2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Robust L_1 Model Reduction for Uncertain Stochastic Systems with State Delay

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Abstract—**This paper investigates the problem of robust L_1 model reduction for continuous-time uncertain stochastic systems with state delay. For a given mean-square stable system, our purpose is to construct reduced-order systems, such that the error system between the two models is mean-square asymptotically stable and has a guaranteed L_1 performance. The peak-to-peak gain criterion is first established for stochastic systems with state delay, and the corresponding model reduction problem is solved by using projection lemma. Sufficient conditions are obtained for the existence of admissible reduced-order models in terms of linear matrix inequalities (LMIs) plus matrix inverse constraints. Since these obtained conditions are not expressed as strict LMIs, the cone complementarity linearization (CCL) method is exploited to cast them into nonlinear minimization problems subject to LMI constraints, which can be readily solved by standard numerical software. In addition, the development of delay-free reduced-order model is also presented. The efficiency of the proposed methods is demonstrated via a numerical example.**

I. Introduction

The problem of model reduction is very important in many areas of engineering. It is often desirable to find a reduced-order model to approximate the original high-order model without significant error introduced. Many important results on model reduction have been reported, which involve various efficient approaches such as the balanced truncation method [1], the optimal Hankel norm approximation method [2]. Very recently, LMI technique has also been introduced to solve the model reduction problem for different classes of systems [3, 4].

In solving the model reduction problem, some performance indices are usually introduced to evaluate the error between the original system and the reduced-order system. Some widely used performances are H_∞, L_2-L_∞ and H_2. These performances assume the input signal to be energy bounded, and therefore the model reduction methods upon these performances are most suitable for systems with inputs that belong to L_2 space. If the input signal is assumed to be bounded on magnitude only, the minimization of peak-to-peak gain, which computes the worst-case peak value of the error with persistent bounded input, appears to be more adequate as performance criterion in the construction of reduced-order models. The model reduction problem based on such a performance is usually also called peak-to-peak model reduction or L_1 model reduction. However, it seems that little effort has been made toward solving the L_1 model reduction problem, except that in [2].

During the last decades stochastic systems received much attention. For the model reduction problem, based on the theory of stochastic realization, an algorithm for obtaining reduced-order models was proposed in [5]. However, to the best of the authors' knowledge, the L_1 model reduction problem has not been solved for stochastic uncertain systems, either with or without time delays.

In this paper, we are interested in the problem of L_1 model reduction for linear continuous-time uncertain stochastic systems with state delay. For a given mean-square stable system, attention is focused on the construction of reduced-order models, which guarantee the error system to be mean-square asymptotically stable and has a prescribed L_1 performance constraint. The L_1 performance criterion is first established for stochastic systems with state delay, upon which sufficient conditions are obtained for the existence of admissible reduced-order models in terms of LMIs with some matrix inverse constraints. Since these obtained conditions are not expressed as strict LMIs, CCL method is exploited to cast them into nonlinear minimization problems subject to LMIs constraints, which can be readily solved by standard software. In addition, the development of delay-free reduced-order models is also presented.

II. Problem Formulation

Consider a continuous-time uncertain stochastic system with state delay

$$\begin{aligned}
&\mathrm{d}x(t) = \left[Ax(t) + A_d x(t-h(t)) + Bu(t)\right]\mathrm{d}t + \left[Mx(t) + M_d x(t-h(t)) + Nu(t)\right]\mathrm{d}w(t) \\
&y(t) = Cx(t) + C_d x(t-h(t)) + Du(t) \\
&x(t) = \phi(t), \quad \forall t \in \left[-\bar{h}, 0\right]
\end{aligned} \tag{1}$$

where $x(t) \in \mathrm{R}^n$ is the state vector; $y(t) \in \mathrm{R}^m$ is the output signal; $u(t) \in \mathrm{R}^q$ is the input which belongs to $L_\infty[0, \infty)$; $w(t)$ is a one-dimensional Brownian motion satisfying $\mathrm{E}\{\mathrm{d}w(t)\} = 0$,

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$E\{dw(t)^2\} = dt$. In addition, $h(t)$ is time-varying bounded time delays satisfying $0 < h(t) \le \bar{h} < \infty, \dot{h}(t) \le \tau < 1$, $\forall t \ge 0$, where $\bar{h}$ and τ are real constant scalars; $\phi(t)$ is given initial vector function that is continuous on the segment $[-\bar{h}, 0]$; B, N, D are known real constant matrices; A, A_d, M, M_d, C, C_d are uncertain matrices that have the following form

$$A = A_0 + \Delta A,\ A_d = A_{d0} + \Delta A_d,\ M = M_0 + \Delta M$$
$$M_d = M_{d0} + \Delta M_d,\ C = C_0 + \Delta C,\ C_d = C_{d0} + \Delta C_d$$

where $A_0, A_{d0}, M_0, M_{d0}, C_0, C_{d0}$ are known constant matrices with appropriate dimensions. $\Delta A, \Delta A_d, \Delta M, \Delta M_d, \Delta C, \Delta C_d$ are real-valued time-varying matrix functions representing norm-bounded parameter uncertainties, satisfying

$$\begin{bmatrix} \Delta A & \Delta A_d \\ \Delta M & \Delta M_d \\ \Delta C & \Delta C_d \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} F \begin{bmatrix} V_1 & V_2 \end{bmatrix} \quad (2)$$

where U_1, U_2, U_3, V_1 and V_2 are constant matrices, and $F \in R^{i \times j}$ is the uncertain matrix satisfying $F^T F \le I$. The parameter uncertainties $\Delta A, \Delta A_d, \Delta M, \Delta M_d, \Delta C$ and ΔC_d are said to be admissible if (2) holds.

