Invariant Measures for Jump-Type Fleming-Viot Processes

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Abstract— We study invariant measures for homogeneous jump-type Fleming-Viot processes. For neutral processes (without selection) we prove the ergodicity of the process. In the case with selection, we study the unicity of the invariant measure via a coupling method.

I. INTRODUCTION

Since the publication of Fleming and Viot's seminal paper [5], invariant measures for continuous Fleming-Viot processes have been studied in various works. For a review of the main results see [3], which gives also a comprehensive bibliography on the subject. But, to our best knowledge, results about invariant measures for jump-type Fleming-Viot processes, introduced by Hiraba [6], have not appeared yet in the literature. This paper tries to fill this gap. We apply here some methods which were developed successfully for continuous Fleming-Viot processes.

II. JUMP-TYPE FLEMING-VIOT PROCESS WITH SELECTION

We start with the definition of the Fleming-Viot process. Let (S,d) be a compact metric space, the space of genetic types. Let $C(S)$ be the Banach space of continuous functions with the norm of the supremum ($\beta \in C(S), ||\beta|| = \sup_{x\in S} |\beta(x)|$). The set of Borel subsets of S will be denoted $\mathcal{B}(S)$. Let $\mathcal{M}(S)$ and $\mathcal{M}_F(S)$ be the space of probability measures and the space of finite Radon measures over $\mathcal{B}(S)$, respectively. For $f \in C(S)$ and $\mu \in \mathcal{M}$ (or $\mathcal{M}_F$) we denote $< f,\mu >= \int_S f(x)\mu(dx)$. To $\mathcal{M}(S)$ and $\mathcal{M}_F(S)$ is given the weak topology. Let $\mathcal{B}(\mathcal{M})$ denote the set of Borel subsets of $\mathcal{M}(S)$. Define also $b\mathcal{B}(S)$ and $b\mathcal{B}(\mathcal{M})$, the vector space of bounded measurable functions on S and on $\mathcal{M}(S)$, respectively.

Let $\mathbb{D} := D([0,\infty[,\mathcal{M}(S))$ be equipped with the Skorokhod topology and $Y_t : \mathbb{D} \to \mathcal{M}(S)$ be the canonical process, $Y_t(\omega) = \omega(t)$. Let the σ-algebra $\mathcal{F}$ be the set of Borel subsets $\mathcal{B}(\mathbb{D})$ and the filtration $\{\mathcal{F}_t\}$ in $\mathbb{D}$ be given by $\mathcal{F}_t := \cap_{\epsilon>0}\mathcal{F}^0_{t+\epsilon}$, where $\mathcal{F}^0_t := \sigma(\{Y_u\} : 0 \le u \le t)$, and $\mathcal{F}_\infty := \vee_{n\in\mathbb{N}}\mathcal{F}_n$.

Let $\mu \in \mathcal{M}(S)$, $g > 0$, $a \ge 0$ and $\nu(du)$ be a measure in $\mathbb{R}$ such that $\int_0^\infty (u \wedge u^2)\nu(du) < \infty$.

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We consider a linear operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset C(S) \to C(S)$ which is the generator of a Feller semigroup $P_t : C(S) \to C(S)$ and plays the role of mutation, and a nonlinear function $F : \mathcal{M}(S) \to \mathcal{M}(S)$ which models selection.

$\{P_\mu^{(\mathcal{L},F,g,a,\nu)} : \mu \in \mathcal{M}(S)\} \subset \mathcal{P}(\mathbb{D})$ is a jump-type Fleming-Viot process if:

i) $P_\mu^{(\mathcal{L},F,g,a,\nu)}[Y(0) = \mu] = 1$,

ii) for $\beta \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned}< \beta, Y_t > &=< \beta, Y_0 > + \int_0^t < \mathcal{L}\beta, Y_s > ds + M_t^c(\beta)\\ &+ \int_0^t \left[\frac{a}{g} + \int_0^\infty \left(\frac{u}{g+u}\right)^2 \nu(du)\right] < \beta, F(Y_s) > ds\\ &+ \int_0^t \int_{\mathcal{M}_F(S)} \frac{<1,\eta>}{g+<1,\eta>} < \beta, \overline{\eta} - Y_{s^-} > \tilde{N}(ds,d\eta),\end{aligned} \tag{1}$$

is an $\mathcal{F}_t$-semimartingale, where $\overline{\eta} = \frac{\eta}{<1,\eta>}$, $\{M^c(\beta)\}_{t\ge0}$ is a continuous martingale with quadratic variation given by

