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On the Use of SoS Methods for Analysis of Connection-Level Stability in the Internet

Ashvin Lakshmikantha, C. L. Beck and R. Srikant

Abstract— **We study connection-level models of file transfer requests in the Internet, where connection arrivals to each route occur according to Poisson processes and the file-sizes have phase-type distributions. We investigate the use of the Sum-of-Squares technique to construct Lyapunov functions that satisfy Foster's condition for stochastic stability.**

I. INTRODUCTION

Over the last few years, it has been observed that TCP-like protocols can be viewed as distributed mechanisms for resource allocation in the Internet. Much attention has been devoted to designing algorithms that are stable when the number of competing sources in the network is fixed; see [6] for a comprehensive survey. In this work, we study the stochastic stability of the network at the connection level, i.e., the dynamics of connection arrivals and departures. This problem was first considered in [8] and generalized in [2]; these papers assume that the connection arrival processes are Poisson and the file sizes are exponentially distributed. We relax the exponential file size distribution assumption and consider a class of file size distributions which are dense in the set of all possible distributions. Note that TCP-compliant streaming flows are not considered in our model (see [3]).

In order to numerically establish connection-level stability of a variety of network topologies, we apply the Sum-of-Squares (SoS) technique to establish the *positive recurrence* of a certain Markov chain; this Markov chain describes the evolution of the number of connections in a congestion-controlled network. In the systems we consider, the state variables are always non-negative. Therefore, to prove stability, it is sufficient to find Lyapunov functions which are copositive and whose time derivatives are conegative[1]. A sufficient condition for $f(\mathbf{x})$ to be copositive is that the function $g(z_1, z_2, \ldots, z_n) = f(z_1^2, z_2^2, \ldots, z_n^2)$ is positive definite. One can then apply the SoS technique directly to the polynomial $g(\mathbf{z})$.

We now describe our connection-level model for the Internet. The SoS technique is then used to establish connection-level stability for linear and star network topologies.

II. MATHEMATICAL NETWORK MODEL

Consider a network consisting of a set of links $\mathcal{L}$ with each link $l \in \mathcal{L}$ having a capacity $c_l > 0$. A number of flows compete for access to these links and each type of flow is associated with a route consisting of a subset of $\mathcal{L}$. The set of all routes (all possible types of flows) is denoted by $\mathcal{R}$. Let n_r denote the number of flows and x_r denote the bandwidth associated with each flow, using route r. All the flows that take a route r are considered type r. The set of rates $\{x_r\}$ is chosen as a function of the number of sources on each route, $\{n_r\}$, by solving an optimization problem which will be described later. Flows arrive to and depart from the network and thus, $\{n_r\}$ changes as a function of time. It is assumed that as the flows arrive and depart, the bandwidths associated with the flows $\{x_r\}$ change instantaneously. Note that we further assume resource allocation in the Internet is accomplished using a congestion controller algorithm that, for our model, converges instantaneously.

Flows of type r arrive according to a Poisson process of rate λ_r. Each file of type r is distributed according to a phase-type distribution (i.e., a Coxian distribution). The number of phases in file type r's distribution is N_r. Each phase consists of an exponential distribution with mean $1/\mu_r$. With the exception of the final phase, after a phase is completed, the file exits the system with probability $(1 - p_r)$ and begins a new exponential phase with probability p_r. After the final phase, the flow exits the system w. p. 1. Thus, the mean file-size distribution is given by

$$\frac{1}{\mu_r}\left(\frac{1 - p_r^{N_r}}{1 - p_r}\right).$$

For any flow $r \in \mathcal{R}$, we denote the number of flows in the i^{th} stage by n_{ir}. The set of rates $\{x_r\}$ at any time t when the number of flows in the network is $\{n_{ri}\}$ is given by the solution to the following optimization problem:

$$\max_{\{x_r\}} \textstyle\sum_r n_r w_r \log x_r$$

$$\text{subject to } \textstyle\sum_{l \in r} n_r x_r \le c_l, \forall l, \text{ and } x_r \ge 0, \forall r,$$

where $n_r = (\sum_{j=1}^{N_r} n_{rj})$.