Here, we are interested in approximating system in (1) by the following reduced-order system

$$\begin{aligned} d\hat{x}(t) &= [\hat{A}\hat{x}(t) + \hat{A}_d\hat{x}(t-h(t)) + \hat{B}u(t)]dt + [\hat{M}\hat{x}(t) + \hat{M}_d\hat{x}(t-h(t)) + \hat{N}u(t)]dw(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{C}_d\hat{x}(t-h(t)) + \hat{D}u(t) \\ \hat{x}(t) &= \rho(t), \quad \forall t \in [-\bar{h}, 0] \end{aligned} \quad (3)$$

where $\hat{x}(t) \in R^l$ is the state vector; $\hat{y}(t) \in R^m$ is output signal; $\rho(t)$ is given initial vector function that is continuous on the segment $[-\bar{h}, 0]$; $\hat{A}$, $\hat{A}_d$, $\hat{B}$, $\hat{M}$, $\hat{M}_d$, $\hat{N}$, $\hat{C}$, $\hat{C}_d$, $\hat{D}$ are appropriately dimensioned matrices to be determined.

Augmenting the model of system (1) to include the states of system (3), we obtain the following error system:

$$\begin{aligned} d\bar{x}(t) &= [\bar{A}\bar{x}(t) + \bar{A}_d\bar{x}(t-h(t)) + \bar{B}u(t)]dt + [\bar{M}\bar{x}(t) + \bar{M}_d\bar{x}(t-h(t)) + \bar{N}u(t)]dw(t) \\ e(t) &= \bar{C}\bar{x}(t) + \bar{C}_d\bar{x}(t-h(t)) + \bar{D}u(t) \\ \bar{x}(t) &= \varphi(t) \quad \forall t \in [-\bar{h}, 0] \end{aligned} \quad (4)$$

where $\bar{x}(t) = \begin{bmatrix} x^T(t) & \hat{x}^T(t) \end{bmatrix}^T$, $\varphi(t) = \begin{bmatrix} \phi^T(t) & \psi^T(t) \end{bmatrix}^T$, $e(t) = y(t) - \hat{y}(t)$.

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & \hat{A}_d \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \bar{M} = \begin{bmatrix} M & 0 \\ 0 & \hat{M} \end{bmatrix}, \bar{M}_d = \begin{bmatrix} M_d & 0 \\ 0 & \hat{M}_d \end{bmatrix} \quad (5)$$

$$\bar{N} = \begin{bmatrix} N^T & \hat{N}^T \end{bmatrix}^T, \bar{C} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \bar{C}_d = \begin{bmatrix} C_d & -\hat{C}_d \end{bmatrix}, \bar{D} = D - \hat{D} \quad (6)$$

Before presenting the main objective of this paper, we first introduce the following definitions for the error system (4), which will be essential for our derivation.

Definition 1: The error system (4) with $u(t) = 0$ is said to be mean-square asymptotically stable if $\lim_{t\to\infty} E\{|\bar{x}(t)|^2\} = 0$ for any initial conditions.

Definition 2: Given a scalar $\gamma > 0$, the error system (4) is said to be mean-square asymptotically stable and have L_1 performance constraint γ if it is mean-square asymptotically stable and under zero initial condition, $\|e(t)\|_{E_\infty} < \gamma \|u(t)\|_{L_\infty}$ for all nonzero $u(t) \in L_\infty[0, \infty)$ (where $\|\cdot\|_{L_\infty} = \sup_{t\ge0}\|\cdot\|$, $\|\cdot\|_{E_\infty} = \sup_t \sqrt{E\{|\cdot|^2\}}$).

Assume the system (1) is mean-square asymptotically stable. Our purpose is to determine reduced-order model (3) such that for all admissible uncertainties and time delays, the error system (4) is mean-square asymptotically stable and has L_1 performance constraint γ.

Before proceeding further, we present the following lemmas which will be used in the proof of our main results.

Lemma 1: Let $L = L^T \in R^{n\times n}$, $S \in R^{n\times m}$, $H \in R^{l\times n}$ be given matrices, and suppose rank(S)<n and rank(H)<n, there exists a matrix Ξ satisfying

$$L + S\Xi H + (S\Xi H)^T < 0$$

if and only if

$$S^\perp L S^{\perp T} < 0,\ H^{T\perp} L H^{T\perp T} < 0 \quad (7)$$

Furthermore, if (7) holds, the solutions of Ξ can be given by

$$\Xi = S_R^+ \Psi H_L^+ + \Phi - S_R^+ S_R \Phi H_L H_L^+$$

With $\Psi = -\Pi^{-1} S_L^T \Lambda H_R^T (H_R \Lambda H_R^T)^{-1} + \Pi^{-1} \Gamma^{\frac{1}{2}} W (H_R \Lambda H_R^T)^{-\frac{1}{2}}$,

$\Lambda = (S_L \Pi^{-1} S_L^T - L)^{-1} > 0$, $\Gamma = \Pi - S_L^T(\Lambda - \Lambda H_R^T (H_R \Lambda H_R^T)^{-1} H_R \Lambda) S_L$.

where Φ, Π, W are any appropriately dimensioned matrices satisfying $\Pi > 0$ and $\|W\| < 1$.

Remark 1: For matrix $S \in R^{n\times m}$, we denote $S^\perp$ as the orthogonal complement, such that $S^\perp S = 0$ and $S^\perp S^{\perp T} > 0$; we denote S^+ as the Moore-Penrose inverse of S; S_L and S_R are any full rank factors of S, that is, $S_L S_R = S$.

Lemma 2: Let $\tilde{U}, \tilde{V}$ and F_1 be real matrices of appropriate dimensions with F_1 satisfying $F_1^T F_1 \le I$. Then, for any scalar $\varepsilon > 0$, there holds $\tilde{U} F_1 \tilde{V} + (\tilde{U} F_1 \tilde{V})^T \le \varepsilon^{-1} \tilde{U}\tilde{U}^T + \varepsilon \tilde{V}^T \tilde{V}$.

III. L_1 Performance Criterion of Stochastic Systems with State Delay

In this section, we will derive an L_1 performance criterion for stochastic time-delay systems, which includes stochastic L_1 performance and delay L_1 performance as special.

