Optimal Control for Singularly Impulsive Dynamical Systems

Nataša A. Kablar
Lola Institute, Kneza Višeslava 70a, 11000 Belgrade, Yugoslavia

Abstract— **Singularly impulsive (or generalized impulsive) dynamical systems are systems which dynamics are characterized by the set of differential, difference and algebraic equations. They represent the class of hybrid systems, where algebraic equations represent constraints that differential and difference equations need to satisfy. For the class of singularly impulsive dynamical systems we present optimal control results. We developed unified framework for hybrid feedback optimal and inverse optimal control involving a hybrid nonlinear-nonquadratic performance functional. It is shown that the hybrid cost functional can be evaluated in closed-form as long as the cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop singularly impulsive system. Furthermore, the Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton-Jacobi-Bellman equation.**

I. Introduction

Singularly impulsive or generalized impulsive dynamical systems has been recently presented in [1]. Dynamics of this systems is characterized with the set of differential, difference and algebraic equations,wherein algebraic equations represents constraints that differential and difference equations need to satisfy. Applications of this class of systems can be found in contact problems.

For the class of nonlinear singularly impulsive dynamical systems [1, 2] we developed optimality results. In doing so, we generalize optimality results developed in [3, 4]. Using the stability results [2], we consider a hybrid feedback optimal control problem over an infinite horizon involving a hybrid nonlinear-nonquadratic performance functional. The performance functional involves a continuous-time cost for addressing performance of the continuous-time system dynamics and a discrete-time cost for addressing performance at the resetting instants. Furthermore, the hybrid cost functional can be evaluated in closed-form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop singularly impulsive system. This Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton-Jacobi-Bellman equation and thus guaranteeing both optimality and stability of the feedback controlled singularly impulsive dynamical system. The overall framework provides the foundation for extending linear-quadratic feedback control methods to nonlinear singularly impulsive dynamical systems. We note that the optimal control framework for singularly impulsive dynamical systems developed herein is quite different from the quasivariational inequality methods for singularly impulsive and hybrid control developed in the literature (e.g.[5-8]). Specifically, quasivariational methods do not guarantee asymptotic stability via Lyapunov functions and do not necessarily yield feedback controllers. In contrast, the proposed approach provides hybrid feedback controllers guaranteeing closed-loop stability via an underlying Lyapunov function.

An important contribution of the paper is to develop unified framework for the analysis and control synthesis of nonlinear singularly impulsive dynamical systems. However, since singularly impulsive dynamical systems involve a hybrid formulation of continuous-time and discrete-time dynamics, this paper also provide a tutorial for optimality for continuous time and singular discrete time dynamical systems which can be viewed as a specialization of singularly impulsive dynamical systems.

The contents of the paper are as follows. In Section 2 we address an optimal control problem with respect to a hybrid nonlinear-nonquadratic performance functional for singularly impulsive dynamical systems. To avoid complexity in solving the hybrid Hamilton-Jacobi-Bellman equation, in Section 3 we specialize the results of Section 2 to address an inverse optimal control problem for nonlinear affine (in the control) singularly impulsive systems. Finally, we draw conclusions in Section 4.

Finally, in this paper we use the following standard notation. Let R denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let R^n denote the set of $n\times 1$ real column vectors, let $R^{n\times m}$ denote the set of $n\times m$ real matrices, let S^n denote the set of $n\times n$ symmetric matrices, and let N^n (resp., P^n) denote the set of $n\times n$ nonnegative (resp., positive) definite matrices, and let I_n or I denote the $n\times n$ identity matrix. Furthermore, $A\geq 0$ (resp., $A>0$)

denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x. Finally, let C^0 denote the set of continuous functions and C^r denote the set of functions with r continuous derivatives.

II. Optimal Control for Singularly Impulsive Dynamical Systems

In this section we consider an optimal control problem for nonlinear singularly impulsive dynamical systems involving notion of optimality with respect to a hybrid nonlinear-nonquadratic performance functional. Specifically, we consider the following singularly impulsive optimal control problem.

Singularly Impulsive Optimal Control Problem. Consider the nonlinear singularly impulsive controlled system given by

$$E_{\mathrm{c}}\dot{x}(t)=F_{\mathrm{c}}(x(t),u_{\mathrm{c}}(t),t),\ u_{\mathrm{c}}(t)\in\mathcal{U}_{\mathrm{c}},\ (t,x(t))\notin\mathcal{S}_x, \quad (1)$$

$$E_{\mathrm{d}}\Delta x(t)=F_{\mathrm{d}}(x(t),u_{\mathrm{d}}(t),t),\ u_{\mathrm{d}}(t)\in\mathcal{U}_{\mathrm{d}},\ (t,x(t))\in\mathcal{S}_x, \quad (2)$$