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A. Lakshmikantha and R. Srikant are with the Dept. of Electrical and Computer Engineering. C. L. Beck is with the Dept. of General Engineering. All authors are affiliated with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. {lkshmknt,beck3,rsrikant }@uiuc.edu

[1] A function f is called copositive if $f(\mathbf{x}) > 0 \ \forall \ \mathbf{x} \in \Re_+^s$. If $-f$ is copositive, then we will call f conegative.

The following is clearly a necessary condition for the Markov chain $\{n_{rj}(t)\}$ to be stable:

$$\sum_{l\in r}\frac{\lambda_r}{\mu_r}\left(\frac{1-p_r^{N_r}}{1-p_r}\right)<c_l\ \forall\ l. \qquad (1)$$

That is, the load on each link should be less than its capacity. We use the term "stable" to indicate that the Markov chain is positive recurrent. To prove that (1) is also sufficient for stability, we first uniformize the continuous-time Markov chain and obtain a discrete-time Markov chain [4]. To obtain the discrete-time chain, we note that the largest rate at which transitions occur out of any state is bounded above by

$$\sum_r \lambda_r + \max_l c_l \sum_r \mu_r N_r.$$

Without loss of generality, we normalize this rate to 1. The equivalent discrete-time chain $\{n_{rj}(k)\}$, $k=0,1,2,\ldots$ then is obtained as follows:

Given any state, the next event is

- an arrival of type r w.p. λ_r,
- a transition from one exponential phase to the next for a connection in progress, i.e., a transition from

  $$(n_{rj}, n_{r,j+1}) \text{ to } (n_{rj}-1, n_{r,j+1}+1)$$

  w.p. $p_r n_{rj} x_{rj}\mu_r$, if $n_{rj}>0$ and $j<N_r$,
- a departure from the network from a phase that is not the last phase, i.e., a transition from

  $$(n_{rj}, n_{r,j+1}) \text{ to } (n_{rj}-1, n_{r,j+1})$$

  w.p. $(1-p_r)n_{rj}x_{rj}\mu_r$, if $n_{rj}>0$ and $j<N_r$,
- a departure from the network from the last phase, i.e., a transition from

  $$n_{rN_r} \text{ to } n_{rN_r}-1$$

  w.p. $n_{rN_r}x_{rN_r}\mu_r$ if $n_{rN_r}>0$, and
- a fictitious event, i.e., no change in the state of the system, w.p. $1-(\sum_r \lambda_r + \sum_{r,j} n_{rj}x_{rj}\mu_r)$.

We use the following lemma, which presents a more stringent sufficient condition than the original Foster-Lyapunov theorem [1, Prop. 5.3], to verify stability of the Markov chain $\{n_{rj}(k)\}$.

Lemma 1: If there exist positive constants $\{\kappa_{rj}\}$, $\{a_{rj}\}$ and C such that

$$V(\mathbf{n}) := \sum_{r,j}\kappa_{rj}n_{rj}^2 \text{ satisfies}$$

$$\begin{array}{l} E[V(\mathbf{n}(k+1)) - V(\mathbf{n}(k))|\mathbf{n}(k)=n] \\ \quad < \ -\epsilon\left(\sum_{r,j}\sum_{j=1}^{N_r} a_{rj}n_{rj}\right) + C, \end{array} \qquad (2)$$

$\forall\ \mathbf{n}\geq 0$, then the Markov chain is positive recurrent. $\diamond$

III. STABILITY ANALYSIS

In this section, we make use of SoS procedures to find Lyapunov functions that satisfy Lemma 1. We consider two types of network topologies, linear and star networks.