$$\ll M^c(\beta) \gg_t = \frac{a}{g}\int_0^t \int_{x\in S}\int_{y\in S} \beta(x)\beta(y)Q(Y(s);dx\times dy)ds \tag{2}$$

with $Q(\mu; dx\times dy) = \delta_{x-y}(dy)\mu(dx) - \mu(dy)\mu(dx)$ and $\tilde{N}$ is a discontinuous $\mathcal{F}_t$-martingale measure corresponding to a random point process N on $\mathbb{R}_+ \times \mathcal{M}_F(S)$, such that $\tilde{N}(t,B) = N(t,B) - \hat{N}(t,B)$, where $\hat{N}$ is the compensator of N given by

$$\hat{N}(ds,d\eta) = \left\{\int_S \left[\int_0^\infty \delta_{u\delta_x}(d\eta)\nu(du)\right] Y_s(dx)\right\} ds\,. \tag{3}$$

To simplify the notation, put

$$k(a,g,\nu) = \frac{a}{g} + \int_0^\infty \left(\frac{u}{g+u}\right)^2 \nu(du), \tag{4}$$

and define

$$Y_{\mathcal{L},F}(t) := Y(t) - Y(0) - \int_0^t [\mathcal{L}^*Y(s) + k(a,g,\nu)F(Y(s))]ds \tag{5}$$

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and, if $F = 0$,

$$Y_{\mathcal{L},0}(t) := Y(t) - Y(0) - \int_0^t [\mathcal{L}^* Y(s)] ds \tag{6}$$

Let $F : \mathcal{M}(S) \to \mathcal{M}(S)$ be given by

$$[F(\mu)](dx) = \int_{y\in S}\int_{z \in S} \sigma(y,z)\mu(dz)Q(\mu; dx \times dy) \tag{7}$$

where $\sigma : S^2 \to \mathbb{R}$ is symmetric bounded function such that $||\sigma(x,y)|| < 1$,

In a previous paper [1], we proved that when the function F satisfies this particular condition then the jump-type Fleming-Viot process is unique. The generator of this process is given as follows.

For $h \in C(S^n)$ define

$$P_t^n h(x) = \left(\prod_{i=1}^n P_t^{(i)}\right) h(x) \tag{8}$$

where $P_t^{(i)}$ is P_t acting on x_i.

Let D^n be the algebra generated by $\{\beta_1(x_1)\cdots\beta_n(x_n) : \beta_i \in \mathcal{D}(\mathcal{L}), i = 1, \dots, n\}$. For $h \in D^n$ let

$$\mathcal{L}^{(n)} h(x) = \sum_{i=1}^n \mathcal{L}_i h(x_1, \dots, x_n) \tag{9}$$

where $\mathcal{L}_i$ is $\mathcal{L}$ acting on x_i.

Let $F_h(\mu) =< h, \mu^n >$. The generator of the jump-type Fleming-Viot process with selection is given by

$$\begin{aligned}
&\mathcal{G}^g F_h(\mu) = \\
&< \mathcal{L}^{(n)} h, \mu^n > + \frac{a}{2g}\sum_{j\neq k}\Big(< \Theta_{j;k} h, \mu^{n-1} > - < h, \mu^n >\Big) \\
&+ \sum_{m=2}^n B_{m,n}[g] \sum_{(j_1,\{j_2,\dots,j_m\})} \Big(< \Theta_{j_1;j_2,\dots,j_m} h, \mu^{n-m+1} > \\
&\quad - < h, \mu^n >\Big) \\
&+ \overline{\sigma} k(a,g,\nu) \sum_{j=1}^n (< K_{jn} h, \mu^{n+2} > - < h, \mu^n >) \\
&+ \overline{\sigma} n k(a,g,\nu) < h, \mu^n >
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\Theta_{j_1;j_2,\dots,j_m}[h(x_1,\dots,x_n)] = h(x_1,\dots,x_{j_2-1},x_{j_1},x_{j_2+1}, \\
&\dots, x_{j_m-1}, x_{j_1}, x_{j_m+1}, \dots, x_n),
\end{aligned} \tag{11}$$

$$\sum_{(j_1,\{j_2,\dots,j_k\})} := k \sum_{j_1=1}^{n-k+1} \sum_{j_2=j_1+1}^{n-k+2} \cdots \sum_{j_k=j_{k-1}+1}^{n}, \tag{12}$$

$$B_{m,n}[g] := \frac{1}{m}\int_0^{+\infty} \frac{u^m [g]^{n-m}}{[g+u]^n} \nu(du), \tag{13}$$

$$\begin{aligned}
&(K_{jn} h)(x_1, \dots, x_{n+2}) = \\
&\frac{\sigma(x_j, x_{n+1}) - \sigma(x_{n+1}, x_{n+2})}{\overline{\sigma}} h(x_1,\dots,x_n)
\end{aligned} \tag{14}$$

with $\overline{\sigma}$ a constant satisfying

$$\overline{\sigma} \geq \sup_{x,y,z \in S} |\sigma(x,y) - \sigma(y,z)|, \tag{15}$$