where $x(t_0) = x_0$, $x(t_f) = x_f$, $t \geq 0$, $x(t) \in \mathcal{D} \subseteq R^n$ is the state vector , $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}} \subseteq R^{m_{\mathrm{c}}} \times R^{m_{\mathrm{d}}}$, $t \in [t_0, t_f]$, $k \in \mathcal{N}_{[t_0,t_f)}$, is the hybrid control input, $x(t_0) = x_0$ is given, $x(t_f) = x_f$ is fixed, $F_{\mathrm{c}} : \mathcal{D} \times \mathcal{U}_{\mathrm{c}} \times R \to R^n$ is Lipschitz continuous and satisfies $F_{\mathrm{c}}(0, 0, 0) = 0$, $F_{\mathrm{d}} : \mathcal{D} \times \mathcal{U}_{\mathrm{d}} \times R \to R^n$ is continuous and satisfies $F_{\mathrm{d}}(0, 0, 0) = 0$, and $\mathcal{S}_x \subset 0, \infty) \times R^n$ is the resetting set [1]. Matrices E_{c}, E_{d} may be singular matrices. Then determine the control inputs $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}, t \in [t_0, t_f], k \in \mathcal{N}_{[t_0,t_f)}$, such that the hybrid performance functional

$$J(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot), t_0) = \int_{t_0}^{t_f} L_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), t)\mathrm{d}t + \sum_{k\in\mathcal{N}_{[t_0,t_f)}} L_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k), t_k), \quad (3)$$

is minimized, where $L_{\mathrm{c}} : \mathcal{D} \times \mathcal{U}_{\mathrm{c}} \times R \to R$ and $L_{\mathrm{d}} : \mathcal{D} \times \mathcal{U}_{\mathrm{d}} \times R \to R$ are given.

Next, we state a hybrid version of Bellman's principle of optimality [2] which provides necessary and sufficient conditions, for a given hybrid control $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}, t_0, k \in \mathcal{N}_{[t,t_0)}$, for minimizing the performance functional (3). Furthermore, we relax Assumption A1 of [1] . This is due to the fact that if $(0, x_0) \in \mathcal{S}_x$, then there can be a cost associated with the initial system reset to x_0^+.

Let $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}, t \in [t_0, t_f], k \in \mathcal{N}_{[t_0,t_f)}$, be an optimal hybrid control that generates the trajectory $x(t), t \in [t_0, t_f]$, with $x(t_0) = x_0$. Then the trajectory $x(\cdot)$ from (t_0, x_0) to (t_f, x_f) is optimal if and only if for all $t^{'}, t^{''} \in [t_0, t_f]$, the portion of the trajectory $x(\cdot)$ going from $(t', x^{'})$ to $t^{''}, x(t^{''})$ optimizes the same cost functional over $[t', t'']$, where $x(t') = x_1$ is a point on the optimal trajectory generated by $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t))$, t$\in [t_0, t')$.

Next, let $(u_{\mathrm{c}}^*(t), u_{\mathrm{d}}^*(t_k)), t \in [t_0, t_f], k \in \mathcal{N}_{[t_0,t_f)}$, solve the Singularly Impulsive Optimal Control Problem and define the optimal cost $J^*(x_0, t_0) = (x_0, u_{\mathrm{c}}^*(\cdot), u_{\mathrm{d}}^*(\cdot), t_0)$. Furthermore, define, for $p \in R^n$ and $q \in R^n$, the Hamiltonians $H_{\mathrm{c}}(x, u_{\mathrm{c}}, p, t) = L_{\mathrm{c}}(x, u_{\mathrm{c}}, t) + p^{\mathrm{T}} F_{\mathrm{c}}(x, u_{\mathrm{c}}, t))$ and $H_{\mathrm{d}}(x, u_{\mathrm{d}}, q, t_k) = L_{\mathrm{d}}(x, u_{\mathrm{d}}, t_k) + q(E_{\mathrm{d}}x + F_{\mathrm{d}}(x, u_{\mathrm{d}}, t_k)) - q(x)$.

Theorem 2.1: Let $J^*(x, t)$ denote the minimal cost for the Singularly Impulsive Optimal Control Problem with $x_0 = x$ and $t_0 = t$ and assume that $J^*(\cdot, \cdot)$ is C^1 in x. Then

$$0 = \frac{\partial J^*(x(t),t)}{\partial t} + \min_{u_{\mathrm{c}}(\cdot)\in\mathcal{U}_{\mathrm{c}}} H_{\mathrm{c}}(x(t), u_{\mathrm{c}}(t), p(t), t), \quad (t, x(t)) \notin \mathcal{S}_x, \quad (4)$$

$$0 = \min_{u_{\mathrm{d}}(\cdot)\in\mathcal{U}_{\mathrm{d}}} H_{\mathrm{d}}(x(t), u_{\mathrm{d}}(t), q(t), t), \quad (t, x(t)) \in \mathcal{S}_x, \quad (5)$$

where $p(t) = (\frac{\partial J^*(x(t),t)}{\partial t})^{\mathrm{T}}$ and $q(t) = J^*(x(t), t)$. Furthermore, if $(u_{\mathrm{c}}^*(\cdot), u_{\mathrm{d}}^*(\cdot))$ solves the Singularly Impulsive Optimal Control Problem, then

$$0 = \frac{\partial J^*(x(t),t)}{\partial t} + H_{\mathrm{c}}(x(t), u_{\mathrm{c}}^*(t), p(t), t), \quad (t, x(t)) \notin \mathcal{S}_x, \quad (6)$$

$$0 = H_{\mathrm{d}}(x(t), u_{\mathrm{d}}^*(t), q(t), t), \quad (t, x(t)) \in \mathcal{S}_x, \quad (7)$$

Proof: Identical to the proof of the corresponding theorem of [4].