A. Linear Network Topologies

Example 1: Consider a linear network consisting of two links with unit link capacities as shown in Figure 1. There are three routes in the network: 0, 1 and 2. Route 0 consists of both the links while route 1 uses link 1 exclusively and route 2 uses link 2 exclusively. The number of phases associated with each route is given by: $N_0=2, N_1=1, N_2=1$. From (1), a necessary condition for stability is

$$\begin{array}{rcl} \frac{\lambda_0}{\mu_0}(1+p)+\frac{\lambda_1}{\mu_1} & < & 1, \\ \frac{\lambda_0}{\mu_0}(1+p)+\frac{\lambda_2}{\mu_2} & < & 1. \end{array} \qquad (3)$$

To show (3) is also sufficient, we consider a quadratic Lyapunov function of the form:

$$V(\mathbf{n}(k)) = \frac{\kappa_0}{\mu_0}n_{01}^2(k)+\frac{\kappa_{02}}{\mu_0}n_{02}^2(k)+\frac{\kappa_1}{\mu_1}n_{11}^2(k)+\frac{\kappa_2}{\mu_2}n_2^2(k).$$

Denoting $\frac{\lambda_r}{\mu_r}$ by ρ_r, we have

$$\begin{array}{l} E[V(\mathbf{n}(k+1)) - V(\mathbf{n}(k))|\mathbf{n}(k)=n] \\ \quad = \ \sum_{r=01,1,2}\kappa_r\rho_r((n_r+1)^2-n_r^2) \\ \qquad +\kappa_{02}pn_{01}x_0 I_{n_{01}>0}((n_{02}+1)^2-n_{02}^2) \\ \qquad +\sum_r \kappa_r x_r n_r (n_r-1)^2 I_{n_r>0} \\ \qquad -\sum_r \kappa_r x_r n_r (n_r^2) I_{n_r>0} \\ \\ \quad = \ \sum_{r=01,1,2}\kappa_r\rho_r(1+2n_r) \\ \qquad +\kappa_{02}px_0n_{01}(1+2n_{02}) \\ \qquad +\sum_r \kappa_r x_r n_r(1-2n_r)(1-I_{n_r=0}) \\ \\ \quad < \ 2\sum_{r=01,1,2}\kappa_r(\rho_r - x_r n_r)n_r \\ \qquad +2\kappa_{02}(px_0n_{01}-x_0n_{02})n_{02}+\mathcal{C} \end{array} \qquad (4)$$

where $\mathcal{C}$ is an appropriately defined constant. Given the state of the system $\mathbf{n}$, the rates allocated to each flow can be calculated by solving the optimization problem:

$$\max_{\{x_r\geq 0\}} n_0w_0\log x_0 + n_1w_1\log x_1 + n_2w_2\log x_2$$

subject to

$$\begin{array}{rcl} n_0x_0+n_1x_1 & \leq & 1, \\ n_0x_0+n_2x_2 & \leq & 1. \end{array}$$

The optimal rates are given by

$$n_rx_r = \left\{\begin{array}{ll} \frac{n_0w_0}{\sum_s n_sw_s} & \text{if } r=0, \\ \frac{n_1w_1+n_2w_2}{(\sum_s n_sw_s)} & \text{if } r=1,2. \end{array}\right. \qquad (5)$$

Substituting for $\{x_r\}$ in (4) we get,

$$\begin{array}{l} E(V(n(k+1)) - V(n(k))|\mathbf{n}(k)) \\ \quad < \ \frac{2}{\sum_r n_rw_r}\Big(\sum_{r=01,1,2}\kappa_r\rho_i n_i(\sum_r n_rw_r) \\ \qquad +\kappa_{02}pw_0n_0n_{01} - (\kappa_{01}n_0n_{01}w_0+ \\ \qquad \kappa_{02}n_0n_{02}w_0+\kappa_1n_1^2w_1+\kappa_2n_2^2w_2)\Big)+\mathcal{C}. \end{array}$$

To apply Lemma 1, we need to show that there exists $\epsilon>0$ and $\{\kappa_i>0\}$ such that

$$\begin{array}{l} (\sum_{r=01,1,2}\kappa_r\rho_rn_r)(\sum_r n_rw_r)+\kappa_{02}pw_0n_0n_{01} \\ -(\sum_{i=1,2}\kappa_{0i}n_0n_{0i}w_0+\sum_{j=1,2}\kappa_jn_j^2w_j) \\ \qquad\qquad\qquad\qquad \leq \ -\epsilon\sum n_r^2 \end{array}$$