Theorem 1: Consider system (1). If there exist matrix $0 < P \in R^{(n+l)\times(n+l)}$, and scalars $0 < \alpha < 1, \beta \in R^+, \xi \in R^+$, and $\mu \in R$ satisfying

$$\begin{bmatrix} -P & P\bar{M} & P\bar{M}_d & P\bar{N} \\ * & \bar{A}^T P + P\bar{A} + \xi P + \alpha\beta P & P\bar{A}_d & P\bar{B} \\ * & * & -(1-\tau)\xi P + (1-\alpha)\beta P & 0 \\ * & * & * & -\mu I \end{bmatrix} < 0 \quad (8)$$

$$\begin{bmatrix} -\alpha\beta P & 0 & 0 & \bar{C}^T \\ * & -(1-\alpha)\beta P & 0 & \bar{C}_d^T \\ * & * & -(\gamma-\mu)I & \bar{D}^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \quad (9)$$

Then, the error system (4) is mean-square asymptotically stable and the maximal peak-to-peak gain is smaller than γ.

Proof: First, choose a Lyapunov functional candidate for the error system in (4)

$$V(\bar{x}(t),t)=\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)+\xi\int_{-h(t)}^{t}\bar{x}^{\mathrm{T}}(s)P\bar{x}(s)\mathrm{d}s$$

where P is real symmetric positive definite matrices to be determined. Then by making use of Itô differential rule, along the solution of system (4) we obtain the stochastic differential as

$$\mathrm{d}V(\bar{x}(t),t)=\mathrm{L}V(\bar{x}(t),t)\mathrm{d}t+2\bar{x}^{\mathrm{T}}(t)P[\overline{M}\bar{x}(t)+\overline{M}_d\bar{x}(t-h(t))+\overline{N}u(t)]\mathrm{d}w(t)$$

where

$$\begin{aligned}\mathrm{L}V(\bar{x}(t),t)&=\xi\,\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)-(1-\dot{h}(t))\xi\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\\&+2\bar{x}^{\mathrm{T}}(t)P[\overline{A}\bar{x}(t)+\overline{A}_d\bar{x}(t-h(t))+\overline{B}u(t)]\\&+[\overline{M}\bar{x}(t)+\overline{M}_d\bar{x}(t-h(t))+\overline{N}u(t)]^{\mathrm{T}}P[\overline{M}\bar{x}(t)+\overline{M}_d\bar{x}(t-h(t))+\overline{N}u(t)]\\&\le\bar{x}^{\mathrm{T}}(t)[\overline{A}^{\mathrm{T}}P+P\overline{A}+\xi P+\overline{M}^{\mathrm{T}}P\overline{M}]\bar{x}(t)+2\bar{x}^{\mathrm{T}}(t)[P\overline{A}_d+\overline{M}^{\mathrm{T}}P\overline{M}_d]\bar{x}(t-h(t))\\&+\bar{x}^{\mathrm{T}}(t-h(t))[-(1-\tau)\xi P+\overline{M}_d^{\mathrm{T}}P\overline{M}_d]\bar{x}(t-h(t))\\&+2\bar{x}^{\mathrm{T}}(t)[P\overline{B}+\overline{M}^{\mathrm{T}}P\overline{N}]u(t)+2\bar{x}^{\mathrm{T}}(t-h(t))\overline{M}_d^{\mathrm{T}}P\overline{N}u(t)+u^{\mathrm{T}}(t)\overline{N}^{\mathrm{T}}P\overline{N}u(t)\end{aligned}$$

Therefore, when assuming zero input $u(t)$=0, it follows that

$$\mathrm{L}V(\bar{x}(t),t)\le\left[\bar{x}^{\mathrm{T}}(t)\quad\bar{x}^{\mathrm{T}}(t-h(t))\right]\Delta\begin{bmatrix}\bar{x}(t)\\\bar{x}(t-h(t))\end{bmatrix}\tag{10}$$

where $\Delta=\begin{bmatrix}\overline{A}^{\mathrm{T}}P+P\overline{A}+\xi P+\overline{M}^{\mathrm{T}}P\overline{M} & P\overline{A}_d+\overline{M}^{\mathrm{T}}P\overline{M}_d\\ * & -(1-\tau)\xi P+\overline{M}_d^{\mathrm{T}}P\overline{M}_d\end{bmatrix}$.

By Schur complement, (8) implies the negative definiteness of Δ. This together with (10) implies that for all $\left[\bar{x}^{\mathrm{T}}(t)\quad\bar{x}^{\mathrm{T}}(t-h(t))\right]^{\mathrm{T}}\neq0$, we have $\mathrm{L}V(\bar{x}(t),t)<0$. Then, by Definition 1 and [8, 9], the error system (4) with $u(t)$=0 is guaranteed to be mean-square asymptotically stable.

Now our task is to establish the L_1 performance. First inequality (8) implies that

$$\mathrm{L}V(\bar{x}(t),t)+\alpha\beta\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)+(1-\alpha)\beta\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))-\mu u^{\mathrm{T}}(t)u(t)<0$$

that is

$$\begin{aligned}&\mathrm{E}\{\mathrm{L}V(\bar{x}(t),t)\}+\alpha\beta\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}\\&+(1-\alpha)\beta\mathrm{E}\{\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\}-\mu u^{\mathrm{T}}(t)u(t)<0\end{aligned}\tag{11}$$

Assume zero initial condition (that is, $\bar{x}(t)=0$ for $t\in[-\bar{h},0]$, then we have $V(\bar{x}(t),t)|_{t=0}=0$) and $u\in L_\infty$ with $\|u\|_{L_\infty}\le1$. Then, by reason of $P>0$, we obtain (pointwise in $t\ge0$) that

$$\alpha\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}+(1-\alpha)\mathrm{E}\{\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\}\le\mu/\beta\tag{12}$$

where α is adding-weight operator.