Next, we provide a converse result to Theorem 2.1.

Theorem 2.2: Suppose there exists a C^1 function $V : \mathcal{D} \times R \to R$ and an optimal control $(u_{\mathrm{c}}^*(\cdot), u_{\mathrm{d}}^*(\cdot))$ such that $V(x(t_f), t_f) = 0$,

$$0 = \tfrac{\partial V(x,t)}{\partial t} + H_{\mathrm{c}}(x, u_{\mathrm{c}}^*(t), \tfrac{\partial V^{\mathrm{T}}(x,t)}{\partial x}, t), \quad (t, x) \notin \mathcal{S}_x, \quad (8)$$

$$0 = H_{\mathrm{d}}(x, u_{\mathrm{d}}^*(t), V(x), t), \quad (t, x) \in \mathcal{S}_x, \quad (9)$$

$$H_{\mathrm{c}}(x, u_{\mathrm{c}}^*(t), \tfrac{\partial V^{\mathrm{T}}(x,t)}{\partial x}, t) \leq H_{\mathrm{c}}(x, u_{\mathrm{c}}(t), \tfrac{\partial V^{\mathrm{T}}(x,t)}{\partial x}, t), \quad (t, x) \notin \mathcal{S}_x,\ u_{\mathrm{c}}(\cdot) \in \mathcal{U}_{\mathrm{c}}, \quad (10)$$

$$H_{\mathrm{d}}(x, u_{\mathrm{d}}^*(t), V(x), t) \leq H_{\mathrm{d}}(x, u_{\mathrm{d}}(t), V(x), t), \quad (t, x) \in \mathcal{S}_x,\ u_{\mathrm{d}}(\cdot) \in \mathcal{U}_{\mathrm{d}}. \quad (11)$$

Then $(u_{\mathrm{c}}^*(\cdot), u_{\mathrm{d}}^*(\cdot))$ solves the Singularly Impulsive Control Problem, that is,

$$J^*(x_0, t_0) = J(x_0, u_{\mathrm{c}}^*(\cdot), u_{\mathrm{d}}^*(\cdot), t_0) \leq J(x_0, u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot), t_0), \quad (u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}, \quad (12)$$

and

$$J^*(x_0,t_0) = V(x_0,t_0). \tag{13}$$

Proof: Identical to the proof of the corresponding theorem of [4].

Next, we use Theorem 2.2 to characterize optimal hybrid *feedback* controllers for nonlinear singularly impulsive dynamical systems. In order to obtain time-invariant controllers, we restrict our attention to state-dependant singularly-impulsive dynamical systems with optimality notions over the infinite horizon with an infinite number of resetting times. To address the optimal nonlinear feedback control problem let $\phi_\mathrm{c} : \mathcal{D} \to \mathcal{U}_\mathrm{c}$ be such that $\phi_\mathrm{c}(0) = 0$ and let $\phi_\mathrm{d} : \mathcal{D} \to \mathcal{U}_\mathrm{d}$ be such that $\phi_\mathrm{d}(0) = 0$. If $(u_\mathrm{c}(t), u_\mathrm{d}(t_k)) = (\phi_\mathrm{c}(x(t)), \phi_\mathrm{d}(x(t_k)))$, where $x(t)$, $t \geq 0$, satisfies (1), (2), then $(u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot))$ is a *hybrid feedback control*. Given the hybrid feedback control $(u_\mathrm{c}(t), u_\mathrm{d}(t_k)) = (\phi_\mathrm{c}(x(t)), \phi_\mathrm{d}(x(t_k)))$, the closed-loop state-dependent singularly impulsive dynamical system has the form

$$E_\mathrm{c}\dot{x}(t) \;=\; F_\mathrm{c}(x(t), \phi_\mathrm{c}(x(t))), \quad x(t_0) = x_0,\, x(t) \notin \mathcal{Z}_x, \tag{14}$$

$$E_\mathrm{d}\Delta x(t) \;=\; F_\mathrm{d}(x(t), \phi_\mathrm{d}(x(t))), \quad x(t) \in \mathcal{Z}_x. \tag{15}$$

Now, we present the main theorem for characterizing hybrid feedback controllers that guarantee closed-loop stability and minimize a hybrid nonlinear-nonquadratic performance functional over the infinite horizon. For the statement of this result, recall that with $\mathcal{S}_x[0,\infty)\times\mathcal{Z}_x$ it follows from Assumptions A2 and A3 of [1] that the resetting times $t_k(= \tau_k(x_0))$ are well defined and distinct for every trajectory of (14), (15). Furthermore, define the set of asymptotically stabilizing hybrid controllers by

$$\mathcal{C}(x_0) = \{\, (u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)) : (u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)) \text{ is admissible and zero solution, } x(t) \equiv 0, \text{ to (14) is as. stable}\} \tag{16}$$