In other words, we need to show the polynomial $\mathcal{F}(\mathbf{n})$ is conegative, where

$$\begin{aligned}&\mathcal{F}(\mathbf{n}) = \\ &\quad (\textstyle\sum_{r=01,1,2} \kappa_r \rho_r n_r)(\sum_r n_r w_r) + \kappa_{02} p w_0 n_0 n_{01} \\ &\quad -(\textstyle\sum_{i=1,2} \kappa_{0i} n_0 n_{0i} w_0 + \sum_{j=1,2} \kappa_j n_j^2 w_j) \\ &\quad +\epsilon \textstyle\sum_r n_r^2)\end{aligned}$$

By substituting $n_i = z_i^2$, it can be seen that $\mathcal{F}(\mathbf{n})$ is conegative when the following inequalities are satisfied:

$$\begin{bmatrix} b_1-\epsilon & -\frac{b_5}{4}-y_1 & -\frac{b_6}{4}-y_2 & -\frac{b_7}{4}-y_3 \\ -\frac{b_5}{4}-y_1 & b_2-\epsilon & -\frac{b_8}{4}-y_4 & -\frac{b_9}{4}-y_5 \\ -\frac{b_6}{4}-y_2 & -\frac{b_8}{4}-y_4 & b_3-\epsilon & \frac{b_{10}}{4}-y_6 \\ -\frac{b_7}{4}-y_3 & -\frac{b_9}{4}-y_5 & \frac{b_{10}}{4}-y_6 & b_4-\epsilon \end{bmatrix} > 0,$$

$$\begin{aligned} \tfrac{1}{2}b_5 + 2y_1 &> 0 \\ \tfrac{1}{2}b_6 + 2y_2 &> 0 \\ \tfrac{1}{2}b_7 + 2y_3 &> 0 \\ \tfrac{1}{2}b_8 + 2y_4 &> 0 \\ \tfrac{1}{2}b_9 + 2y_5 &> 0 \\ \tfrac{1}{2}b_{10} + 2y_6 &> 0, \end{aligned}$$

where

$$\begin{array}{ll} b_1 = \kappa_{01} w_0 (1-\rho_0) & b_2 = \kappa_{02} w_0 \\ b_3 = \kappa_1 w_1 (1-\rho_1) & b_4 = \kappa_2 w_2 (1-\rho_2) \\ b_5 = (\kappa_{02} p + \kappa_{01}\rho_0) & b_6 = (\kappa_{01}\rho_0 w_1 + \kappa_1 \rho_1 w_0) \\ b_7 = (\kappa_{01}\rho_0 w_2 + \kappa_2\rho_2 w_0) & b_8 = \kappa_1 \rho_1 w_0 \\ b_9 = \kappa_2 \rho_2 w_0 & \end{array}$$

and $b_{10} = \kappa_2 w_1 (1-\rho_2) + \kappa_1 w_2 (1-\rho_1)$.
Therefore to show the system is stable, we have to find parameters $\{\kappa_i\}, \{y_i\}$ such that the above linear matrix inequalities (LMIs) are satisfied (see, for example [7]).

A set of numerical experiments have been completed for a large variety of system parameter values. A small subset of these parameter values are given in Table I. In all experiments described herein, $\mu_r = 1\ \forall r$; the arrival rates $\{\lambda_r\}$ range from 0 to 0.99; the probability of re-entry ranges from 0.1 to 0.99; and ϵ is set to 0.001. The link utilization is set to 0.99 [2] i.e.,

$$\sum_{l \in r} \frac{\lambda_r}{\mu_r}\left(\frac{1-p_r^{N_r}}{1-p_r}\right) = 0.99 c_l\ \forall\ l.$$

The resulting coefficients for the Lyapunov functions obtained via solution of the above LMI feasibility problem are given in Table II.