III. THE DUAL PROCESS

In this section we describe the dual process. It will aid us to prove weak ergodicity of the jump-type Fleming-Viot process without selection.

Let us define an operator

$$\mathcal{H}^g : C(\cup_{n=1}^\infty C(S^n)) \to C(\cup_{n=1}^\infty C(S^n)),$$

such that

$$\mathcal{H}^g F_\mu(h) := \mathcal{G}^g F_h(\mu) - \overline{\sigma} n k(a,g,\nu) F_h(\mu) \tag{16}$$

$\mathcal{H}^g$ is the generator of the dual process which we construct as follows.

Set

$$\gamma^* = \overline{\sigma} k(a,g,\nu) \tag{17}$$

$$\gamma_{2,n}^0 = \frac{a}{2g} + B_{2,n}(g) \tag{18}$$

$$\gamma_{m,n}^0 = B_{m,n}(g), \quad m = 3, \dots, n. \tag{19}$$

Let $N \in \mathbb{N}$ and $M = \{M(s) : s \geq 0\}$ be a continuous-time Markov chain over $\mathbb{N}$ which has transition intensities

$$q_{N,N+2} = N\gamma^*, \tag{20}$$

$$q_{N,N-d} = \sum_{(j_1;\{j_2,\dots,j_{d+1}\})} \gamma_{d+1,N}^0, \tag{21}$$

$d = 1, \dots, N-1$. Set $\{\tau_k\}_{k\geq 0}$ to be the sequence of jump times of M, $\tau_0 = 0$. Let Γ_i be a sequence of random operators which, given M, are conditionally independent and satisfy

$$P[\Gamma_k = K_{jN}|M] = \frac{1}{N} \mathbb{I}_{[M(\tau_k-)=N, M(\tau_k)=N+2]}, \tag{22}$$

for $j = 1, \dots, N$ and

$$\begin{aligned}
&P[\Gamma_k = \Theta_{j_1;j_2,\dots,j_{d+1}}|M] = \\
&\frac{1}{\sum_{(j_1;\{j_2,\dots,j_{d+1}\})} 1} \mathbb{I}_{[M(\tau_k-)=N, M(\tau_k)=N-d]},
\end{aligned} \tag{23}$$

for $d = 1, \dots, N-1$.

Let $h \in D^N$, $M(0) = N$ and $W(0) = h$. Then the dual process, $W(t)$, is given by

$$P_{t-\tau_k}^{M(\tau_k)} \Gamma_k P_{\tau_k - \tau_{k-1}}^{M(\tau_{k-1})} \Gamma_{k-1} \cdots \Gamma_1 P_{\tau_1}^{M(0)} W(0), \tag{24}$$

$\tau_k \leq t < \tau_{k+1}$, $k = 0, 1, \dots$.

The proof of the following proposition is found in [1].

Proposition 3.1: The following duality identity holds:

$$\begin{aligned}
&E[< h, Y(t)^N >] = \\
&E\left[< W(t), \mu^{M(t)} > \exp\{\gamma^* \int_0^t M(s) ds\}\right],
\end{aligned} \tag{25}$$

where $W(0) = h$ and $Y(0) = \mu$.

Proposition 3.2: Suppose that the semigroup P_t has an invariant measure π. Then, for the neutral jump-type Fleming-Viot process,

$$\lim_{t\to\infty} E[< h, Y(t)^N >] = E\left[< W(\tau_{N-1}), \pi >\right], \quad (26)$$

for each $N \geq 1$ and $h \in C(S)$.

Proof: The proof is based essentially on the duality relation (25). See [3].

IV. STATIONARY DISTRIBUTION FOR THE HOMOGENEOUS CASE WITHOUT SELECTION

In order to prove existence and uniqueness of an invariant measure for the jump-type Fleming-Viot process without selection, we will prove first that it is a Feller process.