Theorem 2.3: Consider the nonlinear controlled singularly impulsive system (14), (15) with hybrid performance functional

$$J(x_0, u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)) \;=\; \int_0^\infty L_\mathrm{c}(x(t), u_\mathrm{c}(t))\mathrm{d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} L_\mathrm{d}(x(t_k), u_\mathrm{d}(t_k)). \tag{17}$$

where $(u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot))$ is an admissible control. Assume there exists a C^1 function $V : \mathcal{D} \to R$ and a hybrid control law $\phi_\mathrm{c} : \mathcal{D} \to \mathcal{U}_\mathrm{c}$ and $\phi_\mathrm{d} : \mathcal{D} \to \mathcal{U}_\mathrm{d}$ such that $V(0) = 0, V(x) > 0$, $x \neq 0$, $\phi_\mathrm{c}(0) = 0$, $\phi_\mathrm{d}(0) = 0$, and

$$V'(x)F_\mathrm{c}(x, F_\mathrm{c}(x, \phi_\mathrm{c}(x)) \leq 0, \quad x \notin \mathcal{Z}_x,\, x \neq 0, \tag{18}$$

$$V(E_\mathrm{d}x + F_\mathrm{d}(x, \phi_\mathrm{d}(x))) - V(x) \leq 0, \quad x \in \mathcal{Z}_x, \tag{19}$$

$$H_\mathrm{c}(x, \phi_\mathrm{c}(x)) = 0, \quad x \notin \mathcal{Z}_x, \tag{20}$$

$$H_\mathrm{d}(x, \phi_\mathrm{d}(x)) = 0, \quad x \in \mathcal{Z}_x, \tag{21}$$

$$H_\mathrm{c}(x, u_\mathrm{c} \geq 0, \quad x \notin \mathcal{Z},\, u_\mathrm{c} \in \mathcal{U}_\mathrm{c} \tag{22}$$

$$H_\mathrm{d}(x, u_\mathrm{d}) \geq 0, \quad x \in \mathcal{Z},\, u_\mathrm{d} \in \mathcal{U}_\mathrm{d} \tag{23}$$

where

$$H_\mathrm{c}(x, u_\mathrm{c}) = L_\mathrm{c}(x, u_\mathrm{c}) + V'(x)F_\mathrm{c}(x, u_\mathrm{c}), \tag{24}$$

$$H_\mathrm{d}(x, u_\mathrm{d}) = L_\mathrm{d}(x, u_\mathrm{d}) + V(E_\mathrm{d}x + F_\mathrm{d}(x, u_\mathrm{d})) - V(E_\mathrm{d}x). \tag{25}$$

Then, with the hybrid feedback control $(u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot))$, there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that if $x_0 \in \mathcal{D}_0$, the zero solution $x(t) \equiv 0$ of the closed-loop system (14), (15) is locally asymptotically stable. Furthermore,

$$\mathcal{J}(x_0, \phi_\mathrm{c}(x(\cdot)), \phi_\mathrm{d}(x(\cdot))) = V(x_0), \qquad x_0 \in \mathcal{D}_0. \tag{26}$$

In addition, if $x_0 \in \mathcal{D}_0$ then the hybrid feedback control $(u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)) = (\phi_\mathrm{c}(x(\cdot)), \phi_\mathrm{d}(x(\cdot)))$ minimizes $J(x_0, u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)$ in the sense that

$$J(x_0, \phi_\mathrm{c}(x(\cdot)), \phi_\mathrm{d}(x(\cdot))) = \min_{(u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot))} J(x_0, u_\mathrm{c}(\cdot), u_\mathrm{d}(\cdot)).$$

Finally, if $\mathcal{D} = R^n, \mathcal{U}_\mathrm{c} = R^{m_\mathrm{c}}, \mathcal{U}_\mathrm{d} = R^{m_\mathrm{d}}$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \equiv 0$ of the closed-loop system (14), (15) is globally asymptotically stable.

Proof: Local and global asymptotic stability is a direct consequence of (18) and (19) by applying Theorem 3.2 of [2] to the closed-loop system (14), (15). Conditions (26) and (**??**) are a direct consequence of Theorem 2.2, with $V(x,t) = V(x)$, $t_0 = 0$, $t_f \to \infty$, and using the fact that $\lim_{t\to\infty} V(x(t)) = 0$ and $\lim_{k\to\infty} V(x(t_k)) = 0$.

Remark 2.1: Theorem 2.3 guarantees optimality with respect to the set of admissible stabilizing hybrid controllers $\mathcal{C}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{C}(x_0)$ is not required. In addition, the optimal stabilizing hybrid *feedback* control law $(u_\mathrm{c}, u_\mathrm{d}) = (\phi_\mathrm{c}(x), \phi_\mathrm{d}(x))$ is independent of the initial condition x_0.