Example 2: The same network topology is now considered, however, the number of phases associated with each route is as follows: $N_0 = 1, N_1 = 2, N_2 = 2$. Again from (1), a necessary condition for stability is

$$\begin{aligned} \tfrac{\lambda_0}{\mu_0} + \tfrac{\lambda_1}{\mu_1}(1+p) &< 1, \\ \tfrac{\lambda_0}{\mu_0} + \tfrac{\lambda_2}{\mu_2}(1+p) &< 1. \end{aligned} \tag{6}$$

[2]It is easy to show that if the utilization is less than 0.99 then one can always find $\{\lambda_r\}$ with utilization equal to 0.99 such that the Lyapunov function associated with these parameters is als a Lyapunov function for the original problem

λ_0	λ_1	λ_2	p
0.80	0.11	0.11	0.10
0.17	0.77	0.77	0.32
0.50	0.22	0.22	0.55
0.37	0.33	0.33	0.77
0.28	0.44	0.44	0.99

TABLE I

EXAMPLE 1- SYSTEM PARAMETER VALUES

κ_{01}	κ_{02}	κ_1	κ_2
18.12	155.88	113.65	113.65
93.85	163.36	15.61	15.61
23.00	27.57	33.40	33.40
31.33	23.11	20.90	20.90
47.79	24.07	15.15	15.15

TABLE II

EXAMPLE 1-LYAPUNOV FUNCTION COEFFICIENTS

The optimal resource allocation does not change with the number of phases and hence the rates $\{x_r\}$ are given by (5). To verify that (6) is sufficient for stability, we consider the following Lyapunov function.

$$V(k) = \frac{\kappa_0}{\mu_0} n_0^2 + \frac{\kappa_{11}}{\mu_1} n_{11}^2 + \frac{\kappa_{12}}{\mu_1} n_{12}^2 + \frac{\kappa_{21}}{\mu_2} n_{21}^2 + \frac{\kappa_{22}}{\mu_2} n_{22}^2$$

As in Example 1, it is possible to convert the Lyapunov function search into a LMI (solved, for example, using [7]). A small subset of the system parameter values considered are given in Table III, and corresponding Lyapunov function coefficients are given in Table IV.

λ_0	λ_1	λ_2	p
0.33	0.60	0.60	0.10
0.44	0.42	0.42	0.32
0.11	0.57	0.57	0.55
0.99	0.00	0.00	0.77
0.77	0.11	0.11	0.99

TABLE III

EXAMPLE 2- SYSTEM PARAMETER VALUES

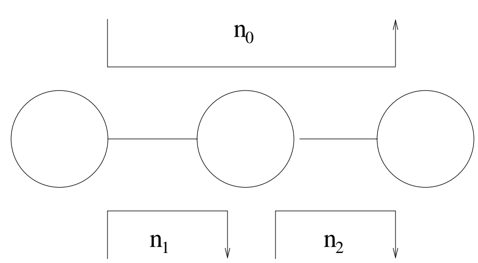

Fig. 1. Linear Network

κ_0	κ_{11}	κ_{12}	κ_{21}	κ_{22}
8.42	5.02	32.72	5.02	32.72
5.30	7.27	14.85	7.27	14.85
18.77	6.30	7.37	6.30	7.37
3.31	675.14	485.28	675.14	485.28
3.64	45.38	24.23	45.38	24.23

TABLE IV

EXAMPLE 2-LYAPUNOV FUNCTION COEFFICIENTS

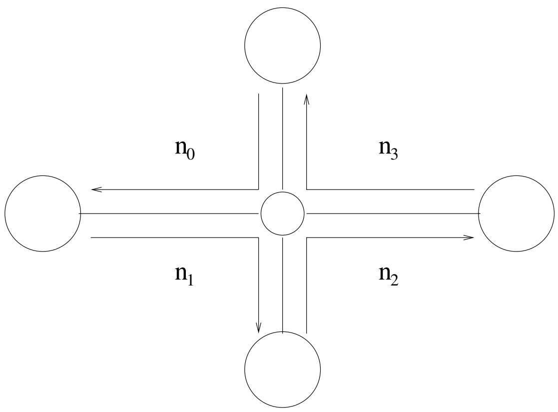

Fig. 2. Star Network Topology

B. Star Network Topologies

We now consider the star network topology[3] which is comprised of 4 links, with unit capacities, as shown in Figure (2). There are 4 routes in the network : 0, 1, 2 and 3. The number of phases associated with each route is: $N_0 = 1, N_1 = 2, N_2 = 2, N_3 = 1$. From (1) a necessary condition for stability is