Theorem 4.1: The jump-type Fleming-Viot process without selection is a Feller-Markov process.

Proof: It follows along the same lines as in [2], bearing in mind that the semigroup $\{T_t\}$ associated to the jump-type Fleming-Viot process is given by (see [6])

$$T_t F_h(\mu) = \int_{S^n} (V_t h)(x) \mu^n(dx) \quad (27)$$

where

$$\begin{aligned} V_t h(x) = {} & \exp\left[-\sum_{m=2}^{n} \gamma_{m,n}\right] P_t^n h(x) \\ & + \sum_{m=2}^{n} \exp\left[-\sum_{k\neq m; 2\leq k\leq n} \gamma_{k,n}\right] \\ & \times \sum_{(j_1,\{j_2,\dots,j_m\})} \int_0^t \gamma^0_{m,n} \exp[-\gamma_{m,n}] \\ & \times P_u^n(\Theta_{j_1;j_2,\dots,j_m}(V_{t-u}h))(x) du \quad (28) \end{aligned}$$

is generated by $\mathcal{G}^g$.

Remark 1: The main difficulty in dealing with processes which include selection, vis-à-vis existence of invariant measure, has to do with the change in the infinitesimal generator, which makes it hard to prove that such processes are Feller.

Theorem 4.2: If there is a unique $\pi \in \mathcal{M}(S)$ such that, for $\beta \in \mathcal{D}(\mathcal{L})$,

$$< \mathcal{L}\beta, \pi >= 0 \quad (29)$$

then there exists a unique stationary distribution $\Pi \in \mathcal{P}(\mathcal{M}(S))$ for the homogeneous jump-type Fleming-Viot process without selection.

Proof: The existence is due to the fact that the homogeneous jump-type Fleming-Viot process has the Feller property. For uniqueness, note that the range of $\lambda - \mathcal{L}^{(n)}$ contains $C(S^n)$ for all $\lambda > 0$ and, for $h \in C(S^n)$,

$$R^{(n)}\, h = [\lambda_n - \mathcal{L}^{(n)}]^{-1} h = \int_0^\infty e^{-\lambda_n s} P_s^n h \, ds. \quad (30)$$

where $\lambda_n = \frac{a}{g} n()2 + \sum_{m=2}^n B_{m,n}[g]mn()m$. The rest of the proof will follow as in [3].

Corollary 1: The homogeneous jump-type Fleming-Viot process without selection is weakly ergodic.

Proof: From (26) we obtain

$$\lim_{t\to\infty} E[F_h(Y_t)] = \int_{\mathcal{M}(S)} F_h(\mu)\Pi(d\mu) \quad (31)$$

By the property of convergence determining of the collection of functions of the form $F_h =< h, \mu^n >$, we have the result.

V. COUPLING HOMOGENEOUS JUMP-TYPE FLEMING-VIOT PROCESSES

Our aim is to obtain a coupling for the homogeneous jump-type Fleming-Viot process in order to study the stationary distribution for the homogeneous jump-type Fleming-Viot process with selection. We start with the description of a successful Markov coupling, entirely inspired by that from [4]. Let $\tilde{S}$ be a compact metric space, $\rho_i : \tilde{S} \to S$, for $i = 1, 2$, and $\rho : S \times S \to \tilde{S}$ be Borel measurable mappings such that $(\rho_i \circ \rho)(x_1, x_2) = x_i$, and $\tilde{\mathcal{L}}$ an operator on $b\mathcal{B}(\tilde{S})$. $(\tilde{\mathcal{L}}, \rho_1, \rho_2, \rho)$ determines a successful Markov coupling for $\mathcal{L}$ if

1) the martingale problem for $\tilde{\mathcal{L}}$ is well posed,
2) for each $\beta \in \mathcal{D}(\mathcal{L})$, $\beta \circ \rho_i \in \mathcal{D}(\tilde{\mathcal{L}})$ and $\tilde{\mathcal{L}}(\beta \circ \rho_i) = (\mathcal{L}\beta) \circ \rho_i$ for $i = 1, 2$, and
3) for each solution X of the $D_{\tilde{S}}[0, \infty[$-martingale problem for $\tilde{\mathcal{L}}$, there exists a random time τ such that

$$P(\{\tau < \infty,\ \rho_1 \circ X(\tau + t) = \rho_2 \circ X(\tau + t) \text{ for all } t \geq 0\}) = 1. \quad (32)$$

Define $\hat{\rho}_i : \mathcal{M}(\tilde{S}) \to \mathcal{M}(S)$, $i = 1, 2$, by $\hat{\rho}_i(\mu) = \mu\rho_i^{-1}$, and $\hat{\rho} : \mathcal{M}(S) \times \mathcal{M}(S) \to \mathcal{M}(\tilde{S})$ by $\hat{\rho}(\mu_1, \mu_2) = (\mu_1 \times \mu_2)\rho^{-1}$. Let $\tilde{\mathcal{G}}$ be the generator for the jump-type Fleming-Viot process in $\mathcal{M}(\tilde{S})$ without selection, with mutation operator $\tilde{\mathcal{L}}$.