Next, we specialize Theorem 2.3 to linear singularly impulsive systems. For the following result let $A_\mathrm{c} \in R^{n\times n}, B_\mathrm{c} \in R^{n\times m_\mathrm{c}}, A_\mathrm{d} \in R^{n\times n}, B_\mathrm{d} \in R^{n\times m_\mathrm{d}}, R_{1\mathrm{c}} \in R^{n\times n}, R_{2\mathrm{c}} \in R^{m_\mathrm{c}\times m_\mathrm{c}}, R_{1\mathrm{d}} \in R^{n\times n}$ and $R_{2\mathrm{d}} \in R^{m_\mathrm{d}\times m_\mathrm{d}}$ be given, where $R_{1\mathrm{c}}, R_{2\mathrm{c}}, R_{1\mathrm{d}}$, and $R_{2\mathrm{d}}$ are positive definite.

Corollary 2.1: Consider the linear controlled singularly impulsive system

$$E_c\dot{x}(t)=A_c x(t)+B_c u_c(t), \quad x(0)=x_0, x(t)\notin\mathcal{Z}_x, \tag{27}$$

$$E_d\Delta x(t)=(A_d-E_d)x(t)+B_d u_d(t), \quad x(t)\in\mathcal{Z}_x, \tag{28}$$

with quadratic hybrid performance functional

$$J(x_0,u_c(\cdot),u_d(\cdot))=\int_0^\infty[x^T(t)R_{1c}x(t)+u_c^T(t)R_{2c}u_c(t)]dt + \sum_{k\in\mathcal{N}_{[0,\infty)}}[x^T(t_k)R_{1d}x(t_k)+u_d^T(t_k)R_{2d}u_d(t_k), \tag{29}$$

where $(u_c(\cdot),u_d(\cdot))$ is an admissible hybrid control. Furthermore, assume there exists a positive-definite matrix $P\in R^{n\times n}$ such that

$$0 = x^T(A_c^T PE_c+E_c^T PA_c+R_{1c}-PB_cR_{2c}^{-1}B_c^T P)x, \quad x\notin\mathcal{Z}_x, \tag{30}$$

$$0 = x^T(A_d^T PA_d+E_d^T PE_d+R_{1d}-A_d^T PB_d(R_{2d}+B_d^T PB_d)^{-1}B_d^T PA_d)x, \quad x\in\mathcal{Z}_x. \tag{31}$$

Then, the zero solution $x(t)\equiv 0$ to (27), (28) is globally asymptotically stable with the hybrid feedback controller

$$u_c=\phi_c(x)=-R_{2c}^{-1}B_c^T Px, \quad x\notin\mathcal{Z}_x, \tag{32}$$

$$u_d=\phi_d(x)=-(R_{2d}+B_d^T PB_d)^{-1}B_d^T PA_d x, \ x\in\mathcal{Z}_x, \tag{33}$$

and

$$J(x_0,\phi_c(\cdot),\phi_d(\cdot))=x_0^T E_c^T PE_c x_0, \quad x_0\in R^n. \tag{34}$$

Furthermore,

$$J(x_0,\phi_c(\cdot),\phi_d(\cdot))=\min_{(u_c(\cdot),u_d(\cdot))\in\mathcal{C}(x_0)} J(x_0,u_c(\cdot),u_d(\cdot)), \tag{35}$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing controllers for (27), (28) and $x_0\in R^n$.

Proof: The result is a direct consequence of Theorem 2.3 with $F_c(x,u_c)=A_cx+B_cu_c$, $L_c(x,u_c)=x^TR_{1c}x+u_c^TR_{2c}u_c$, $for\, x\notin\mathcal{Z}_x$, $F_d(x,u_d)=(A_d-E_d)x+B_du_d$, $L_d(x,u_d)=x^TR_{1d}x+u_d^TR_{2d}u_d$, $for\, x\in\mathcal{Z}_x$, $V(x)=x^TPx$, with argument $E_cx, E_dx, \mathcal{D}=R^n$, and $\mathcal{U}_c\times\mathcal{U}_d=R^{m_c}\times R^{m_d}$. Specifically, it follows from (30) that $H_c(x,\phi_c(x))=0, x\notin\mathcal{Z}_x$, and hence $V'(x)F_c(x,\phi_c(x))<0$ for all $x\neq 0$ and $x\notin\mathcal{Z}_x$. Similarly, it follows from (31) that $H_d(x,\phi_d(x))=0$, $x\in\mathcal{Z}_x$, and hence $V(E_dx+F_d(x,\phi_d(x))-V(x)<0$ for all $x\neq 0$ and $x\in\mathcal{Z}_x$. Thus, $H_c(x,u_c)=H_c(x,u_c)-H_c(x,\phi_c(x))=[u_c-\phi_c(x)]^TR_{2c}[u_c-\phi_c(x)]\geq 0$, $x\notin\mathcal{Z}_x$, and $H_d(x,u_d)=H_d(x,u_d)-H_d(x,\phi_d(x))=[u_d-\phi_d(x)]^T(R_{2d}+B_d^TPB_d)[u_d-\phi_d(x)]\geq 0$, $x\in\mathcal{Z}_x$, so that all conditions of Theorem 2.3 are satisfied. Finally, since $V(\cdot)$ is radially unbounded, the zero solution $x(t)\equiv 0$ to (27), (28) with $u_c(t)=\phi_c(x(t))=-R_{2c}^{-1}B_c^TPx(t), x(t)\in\mathcal{Z}_x$, and $u_d(t)=\phi_d(x(t))=-(R_{2d}+B_d^TPB_d)^{-1}B_d^TPA_dx(t), x(t)\in$, is globally asymptotically stable.