$$\begin{aligned} \frac{\lambda_0}{\mu_0} + \frac{\lambda_1}{\mu_1} &< 1, \\ \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}(1+p) &< 1, \\ \frac{\lambda_2}{\mu_2}(1+p) + \frac{\lambda_3}{\mu_3}(1+p) &< 1, \\ \frac{\lambda_0}{\mu_0} + \frac{\lambda_3}{\mu_3}(1+p) &< 1. \end{aligned} \tag{7}$$

Given the number of flows in the system, the optimal allocation of the rates $\{x_r\}$ can be found by solving the problem:

$$\max_{\{x_r \geq 0\}} n_0 w_0 \log x_0 + n_1 w_1 \log x_1 + n_2 w_2 \log x_2 + n_3 w_3 \log x_3$$

subject to :

$$\begin{aligned} n_0x_0 + n_1x_1 &\leq 1, \\ n_1x_1 + n_2x_2 &\leq 1, \\ n_2x_2 + n_3x_3 &\leq 1, \\ n_0x_0 + n_3x_3 &\leq 1. \end{aligned}$$

Solving the above problem we get

$$\begin{aligned} n_0x_0 &= \frac{n_0w_0 + n_2w_2}{\sum_i n_iw_i} = n_2x_2, \\ n_1x_1 &= \frac{n_1w_1 + n_3w_3}{\sum_i n_iw_i} = n_3x_3. \end{aligned}$$

[3]This topology is motivated by the fact that the current Internet can be modelled as a star network since the core network links are relatively uncongested and the congestion occurs mainly on the access links.

To show that (7) is sufficient for stability, using Lemma 1, we consider the Lyapunov function:

$$V(k) = \frac{\kappa_0}{\mu_0}n_0^2 + \frac{\kappa_1}{\mu_1}n_1^2 + \frac{\kappa_{21}}{\mu_2}n_{21}^2 + \frac{\kappa_{22}}{\mu_2}n_{22}^2 + \frac{\kappa_{31}}{\mu_3}n_{31}^2 + \frac{\kappa_{32}}{\mu_3}n_{32}^2$$

As in Example 1, the search for such Lyapunov functions can be converted to a series of LMI feasibility problems. A selection of parameters for which the system has been studied are given in Table V and the corresponding Lyapunov function coefficients are given in Table VI.

λ_0	λ_1	λ_2	λ_3	p
0.37	0.62	0.33	0.57	0.10
0.10	0.89	0.08	0.67	0.32
0.46	0.53	0.29	0.35	0.55
0.72	0.27	0.41	0.15	0.77
0.90	0.09	0.45	0.05	0.99

TABLE V

STAR NETWORK :SYSTEM PARAMETER VALUES

κ_0	κ_1	κ_{21}	κ_{22}	κ_{31}	κ_{32}
8.11	4.71	8.71	46.07	5.09	35.21
26.25	2.98	37.01	54.56	5.02	11.82
5.13	4.37	10.65	11.55	9.21	10.08
3.17	8.70	9.37	6.84	21.72	15.35
2.75	29.35	11.00	5.58	74.75	41.3

TABLE VI

STAR NETWORK : LYAPUNOV FUNCTION COEFFICIENTS

In this paper, we have presented a technique to verify the stochastic stability of an arbitrary network. In the limited numerical evalutations that we have conducted so far, the necessary condition (1) appears to be sufficient. Further, it seems that Lyapunov functions of quadratic form are sufficient to establish stability. Further work is required to verify whether the condition given by (1) holds for any $\alpha-$fair allocation [5].

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