Analyzing (12) farther by reduction to absurdity, now, we provide a contrary conclusion to (12). For $t\in[0,\infty)$, propose that exist the points satisfying

$$\alpha\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}+(1-\alpha)\mathrm{E}\{\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\}>\mu/\beta\tag{13}$$

By reason of that α is adding-weight operator, based on the hypothesis (13), there exists the points satisfying

$$\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}>\mu/\beta\quad\text{or}\quad\mathrm{E}\{\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\}>\mu/\beta$$

or

$$\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}>\mu/\beta\quad\text{and}\quad\mathrm{E}\{\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))\}>\mu/\beta$$

Since the system (4) is mean-square asymptotically stable, and for peak-bounded input, as $t\to\infty$, $\mathrm{E}\{\bar{x}(t)\}$ should be bounded, we put forward a function $f(t)=\mathrm{E}\{\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)\}$ that must appear a peak value. We propose the peak point of $f(t)$ is lie in the point $t=t^*$. So the expectation of the differential of Lyapunov function at the point as $t=t^*$ can be obtained.

$$\begin{aligned}\mathrm{E}\{\mathrm{d}V(\bar{x}(t))\}|_{t=t^*}&=\mathrm{E}\{\mathrm{d}(\bar{x}^{\mathrm{T}}(t)P\bar{x}(t))\}|_{t=t^*}+\mathrm{E}\{\xi\bar{x}^{\mathrm{T}}(t^*)P\bar{x}(t^*)\}\mathrm{d}t^*\\&-\mathrm{E}\{(1-\tau)\xi\bar{x}^{\mathrm{T}}(t^*-h(t))P\bar{x}(t^*-h(t))\}\mathrm{d}t^*\end{aligned}$$

By reason of $\mathrm{E}\{\bar{x}^{\mathrm{T}}(t^*)P\bar{x}(t^*)\}>\mathrm{E}\{\bar{x}^{\mathrm{T}}(t^*-h(t^*))P\bar{x}(t^*-h(t^*))\}$, $\mathrm{E}\{\mathrm{d}(\bar{x}^{\mathrm{T}}(t)P\bar{x}(t))\}|_{t=t^*}=0$, yields $\mathrm{E}\{\mathrm{d}V(\bar{x}(t))\}|_{t=t^*}>0$. Furthermore, by reason of $\mathrm{E}\{\mathrm{d}V(\boldsymbol{x}(t),t)\}=\mathrm{E}\{\mathrm{L}V(\boldsymbol{x}(t),t)\}\mathrm{d}t$, yields $\mathrm{E}\{\mathrm{L}V(\boldsymbol{x}(t),t)\}|_{t=t^*}>0$. According to the hypothesis (13), yields

$$\begin{aligned}&\mathrm{E}\{\mathrm{L}V(\bar{x}(t^*),t^*)\}+\alpha\beta\mathrm{E}\{\bar{x}^{\mathrm{T}}(t^*)P\bar{x}(t^*)\}\\&+(1-\alpha)\beta\mathrm{E}\{\bar{x}^{\mathrm{T}}(t^*-h(t^*))P\bar{x}(t^*-h(t^*))\}-\mu u^{\mathrm{T}}(t^*)u(t^*)>0\end{aligned}\tag{14}$$

The inequality (14) is in contradiction with the inequality (11), so the hypothesis (13) is not tenable. We can conclude that the inequality (12) is accurate.

Performing a Schur complement operation to (9) yields that there exists certain $\kappa>0$ satisfying

$$\begin{bmatrix}-\alpha\beta P & 0 & 0\\ 0 & -(1-\alpha)\beta P & 0\\ 0 & 0 & -(\gamma-\mu)I\end{bmatrix}+\frac{1}{\gamma-\kappa}\left[\overline{C}\quad\overline{C}_d\quad\overline{D}\right]^{\mathrm{T}}\left[\overline{C}\quad\overline{C}_d\quad\overline{D}\right]<0$$

Pointwise in $t\ge0$ and for all $\|u\|_{L_\infty}\le1$, we can obtain

$$\begin{aligned}&\mathrm{E}\{|e(t)|^2\}=\mathrm{E}\{e^{\mathrm{T}}(t)e(t)\}\\&\le(\gamma-\kappa)\mathrm{E}\{\alpha\beta\bar{x}^{\mathrm{T}}(t)P\bar{x}(t)+(1-\alpha)\beta\bar{x}^{\mathrm{T}}(t-h(t))P\bar{x}(t-h(t))+(\gamma-\mu)u^{\mathrm{T}}(t)u(t)\}\\&\le\gamma(\gamma-\kappa)\end{aligned}$$

Taking the supremum over $t\ge0$ yields $\|e(t)\|_{\mathrm{E}_\infty}<\gamma\|u(t)\|_{L_\infty}$ for all nonzero $u\in L_\infty$. Consequently, the conditions (8) and (9) ensure that the error system (4) is mean-square asymptotically stable and the maximal peak-to-peak gain is smaller than γ and the proof is completed.

Remark 2: Note that conditions (8) and (9) are LMIs when α, β and ξ are fixed. β must lies inside the interval $(0,c)$ (where $c=\min\left[(-2(\max Re(\lambda(A)))-\xi)/\alpha,\ (1-\tau)\xi/(1-\alpha)\right]$) to assure a positive definite solution to (8). In addition, a three dimensional line search on α, β and ξ must be performed in order to obtain tighter bound γ.

Remark 3: It is important to note that the L_1 performance criterion obtained in Theorem 1 includes several versions as its special cases. In the following, we will present L_1 performance criterion for delay systems without taking into account the stochastic issue, and for stochastic systems without considering the delay factor.

Corollary 1: Consider the following time-delay system

$$\begin{aligned}\dot{\bar{x}}(t)&=\overline{A}\bar{x}(t)+\overline{A}_d\bar{x}(t-h(t))+\overline{B}u(t)\\e(t)&=\overline{C}\bar{x}(t)+\overline{C}_d\bar{x}(t-h(t))+\overline{D}u(t)\\\bar{x}(t)&=\varphi(t)\quad\forall t\in[-\bar{h},0]\end{aligned}\tag{15}$$

If there exist matrices $0<P\in R^{(n+l)\times(n+l)}$, and scalars $0<\alpha<1$, $\beta\in R^+$, $\xi\in R^+$ and $\mu\in R$ satisfying

$$\begin{bmatrix}\overline{A}^{\mathrm{T}}P+P\overline{A}+\xi P+\alpha\beta P & P\overline{A}_d & P\overline{B}\\ * & -(1-\tau)\xi P+(1-\alpha)\beta P & 0\\ * & * & -\mu I\end{bmatrix}<0$$

$$\begin{bmatrix} -\alpha\beta P & 0 & 0 & \overline{C}^{\mathrm{T}} \\ * & -(1-\alpha)\beta P & 0 & \overline{C}_d^{\mathrm{T}} \\ * & * & -(\gamma-\mu)I & \overline{D}^{\mathrm{T}} \\ * & * & * & -\gamma I \end{bmatrix} < 0$$

Then, the time-delay system (15) is asymptotically stable and $\|e(t)\|_{L_\infty} < \gamma \|u(t)\|_{L_\infty}$.