Remark 2.2: The optimal hybrid feedback control $(\phi_c(x),\phi_d(x))$ in Corollary 2.1 is derived using the properties of $H_c(x,u_c)$ and $H_d(x,u_d)$ as defined in Theorem 2.3. Specifically, since $H_c(x,u_c)=x^TR_{1c}x+u_c^TR_{2c}u_c+x^T(A_c^TPE_c+E_c^TPA_c)x+2x^TPB_cu_c$, $x\notin\mathcal{Z}_x$, and $H_d(x,u_d)=x^TR_{1d}x+u_d^TR_{2d}u_d+(A_dx+B_du_d)P(A_dx+B_du_d)-x^TE_d^TPE_dx$, $x\in\mathcal{Z}_x$, it follows that $\frac{\partial^2 H_c}{\partial u_c^2}=R_{2c}>0$ and $\frac{\partial^2 H_d}{\partial u_d^2}=R_{2d}+B_d^TPB_d>0$. Now, $\frac{\partial H_c}{\partial u_c}=2R_{2c}+2B_c^TPx=0, x\notin\mathcal{Z}_x$, and $\frac{\partial H_d}{\partial u_d}=2(R_{2d}+B_d^TPB_d)u_d+2B_d^TPA_dx=0, x\in\mathcal{Z}_x$, give the unique global minimum of $H_c(x,u_c)$, $x\notin\mathcal{Z}_x$, and $H_d(x,u_d), x\in\mathcal{Z}_x$, respectively. Hence, since $\phi_c(x)$ minimizes $H_c(x,u_c)$ on $x\notin\mathcal{Z}_x$ and $\phi_d(x)$ minimizes $H_d(x,u_d)$ on $x\in\mathcal{Z}_x$, it follows that $\phi_c(x)$ satisfies $\frac{\partial H_c}{\partial u_c}=0$ and $\phi_d(x)$ satisfies $\frac{\partial H_d}{\partial u_d}=0$, or, equivalently, $\phi_c(x)=-R_{2c}^{-1}B_c^TPx$, $x\notin\mathcal{Z}_x$, and $\phi_d(x)=-(R_{2d}+B_dPB_d)^{-1}B_d^TPA_dx$, $x\in\mathcal{Z}_x$

Remark 2.3: For given R_{1c}, R_{2c}, R_{1d}, and R_{2d}, (30) and (31) can be solved using constrained nonlinear programming methods using the structure of $\mathcal{Z}_x$. For example, in the case where $\mathcal{Z}_x$ is characterized by the hyperplane $\mathcal{Z}_x=\{x\in R^n : Hx=0\}$, where $H\in R^{m\times n}$, it follows that (31) holds when $x\in\mathcal{N}(H)$ and (30) holds when $x\in[\mathcal{N}(H)]^\perp=\mathcal{R}(H)^T$, where $\mathcal{N}$ denotes the null space of H and $\mathcal{R}(H^T)$ denotes the range space of H^T. Now, reformulating $\mathcal{Z}_x$ as $\{x\in R^n : Ex=0\}$, where E is an elementary matrix composed of zeroes and ones such that the columns of E span the nullspace of H, and using the fact that $P>0$, (30) and (31) will hold for $P>0$ with a specific internal matrix structure. This of course reduces the number of free elements in P satisfying (30) and (31). Alternatively, to avoid complexity in solving (30) and (31), an inverse optimal control problem can be solved wherein R_{1c}, R_{2c}, R_{1d}, and R_{2d} are arbitrary. In this case, (30) and (31) are implied by

$$0 = A_c^TPE_c+E_c^TPA_c+R_{1c}-PB_cR_{2c}B_c^TP, \tag{36}$$

$$0 = A_d^TPA_d-E_d^TPE_d+R_{1d}-A_d^TPB_d(R_{2d}+B_d^TPB_d)^{-1}B_d^TPA_d. \tag{37}$$

Since R_{1c}, R_{2c}, R_{1d}, and R_{2d} are arbitrary, (36) and (37) can be cast as an LMI [5] feasibility problem involving

$$P>0, [\begin{array}{cc} A_c^TPE_c+E_c^TPA_c & PB_c \\ B_c^TP & -R_{2c} \end{array}]<0,$$

$$\begin{bmatrix} A_{\rm d}^{\rm T}PA_{\rm d}-E_{\rm d}^{\rm T}PE_{\rm d} & A_{\rm d}^{\rm T}PB_{\rm d} \\ B_{\rm d}^{\rm T}PA_{\rm d} & -(R_{2\rm d}+B_{\rm d}^{\rm T}PB_{\rm d}) \end{bmatrix} < 0. \quad (38)$$

III. Inverse Optimal Control for Nonlinear Affine Singularly Impulsive Systems

In this section we specialize Theorem 2.3 to affine systems. The controllers obtained are predicated on an *inverse optimal hybrid control problem*. In particular, to avoid the complexity in solving steady-state hybrid Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parametrize a family of stabilizing hybrid controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear singularly impulsive system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints.