Theorem 5.1: Let $\tilde{Y}$ be a homogeneous jump-type Fleming-Viot process with type space $\tilde{S}$, mutation operator $\tilde{\mathcal{L}}$ with corresponding semigroup $\{\tilde{P}_t\}$ on $b\mathcal{B}(\tilde{S})$. Let $D \subset \tilde{S}$ be closed. Suppose that $\tilde{P}_t \mathbf{I}_D \geq \mathbf{I}_D$ for all $t \geq 0$, $\lim_{t\to\infty} \tilde{P}_t \mathbf{I}_D(x) = 1$ for each $x \in \tilde{S}$. Define $\tau = \inf\{t \geq 0 :< \mathbf{I}_D, \tilde{Y}_t >= 1\}$. Then $P\{\tau < \infty\} = 1$.

Proof: Let $\lambda > 0$ and $\psi \in b\mathcal{B}(\tilde{S})$. Define $h_{\lambda,\psi} = \int_0^\infty \lambda e^{-\lambda t} \tilde{P}_t \psi dt$. If $h_{\lambda,\psi} \in \mathcal{D}(\tilde{\mathcal{L}})$ then $\tilde{\mathcal{L}} h_{\lambda,\psi} = \lambda(h_{\lambda,\psi} - \psi)$ and

$$\begin{aligned} < h_{\lambda,\psi}, \tilde{Y}_{\tilde{\mathcal{L}},0}(t) > = {} & < h_{\lambda,\psi}, \tilde{Y}_t > - < h_{\lambda,\psi}, \tilde{Y}_0 > \\ & - \int_0^t < \lambda(h_{\lambda,\psi} - \psi), \tilde{Y}_s > ds \quad (33) \end{aligned}$$

is an $\tilde{\mathcal{F}}_t$-martingale. By Ito's formula (see [2]), for $f \in C^2(\mathbb{R})$ we have

$$
\begin{aligned}
V_{\lambda,\psi}(t) &:= f(< h_{\lambda,\psi}, \tilde{Y}_t >) - f(< h_{\lambda,\psi}, \tilde{Y}_0 >) \\
&- \int_0^t < \lambda(h_{\lambda,\psi} - \psi), \tilde{Y}_s > f'(< h_{\lambda,\psi}, \tilde{Y}_s >) ds \\
&- \frac{a}{2g} \int_0^t (< h^2_{\lambda,\psi}, \tilde{Y}_s > - < h_{\lambda,\psi}, \tilde{Y}_s >^2) \\
&\times f''(< h_{\lambda,\psi}, \tilde{Y}_s >) ds \\
&- \int_{s=0}^t \int_{u=0}^\infty \int_{x \in \tilde{S}} f\bigg(\frac{g}{g+u} < h_{\lambda,\psi}, \tilde{Y}_s > \\
&+ \frac{u}{g+u} h_{\lambda,\psi}(x) \bigg) \tilde{Y}_s(dx) \nu(du) ds \\
&+ \int_{u=0}^\infty \nu(du) \int_{s=0}^t f(< h_{\lambda,\psi}, \tilde{Y}_s >) ds \qquad (34)
\end{aligned}
$$

is a $\tilde{\mathcal{F}}_t$-martingale. Now, taking $\psi = \mathbb{I}_D$, under the hipotheses which $\tilde{P}$ satisfies, and assuming $f' \geq 0$, we get

$$
\begin{aligned}
& V^*_{\lambda,D}(t) := f(< h_\lambda, \tilde{Y}_t >) - f(< h_\lambda, \tilde{Y}_0 >) \\
& - \frac{a}{2g} \int_0^t (< h^2_\lambda, \tilde{Y}_s > - < h_\lambda, \tilde{Y}_s >^2) f''(< h_\lambda, \tilde{Y}_s >) ds \\
& - \int_{s=0}^t \int_{u=0}^\infty \int_{x \in \tilde{S}} f\bigg(\frac{g}{g+u} < h_\lambda, \tilde{Y}_s > \\
& + \frac{u}{g+u} h_\lambda(x) \bigg) \tilde{Y}_s(dx) \nu(du) ds \\
& + \int_{u=0}^\infty \nu(du) \int_{s=0}^t f(< h_\lambda, \tilde{Y}_s >) ds \qquad (35)
\end{aligned}
$$