Corollary 2: Consider the following stochastic system

$$\mathrm{d}\overline{x}(t) = [\overline{A}\overline{x}(t) + \overline{B}u(t)]\mathrm{d}t + [\overline{M}\overline{x}(t) + \overline{N}u(t)]\mathrm{d}w(t) \qquad (16)$$
$$e(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

If there exist matrix $0 < P \in R^{(n+l)\times(n+l)}$ and scalars $\beta \in R^+$, and $\mu \in R$ satisfying

$$\begin{bmatrix} -P & P\overline{M} & P\overline{N} \\ * & \overline{A}^{\mathrm{T}}P + P\overline{A} + \beta P & P\overline{B} \\ * & * & -\mu I \end{bmatrix} < 0, \quad \begin{bmatrix} -\beta P & 0 & \overline{C}^{\mathrm{T}} \\ * & -(\gamma-\mu)I & \overline{D}^{\mathrm{T}} \\ * & * & -\gamma I \end{bmatrix} < 0$$

Then, the stochastic system (16) is mean-square asymptotically stable and $\|e(t)\|_{E_\infty} < \gamma \|u(t)\|_{L_\infty}$.

IV. Parameterization of L_1 Reduced-Order Models

The following Theorem provides sufficient conditions for the existence of robust L_1 reduced-order models for stochastic systems with state delay.

Theorem 2: Consider system (1), given $\alpha \in (0,1)$, $\beta \in R^+$, $\xi \in R^+$. Then an admissible L_1 reduced-order system (3) exists if there exists appropriately dimensioned matrices $P > 0$, $X > 0$ and scalars $\mu \in R$, $\varepsilon_i > 0$ ($i = 1,\cdots,6$), $\sigma_1 > 0$, $\sigma_3 > 0$ satisfying

$$\begin{bmatrix} -ZXZ^{\mathrm{T}} & Z\overline{M}_0XZ^{\mathrm{T}} & Z\overline{M}_{d0} & N & 0 & 0 & 0 & 0 & U_2 & U_2 \\ * & \Gamma_{2,2} & Z\overline{A}_{d0} & B & ZXG_1^{\mathrm{T}} & ZXG_1^{\mathrm{T}} & U_1 & U_1 & 0 & 0 \\ * & * & \Gamma_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\mu I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\sigma_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\sigma_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix} < 0 \qquad (17)$$

$$\begin{bmatrix} -X & \overline{M}_0Z^{\mathrm{T}} & \overline{M}_{d0}Z^{\mathrm{T}} & 0 & 0 & K_2 & K_2 \\ * & \Theta_{2,2} & ZP\overline{A}_{d0}Z^{\mathrm{T}} & ZPK_1 & ZPK_1 & 0 & 0 \\ * & * & \Theta_{3,3} & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix} < 0 \qquad (18)$$

$$\begin{bmatrix} -\alpha\beta P + \varepsilon_5 G_1^{\mathrm{T}}G_1 & 0 & 0 \\ * & -(1-\alpha)\beta P + \varepsilon_6 G_2^{\mathrm{T}}G_2 & 0 \\ * & * & -(\gamma-\mu)I \end{bmatrix} < 0 \qquad (19)$$

$$\begin{bmatrix} -\alpha\beta ZPZ^{\mathrm{T}} + \varepsilon_5 V_1^{\mathrm{T}}V_1 & 0 & C_0^{\mathrm{T}} & 0 & 0 \\ * & -(1-\alpha)\beta ZPZ^{\mathrm{T}} + \varepsilon_6 V_2^{\mathrm{T}}V_2 & C_{d0}^{\mathrm{T}} & 0 & 0 \\ * & * & -\gamma I & U_3 & U_3 \\ * & * & * & -\varepsilon_5 I & 0 \\ * & * & * & * & -\varepsilon_6 I \end{bmatrix} < 0 \qquad (20)$$

$$PX = I,\ \varepsilon_1\sigma_1 = 1,\ \varepsilon_3\sigma_3 = 1 \qquad (21)$$

Furthermore, if we can obtain a feasible solution $(P, X, \mu, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \sigma_1, \sigma_3)$ to (17)-(21), then the system matrices of an admissible L_1 reduced-order system (3) can be given by

$$\Xi_1 = \begin{bmatrix} \hat{M} & \hat{M}_d & \hat{N} \\ \hat{A} & \hat{A}_d & \hat{B} \end{bmatrix} = -\Pi_1^{-1}S_1^{\mathrm{T}}\Lambda_1H_1^{\mathrm{T}}(H_1\Lambda_1H_1^{\mathrm{T}})^{-1} + \Pi_1^{-1}\Gamma_1^{\frac{1}{2}}W_1(H_1\Lambda_1H_1^{\mathrm{T}})^{-\frac{1}{2}} \qquad (22)$$

$$\Xi_2 = \begin{bmatrix} \hat{C} & \hat{C}_d & \hat{D} \end{bmatrix} = -\Pi_2^{-1}S_2^{\mathrm{T}}\Lambda_2H_2^{\mathrm{T}}(H_2\Lambda_2H_2^{\mathrm{T}})^{-1} + \Pi_2^{-1}\Gamma_2^{\frac{1}{2}}W_2(H_2\Lambda_2H_2^{\mathrm{T}})^{-\frac{1}{2}} \qquad (23)$$

where

$$\Lambda_1 = (S_1\Pi_1^{-1}S_1^{\mathrm{T}} - L_{u1})^{-1} > 0,\ \Gamma_1 = \Pi_1 - S_1^{\mathrm{T}}(\Lambda_1 - \Lambda_1H_1^{\mathrm{T}}(H_1\Lambda_1H_1^{\mathrm{T}})^{-1}H_1\Lambda_1)S_1$$
$$\Lambda_2 = (S_2\Pi_2^{-1}S_2^{\mathrm{T}} - L_{u2})^{-1} > 0,\ \Gamma_2 = \Pi_2 - S_2^{\mathrm{T}}(\Lambda_2 - \Lambda_2H_2^{\mathrm{T}}(H_2\Lambda_2H_2^{\mathrm{T}})^{-1}H_2\Lambda_2)S_2$$