Consider the state-dependent affine (in the control) singularly impulsive dynamical system

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))u_{\rm c}(t), \quad x(t) \notin \mathcal{Z}_x, \quad (39)$$
$$E_{\rm d}\Delta x(t) = f_{\rm d}(x(t) + G_{\rm d}(x(t))u_{\rm d}(t), \quad x(t) \in \mathcal{Z}_x, \quad (40)$$

where $x(0) = x_0$. Furthermore, we consider performance integrands $L_{\rm c}(x, u_{\rm c})$ and $L_{\rm d}(x, u_{\rm d})$ of the form

$$L_{\rm c}(x, u_{\rm c}) = L_{1\rm c} + u_{\rm c}^{\rm T}R_{2\rm c}(x)u_{\rm c},$$
$$L_{\rm d}(x, u_{\rm d}) = L_{1\rm d} + u_{\rm d}^{\rm T}R_{2\rm d}(x)u_{\rm d}, \quad (41)$$

where $L_{1\rm c} : R^n \to R$ and satisfies $L_{1\rm c}(x) \geq 0$, $x \in R^n$, $R_{2\rm c} : R^n \to P^{m_{\rm c}}$, $L_{1\rm d} : R^n \to R$ and satisfies $L_{1\rm d}(x) \geq 0$, $x \in R^n$, and $R_{2\rm d} : R^n \to P^{m_{\rm d}}$ so that (3) becomes

$$J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) = \int_0^\infty [L_{1\rm c}(x(t)) + u_{\rm c}^{\rm T}(t)R_{2\rm c}(x(t))u_{\rm c}(t)]{\rm d}t$$
$$+ \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{1\rm d}(x(t_k)) + u_{\rm d}(t_k)R_{2\rm d}(x(t_k))u_{\rm d}(t_k)]., \quad (42)$$

Corollary 3.1: Consider the nonlinear singularly impulsive controlled system (39), (40) with performance functional (42). Assume there exists a C^1 function $V : R^n \to R$, and functions $P_{12} : R^n \to R^{1\times m_{\rm d}}$ and $P_2 : R^n \to N^{m_{\rm d}}$ such that $V(0) = 0, V(x) > 0, x \in R^n, x \neq 0,$

$$P_{12}(0) = 0, \quad (43)$$
$$V'(x)[f_{\rm c}(x) - \tfrac{1}{2}G_{\rm c}(x)R_2^{-1}G_{\rm c}^{\rm T}(x)V'^{\rm T}(x)] < 0,$$
$$x \notin \mathcal{Z}_x, \quad x \neq 0, \quad (44)$$
$$V(E_{\rm d}x + f_{\rm d}(x) - \tfrac{1}{2}G_{\rm d}(x)(R_{2\rm d}(x)$$
$$+P_2(x))^{-1}P_{12}^{\rm T}(x)) - V(E_{\rm d}x) \leq 0, \quad x \in \mathcal{Z}_x, \quad (45)$$
$$V(E_{\rm d}x + f_{\rm d}(x) + G_{\rm d}(x)u_{\rm d}) = V(E_{\rm d}x + f_{\rm d}(x))$$
$$+P_{12}(x)u_{\rm d} + u_{\rm d}^{\rm T}P_2(x)u_{\rm d}, \quad (46)$$

where $u_{\rm d}$ is admissible, and

$$V(x) \to \infty \ \ as \ \ \|x\| \to \infty. \quad (47)$$

Then the zero solution $x(t) \equiv 0$ to the closed-loop system

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))\phi_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z}_x, \quad (48)$$
$$E_{\rm d}\Delta x(t) = f_{\rm d}(x(t) + G_{\rm d}(x(t))\phi_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \quad (49)$$

where $x(0) = x_0$,is globally asymptotically stable with the hybrid feedback control law

$$\phi_{\rm c}(x) = \frac{1}{2}R_{2\rm c}^{-1}(x)G_{\rm c}^{\rm T}(x)V'^{\rm T}(x), \quad x \notin \mathcal{Z}_x, \quad (50)$$
$$\phi_{\rm d}(x) = \frac{1}{2}(R_{2\rm d}(x) + P_2(x))^{-1}P_{12}^{\rm T}(x), \quad x \in \mathcal{Z}_x, \quad (51)$$

and performance functional (42), with

$$L_{1\rm c}(x) = \phi_{\rm c}^{\rm T}(x)R_{2\rm c}(x)\phi_{\rm c}(x) - V'(x)f_{\rm c}(x), \quad (52)$$
$$L_{1\rm d}(x) = \phi_{\rm d}^{\rm T}(x)(R_{2\rm d}(x) + P_2(x)) - V(E_{\rm d}x + f_{\rm d}(x))$$
$$+V(E_{\rm d}x), \quad (53)$$

is minimized in the sense that

$$J(x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))) = \min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)} J(x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)),$$
$$x_0 \in R^n.$$

Finally,

$$J(x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))) = V(x_0), \quad x_0 \in R^n. \quad (54)$$

