})t^4/32+(19-40\sqrt{2})t^3/80\\ \quad+(100-17\sqrt{2})t^2/40+(57-160\sqrt{2})t/80\\ \quad+(30-9\sqrt{2})/40,\ t\in[Ti-3T/4,Ti-T/4)\\ i=1,2,\cdots\end{cases}$$

where $T=4\sqrt{2}$ is the period. The reference $r(t)$ is described in Figure 2 (a). Corollary 1 guarantees that the tracking problem of the 2-cart system for this reference is feasible. Moreover, the reference system given by Lemma 6 satisfies the condition in Theorem 1 for

$$\Psi=\begin{bmatrix}1&0&0&0\\0&1&0&0\\-1&-1&1&1\end{bmatrix},\ P=\begin{bmatrix}18&18&5.6\\18&25&8.4\\5.6&8.4&4.8\end{bmatrix}$$

$$\Lambda_1=[5.3\ 8.7\ 4.3],\ \Lambda_2=[5.6\ 9.2\ 4.5]$$

$$\Gamma_1=[-2\ \ -1\ 2\ 1],\ \Gamma_2=[-2.2\ \ -1.1\ 2.1\ 1.1]$$

$$\Omega_1=\Omega_2=X=I_2,\ \Theta_1=\Theta_2=1$$

$$\mu=0.35,\ \kappa=0.60,\ \eta=0.71,\ \iota=0.46$$

$$\gamma=0.71,\ \zeta_1=\zeta_2=0.21,\ \lambda_1=\lambda_2=0.53,$$

and the trajectory tracking is possible for the initial state $x(0)=x_0$ such that $||x_0-\Psi w(0)||_P<\iota$, where $w(0)=[-0.50\ 0.0\ 0.50]^{\mathrm{T}}$. For the initial state $x(0)=[-0.63\ 0.0\ 0.0\ 0.0]^{\mathrm{T}}$ which satisfies this condition, the behavior of the cart system with the input u given by (18) is simulated. The tracking error $z(t)-r(t)$ is depicted in Figure 2 (b), which shows that the tracking output $z(t)$ converges to the reference trajectory $r(t)$. The Lyapunov-like function $V(\bar{\epsilon})=\bar{\epsilon}^{\mathrm{T}}P\bar{\epsilon}$ is depicted in Figure 2 (c), which shows that $V(\bar{\epsilon}(t))$ converges to 0 although it changes discontinuously at $t=1.5$, 4.2 and 7.1. Note that the vertical axis of Figure 2 (c) is on a log scale. This result illustrates the effectiveness of the proposed method.

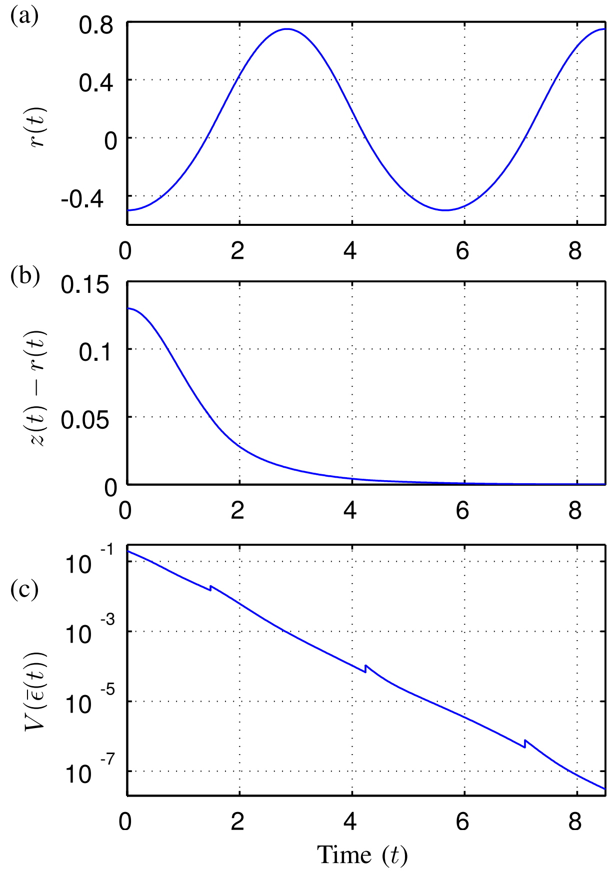

Fig. 2. Simulation result

VIII. Conclusion

We have discussed a trajectory tracking problem for bimodal PWA systems. First, we introduced an error variable and an error system as a generalization of the tracking error and its system. As an error variable, a function switched by the mode of a PWA system was adopted in order to overcome an inherent problem in tracking of PWA systems. Next, we designed a tracking controller which stabilizes the error system using a Lyapunov-like function. Furthermore, the feasibility condition of tracking for SISO PWL systems was simplified. Finally, the numerical example of the 2-cart system illustrated the effectiveness of the proposed method.

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