Π_1, Π_2, W_1, W_2 are any appropriately dimensioned matrices satisfying $\Pi_1 > 0$, $\Pi_2 > 0$ and $\|W_1\| < 1$, $\|W_2\| < 1$.

$$\overline{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}, \overline{A}_{d0} = \begin{bmatrix} A_{d0} & 0 \\ 0 & 0 \end{bmatrix}, \overline{B}_0 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \overline{M}_0 = \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}, \overline{N}_0 = \begin{bmatrix} N \\ 0 \end{bmatrix} \qquad (24)$$

$$\overline{M}_{d0} = \begin{bmatrix} M_{d0} & 0 \\ 0 & 0 \end{bmatrix}, \overline{C}_0 = \begin{bmatrix} C_0 & 0 \end{bmatrix}, \overline{C}_{d0} = \begin{bmatrix} C_{d0} & 0 \end{bmatrix}, \overline{D}_0 = D \qquad (25)$$

$$U = \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, F_1 = -I \qquad (26)$$

$$K_1 = \begin{bmatrix} U_1 \\ 0 \end{bmatrix}, K_2 = \begin{bmatrix} U_2 \\ 0 \end{bmatrix}, K_3 = \begin{bmatrix} U_3 \\ 0 \end{bmatrix}, G_1 = \begin{bmatrix} V_1 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} V_2 & 0 \end{bmatrix}, Z = \begin{bmatrix} I_{n\times n} & 0_{n\times l} \end{bmatrix}$$

$$\Gamma_{2,2} = Z\left(X\overline{A}_0^{\mathrm{T}} + \overline{A}_0X + \xi X + \alpha\beta X\right)Z^{\mathrm{T}}$$
$$\Gamma_{3,3} = -(1-\tau)\xi P + (1-\alpha)\beta P + (\varepsilon_2 + \varepsilon_4)G_2^{\mathrm{T}}G_2$$
$$\Theta_{2,2} = Z\left(\overline{A}_0^{\mathrm{T}}P + P\overline{A}_0 + \xi P + \alpha\beta P\right)Z^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_3)V_1^{\mathrm{T}}V_1$$
$$\Theta_{3,3} = Z\left(-(1-\tau)\xi P + (1-\alpha)\beta P\right)Z^{\mathrm{T}} + (\varepsilon_2 + \varepsilon_4)V_2^{\mathrm{T}}V_2$$

$$L_{u1} = \begin{bmatrix} \Sigma_1 & \overline{M}_0 & \overline{M}_{d0} & \overline{N}_0 \\ * & \Sigma_2 & P\overline{A}_{d0} & P\overline{B}_0 \\ * & * & \Sigma_3 & 0 \\ * & * & * & -\mu I \end{bmatrix} \qquad (27)$$

$$L_{u2} = \begin{bmatrix} -\alpha\beta P + \varepsilon_5 G_1^{\mathrm{T}}G_1 & 0 & 0 & \overline{C}_0^{\mathrm{T}} \\ * & -(1-\alpha)\beta P + \varepsilon_6 G_2^{\mathrm{T}}G_2 & 0 & \overline{C}_{d0}^{\mathrm{T}} \\ * & * & -(\gamma-\mu)I & \overline{D}_0^{\mathrm{T}} \\ * & * & * & -\gamma I + (\varepsilon_5^{-1} + \varepsilon_6^{-1})U_3U_3^{\mathrm{T}} \end{bmatrix} \qquad (28)$$

Where $\Sigma_1 = -P^{-1} + (\varepsilon_3^{-1} + \varepsilon_4^{-1})K_2K_2^{\mathrm{T}}$, $\Sigma_2 = \overline{A}_0^{\mathrm{T}}P + P\overline{A}_0 + (\varepsilon_1^{-1} + \varepsilon_2^{-1})PK_1K_1^{\mathrm{T}}P + \xi P + \alpha\beta P + (\varepsilon_1 + \varepsilon_3)G_1^{\mathrm{T}}G_1$, $\Sigma_3 = -(1-\tau)\xi P + (1-\alpha)\beta P + (\varepsilon_2 + \varepsilon_4)G_2^{\mathrm{T}}G_2$.

$$S_1 = \begin{bmatrix} E_1^{\mathrm{T}} & E_2^{\mathrm{T}}P & 0 & 0 \end{bmatrix}^{\mathrm{T}},\ H_1 = \begin{bmatrix} 0 & U & V & T \end{bmatrix} \qquad (29)$$
$$S_2 = \begin{bmatrix} 0 & 0 & 0 & -I \end{bmatrix}^{\mathrm{T}},\ H_2 = \begin{bmatrix} U & V & T & 0 \end{bmatrix} \qquad (30)$$

Proof: From Theorem 1, we know that there exists a reduced-order system (3) such that the error system (4) is mean-square asymptotically stable and has a guaranteed L_1 performance γ if there exist matrices $P > 0$ and $\mu \in \mathrm{R}$ satisfying (8) and (9). We define the following matrices

$$\Delta\overline{A} = \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \Delta\overline{A}_d = \begin{bmatrix} \Delta A_d & 0 \\ 0 & 0 \end{bmatrix}, \Delta\overline{M} = \begin{bmatrix} \Delta M & 0 \\ 0 & 0 \end{bmatrix}, \Delta\overline{M}_d = \begin{bmatrix} \Delta M_d & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Delta\overline{C} = \begin{bmatrix} \Delta C & 0 \end{bmatrix},\ \Delta\overline{C}_d = \begin{bmatrix} \Delta C_d & 0 \end{bmatrix}.$$