Proof: The result is a direct consequence of Theorem 2.3 with $\mathcal{D} = R^n$, $u_{\rm c} \in R^{m_{\rm c}}$, $u_{\rm d} \in R^{m_{\rm d}}$, $F_{\rm c}(x, u_{\rm c}) = f_{\rm c}(x) + G_{\rm c}(x)u_{\rm c}$, $\phi_{\rm d}(x, u_{\rm d}) = f_{\rm d}(x) + G_{\rm d}(x)u_{\rm d}$, $L_{\rm c}(x, u_{\rm c}) = L_{1\rm c} + u_{\rm c}^{\rm T}R_{2\rm c}(x)u_{\rm c}$ and $L_{\rm d}(x, u_{\rm d}) = L_{1\rm d} + u_{\rm d}^{\rm T}R_{2\rm d}(x)u_{\rm d}$. Specifically, with (41) the Hamiltonians have the form

$$H_{\rm c}(x, u_{\rm c}) = L_{1\rm c}(x) + u_{\rm c}^{\rm T}R_{2\rm c}(x)u_{\rm c} + V'(x)(f_{\rm c}(x)$$
$$+G_{\rm c}(x)u_{\rm c}), \quad x \notin \mathcal{Z}_x, \ u_{\rm c} \in \mathcal{U}_{\rm c}, \quad (55)$$
$$H_{\rm d}(x, u_{\rm d}) = L_{1\rm d}(x) + V(E_{\rm d}x + f_{\rm d}(x)) + P_{12}(x)u_{\rm d}$$
$$+u_{\rm d}^{\rm T}(R_{2\rm d} + P_2(x))u_{\rm d} - V(E_{\rm d}x)$$
$$x \in \mathcal{Z}_x, \quad u_{\rm d} \in \mathcal{U}_{\rm d}. \quad (56)$$

Now, the hybrid feedback control law (50), (51) is obtained by setting $\frac{\partial H_{\rm c}}{\partial u_{\rm c}} = 0$ and $\frac{\partial H_{\rm d}}{\partial u_{\rm d}} = 0$. With (50) and (51) it follows that (44) and (45) imply (18) and (19), respectively. Next, since $V(\cdot)$ is C^1 and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, since by assumption $P_{12}(0) = 0$, it follows that $\phi_{\rm c}(0) = 0$ and $\phi_{\rm d}(0) = 0$. Next, with $L_{1\rm c}(x)$ and $L_{1\rm d}(x)$ given by (52) and (53), respectively, and $\phi_{\rm c}(x)$, $\phi_{\rm d}(x)$ given by (50) and (51), (20) and (21) hold. Finally, since

$$\begin{aligned} H_{\rm c}(x,u_{\rm c}) &= H_{\rm c}(x,u_{\rm c}) - H_{\rm c}(x,\phi_{\rm c}(x)) \\ &= [u_{\rm c} - \phi_{\rm c}(x)]^{\rm T} R_{2\rm c}(x)[u_{\rm c} - \phi_{\rm c}(x)],\ x \notin \mathcal{Z}_x \end{aligned} \quad (57)$$

$$\begin{aligned} H_{\rm d}(x,u_{\rm d}) &= H_{\rm d}(x,u_{\rm d}) - H_{\rm d}(x,\phi_{\rm d}(x)) \\ &= [u_{\rm d} - \phi_{\rm d}(x)]^{\rm T} (R_{2\rm d}(x) + P_2(x))[u_{\rm d} - \phi_{\rm d}(x)], \\ &\qquad x \in \mathcal{Z}_x, \end{aligned} \quad (58)$$

where $R_{2\rm c}(x) > 0$, $x \notin \mathcal{Z}_x$, and $R_{2\rm d}(x) + P_2(x) > 0$, $x \in \mathcal{Z}_x$, conditions (22) and (23) hold. The result now follows as a direct consequence of Theorem 2.3.

Remark 3.1: Note that (44) and (45) are equivalent to

$$\begin{aligned} \dot{V}(x) &= V'(x)[f_{\rm c}(x) + G_{\rm c}(x)\phi_{\rm c}(x)] < 0, \\ &\qquad x \notin \mathcal{Z}_x, \quad x \neq 0, \end{aligned} \quad (59)$$

$$\begin{aligned} \Delta V(E_{\rm d}x) &= V(E_{\rm d}x + f_{\rm d}(x) + G_{\rm d}(x)\phi_{\rm d}(x)) \\ &\quad -V(E_{\rm d}x) \leq 0, \quad x \in \mathcal{Z}_x, \end{aligned} \quad (60)$$

with $\phi_{\rm c}(x)$ and $\phi_{\rm d}(x)$ given by (50) and (51), respectively. Furthermore, conditions (59) and (60) with $V(0) = 0$ and $V(x) > 0$, $x \in R^n$, $x \neq 0$, assure that $V(x)$ is a Lyapunov function for the singularly impulsive closed-loop system (48) and (49).

IV. Conclusion

In this paper we have developed a unified framework for hybrid feedback optimal control over an infinite horizon involving a hybrid nonlinear-nonquadratic performance functional. The overall framework provides the foundation for generalizing optimal linear-quadratic control methods to nonlinear singularly impulsive dynamical systems [1, 2]. Presented results will be base for further work on optimal nonlinear robust *control* that is under the development.

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