is a $\tilde{\mathcal{F}}_t$-submartingale. Then, making $\lambda \to \infty$, and assuming also that f is convex,

$$
\begin{aligned}
& V_t := f(< \mathbb{I}_D, \tilde{Y}_t >) - f(< \mathbb{I}_D, \tilde{Y}_0 >) \\
& - \frac{a}{2g} \int_0^t (< \mathbb{I}_D, \tilde{Y}_s > - < \mathbb{I}_D, \tilde{Y}_s >^2) f''(< \mathbb{I}_D, \tilde{Y}_s >) ds \qquad (36)
\end{aligned}
$$

is a $\tilde{\mathcal{F}}_t$-submartingale.

On the other hand, we can define

$$\tilde{Z}_t = \lim_{s \in \mathbb{Q}, s \to t^+} < \mathbb{I}_D, \tilde{Y}_s >,$$

a right continuous process, which satisfy

$$\tilde{Z}_t \ \geq \ < \mathbb{I}_D, \tilde{Y}_t >$$

almost sure. Take $\omega \in \Omega$ and let $s_n \in \mathbb{Q}$, $s_n \to t^+$, with $\mathbb{Q}$ denoting set of rational numbers as usual. Then

$$\lim_{s_n \to t^+} \omega(s_n) = \omega(t),$$

because $\omega(t^+) = \omega(t)$. Now, since D is closed,

$$\limsup_{s_n} < \mathbb{I}_D, \omega(s_n) > \ \leq \ < \mathbb{I}_D, \omega(t) > .$$

Then, for almost all ω,

$$\limsup_{s_n} < \mathbb{I}_D, \tilde{Y}_{s_n}(\omega) > \ \leq \ < \mathbb{I}_D, \tilde{Y}_t(\omega) >, \qquad (37)$$

and

$$
\begin{aligned}
\tilde{Z}_t(\omega) &= \lim_{s_n \to t^+} < \mathbb{I}_D, \tilde{Y}_{s_n}(\omega) > \\
&\leq \ \limsup_{s_n} < \mathbb{I}_D, \tilde{Y}_{s_n}(\omega) > \\
&\leq \ < \mathbb{I}_D, \tilde{Y}_t(\omega) > \qquad (38)
\end{aligned}
$$

for all $t \geq 0$. That is,

$$P(\{< \mathbb{I}_D, \tilde{Y}_t > \ \geq \ \tilde{Z}_t \text{ for all } t \geq 0\}) = 1. \qquad (39)$$

From here on the proof follows the same steps as in [4].

Now the last two theorems can be proved, mutatis mutantis, by the same method developed in [4]

Theorem 5.2: Suppose that the martingale problem for $\mathcal{G}$, the neutral jump-type Fleming-Viot generator with mutation operator $\mathcal{L}$, is well posed. Assume that $(\tilde{\mathcal{L}}, \rho_1, \rho_2, \rho)$ determines a successful Markov coupling for $\mathcal{L}$. Suppose that the martingale problem for $\tilde{\mathcal{G}}$, the neutral jump-type Fleming-Viot generator with the mutation operator $\tilde{\mathcal{L}}$ is well posed. Then $(\tilde{\mathcal{G}}, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho})$ determines a successful Markov coupling for $\mathcal{G}$. Besides, the following limits apply:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t F(Y_s) ds = \int_{\mathcal{M}(S)} F(\mu) \Pi(\mu), \text{ a.s.} \qquad (40)$$

and

$$\lim_{t \to \infty} \sup_{G \in \mathcal{B}(\mathcal{M}(S))} |P\{Y_t \in G\} - \Pi(G)| = 0 \qquad (41)$$

for each $F \in b\mathcal{B}(\mathcal{M}(S))$, where Π is the unique stationary measure obtained in Theorem 4.2.

The jump-type Fleming-Viot process with selection was obtained in [1] by a Girsanov transformation, which is used in the proof of the following theorem.

Theorem 5.3: Suppose that the conditions in Theorem 5.2 is satisfied. Assuming that a stationary distribution for the homogenous jump-type Fleming-Viot process with selection exists, then it is unique.

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