Then rewrite (5) and (6) in the following form

$\overline{A} = \overline{A}_0 + \Delta\overline{A} + E_2\Xi_1 U$, $\overline{A}_d = \overline{A}_{d0} + \Delta\overline{A}_d + E_2\Xi_1 V$, $\overline{B} = \overline{B}_0 + E_2\Xi_1 T$

$\overline{M} = \overline{M}_0 + \Delta\overline{M} + E_1\Xi_1 U$, $\overline{M}_d = \overline{M}_{d0} + \Delta\overline{M}_d + E_1\Xi_1 V$, $\overline{N} = \overline{N}_0 + E_1\Xi_1 T$

$\overline{C} = \overline{C}_0 + \Delta\overline{C} + F_1\Xi_2 U$, $\overline{C}_d = \overline{C}_{d0} + \Delta\overline{C}_d + F_1\Xi_2 V$, $\overline{D} = \overline{D}_0 + F_1\Xi_2 T$

where $\overline{A}_0, \overline{A}_{d0}, \overline{B}_0, \overline{M}_0, \overline{M}_{d0}, \overline{N}_0, \overline{C}_0, \overline{C}_{d0}, \overline{D}_0, E_1, E_2, F_1, \Xi_1, \Xi_2$, U, V, T are defined in (22)-(26).

According to Lemma 2 we have

$$\Delta\overline{A}^{\mathrm{T}}P + P\Delta\overline{A} \le \varepsilon_1^{-1}PK_1K_1^{\mathrm{T}}P + \varepsilon_1 G_1^{\mathrm{T}}G_1,\ \begin{bmatrix} 0 & P\Delta\overline{A}_d \\ \Delta\overline{A}_d^{\mathrm{T}}P & 0 \end{bmatrix} \le \begin{bmatrix} \varepsilon_2^{-1}PK_1K_1^{\mathrm{T}}P & 0 \\ 0 & \varepsilon_2 G_2^{\mathrm{T}}G_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \Delta\overline{M} \\ \Delta\overline{M}^{\mathrm{T}} & 0 \end{bmatrix} \le \begin{bmatrix} \varepsilon_3^{-1}K_2K_2^{\mathrm{T}} & 0 \\ 0 & \varepsilon_3 G_1^{\mathrm{T}}G_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \Delta\overline{M}_d \\ 0 & 0 & 0 \\ \Delta\overline{M}_d^{\mathrm{T}} & 0 & 0 \end{bmatrix} \le \begin{bmatrix} \varepsilon_4^{-1}K_2K_2^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_4 G_2^{\mathrm{T}}G_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \Delta\overline{C}_d^{\mathrm{T}} \\ 0 & 0 & 0 \\ \Delta\overline{C}_d & 0 & 0 \end{bmatrix} \le \begin{bmatrix} \varepsilon_6 G_2^{\mathrm{T}}G_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_6^{-1}U_3U_3^{\mathrm{T}} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & \Delta\overline{C}^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta\overline{C} & 0 & 0 & 0 \end{bmatrix} \le \begin{bmatrix} \varepsilon_5 G_1^{\mathrm{T}}G_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_5^{-1}U_3U_3^{\mathrm{T}} \end{bmatrix}$$

Noticing (8), (9) can be rewritten as

$$L_{u1} + S_1\Xi_1 H_1 + (S_1\Xi_1 H_1)^{\mathrm{T}} < 0 \tag{31}$$

$$L_{u2} + S_2\Xi_2 H_2 + (S_2\Xi_2 H_2)^{\mathrm{T}} < 0 \tag{32}$$

where $L_{ui}, S_i, H_i, i = 1, 2$ are defined in (27)-(30). By Lemma 1, the necessary and sufficient conditions for LMIs (31) and (32) to have solutions are

$S_1^{\perp}L_{u1}S_1^{\perp\mathrm{T}} < 0$, $H_1^{\mathrm{T}\perp}L_{u1}H_1^{\mathrm{T}\perp\mathrm{T}} < 0$; $S_2^{\perp}L_{u2}S_2^{\perp\mathrm{T}} < 0$, $H_2^{\mathrm{T}\perp}L_{u2}H_2^{\mathrm{T}\perp\mathrm{T}} < 0$.

Note that $S_i^{\perp}$ and $H_i^{\mathrm{T}\perp}$, $i = 1, 2$ can be selected as follows:

$$S_1^{\perp} = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & ZP^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},\ H_1^{\mathrm{T}\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \end{bmatrix}$$

$$S_2^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix},\ H_2^{\mathrm{T}\perp} = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

By Schur complement and the conditions of $P^{-1} = X$, $\varepsilon_1^{-1} = \sigma_1$, $\varepsilon_3^{-1} = \sigma_3$, we obtain that $S_1^{\perp}L_{u1}S_1^{\perp\mathrm{T}} < 0$ and $H_1^{\mathrm{T}\perp}L_{u1}H_1^{\mathrm{T}\perp\mathrm{T}} < 0$ are equivalent to (17) and (18), $S_2^{\perp}L_{u2}S_2^{\perp\mathrm{T}} < 0$ and $H_2^{\mathrm{T}\perp}L_{u2}H_2^{\mathrm{T}\perp\mathrm{T}} < 0$ are equivalent to (19) and (20). From the matrix inequalities (17)-(21), if there exist matrices $P > 0$ satisfying (8) and (9), there exist matrices Ξ_1 and Ξ_2 such that (8) and (9) hold. By Theorem 1, we can obtain that the error system (4) is mean square asymptotically stable and the maximal peak-to-peak gain is less than γ. In addition, all the parameters of the reduced-order models satisfying (8) and (9) can be obtained by (22) and (23). This completes the proof.

It should be noted that the obtained conditions in Theorem 2 are not LMI conditions due to the equations in (21) in despite of a line search on α, β and ξ. However, with the result of CCL algorithm in [6], we can solve this feasibility problem by formulating it into a nonlinear optimization problem subject to LMI constraints.

Problem 1: min $Trace(PX + \varepsilon_1\sigma_1 + \varepsilon_3\sigma_3)$ subject to (17)-(20) and

$$\begin{bmatrix} P & I \\ I & X \end{bmatrix} \ge 0,\ \begin{bmatrix} \varepsilon_1 & 1 \\ 1 & \sigma_1 \end{bmatrix} \ge 0,\ \begin{bmatrix} \varepsilon_3 & 1 \\ 1 & \sigma_3 \end{bmatrix} \ge 0 \tag{33}$$

According to [6], we can readily modify Algorithm 1 in [6] to solve the above nonlinear problem.

Algorithm 1:

1) Given initial constant matrices, the order of the reduced-order model l and prescribed error $\gamma > 0$; Given the max iterative times Ω; α, β and ξ are fixed respectively in the interval $(0, 1)$, $(0, c)$ and $(0, -2(\max Re(\lambda(A))))$.

2) Set k=0 and choose arbitrarily an initial guess $(P, X, \mu, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \sigma_1, \sigma_3)^0$ satisfying (17)-(20) and (33).

3) Solve the following LMI problem

$$\min \quad Trace(PX^k + P^kX) + \varepsilon_1\sigma_1^k + \varepsilon_1^k\sigma_1 + \varepsilon_3\sigma_3^k + \varepsilon_3^k\sigma_3$$

subject to (17)-(20) and (33).

4) Denote $(P, X, \mu, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \sigma_1, \sigma_3)^{k+1}$ as the minimizer and compute the minimum

$$f_{k+1}(P, X, \varepsilon_1, \sigma_1, \varepsilon_3, \sigma_3) = Trace(PX^{k+1} + P^{k+1}X) + \varepsilon_1\sigma_1^{k+1} + \varepsilon_1^{k+1}\sigma_1 + \varepsilon_3\sigma_3^{k+1} + \varepsilon_3^{k+1}\sigma_3$$

5) If $|f_{k+1} - 2(n+l) - 4| < \delta$ where $\delta > 0$ is a sufficiently small prescribed scalar to control the convergence accuracy, then go to 6); otherwise set k=k+1, it has no results and exist as $k > \Omega$, and it goes to 3) as $k \le \Omega$.

6) Construct a reduced-order model based on (22) and (23).

Algorithm 1 can be used to solve the feasibility problem in Theorem 2 for a given constant γ. However, it is not difficult to further modify Algorithm 1 to obtain the minimum value of γ by adding four-dimensional search technique about α, β, ξ and γ.

V. DELAY-FREE REDUCED-ORDER MODEL

In this section, we will consider the problem of L_1 model reduction by delay-free reduced-order models, and use the following reduced-order system to approximate system (4).

$$\begin{aligned} \mathrm{d}\hat{x}(t) &= [\hat{A}\hat{x}(t) + \hat{B}u(t)]\mathrm{d}t + [\hat{M}\hat{x}(t) + \hat{N}u(t)]\mathrm{d}w(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t) \end{aligned} \tag{34}$$

The error system can be given by (4), but different matrices

$$\overline{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \overline{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \overline{B} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \overline{M} = \begin{bmatrix} M & 0 \\ 0 & \hat{M} \end{bmatrix}, \overline{M}_d = \begin{bmatrix} M_d & 0 \\ 0 & 0 \end{bmatrix},$$

$$\overline{N} = \begin{bmatrix} N^T & \hat{N}^T \end{bmatrix}^T, \overline{C} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \overline{C}_d = \begin{bmatrix} C_d & 0 \end{bmatrix}, \overline{D} = D - \hat{D}.$$

The following theorem gives the solution to above problem.

Theorem 3: Consider system (1). Given $\alpha \in (0, 1)$, $\beta \in R^+$, $\xi \in R^+$, then an admissible delay-free L_1 reduced-order model (34) exists if there exist appropriately dimensioned matrices $P > 0$, $X > 0$ and scalars $\mu \in R$, $\varepsilon_i > 0$ ($i = 1, \cdots, 6$),

$\sigma_1 > 0$, $\sigma_3 > 0$ satisfying (17), (19), (21) and

$$\begin{bmatrix} -X & \overline{M}_0 Z^{\mathrm{T}} & \overline{M}_{d0} & 0 & 0 & K_2 & K_2 \\ * & \Theta_{2,2} & ZP\overline{A}_{d0} & ZPK_1 & ZPK_1 & 0 & 0 \\ * & * & \widetilde{\Theta}_{3,3} & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha\beta ZPZ^{\mathrm{T}} + \varepsilon_5 V_1^{\mathrm{T}} V_1 & 0 & C_0^{\mathrm{T}} & 0 & 0 \\ * & -(1-\alpha)\beta P + \varepsilon_6 G_2^{\mathrm{T}} G_2 & \overline{C}_{d0}^{\mathrm{T}} & 0 & 0 \\ * & * & -\gamma I & U_3 & U_3 \\ * & * & * & -\varepsilon_5 I & 0 \\ * & * & * & * & -\varepsilon_6 I \end{bmatrix} < 0$$

Furthermore, if we can obtain a feasible solution $(P, X, \mu, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \sigma_1, \sigma_3)$ to the above conditions, then the matrices of an admissible delay-free L_1 reduced-order model (34) can be given by

$$\widetilde{\Xi}_1 = \begin{bmatrix} \hat{M} & \hat{N} \\ \hat{A} & \hat{B} \end{bmatrix} = -\Pi_1^{-1} S_1^{\mathrm{T}} \Lambda_1 H_1^{\mathrm{T}} (H_1 \Lambda_1 H_1^{\mathrm{T}})^{-1} + \Pi_1^{-1} \Gamma_1^{\frac{1}{2}} W_1 (H_1 \Lambda_1 H_1^{\mathrm{T}})^{-\frac{1}{2}}$$

$$\widetilde{\Xi}_2 = \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} = -\Pi_2^{-1} S_2^{\mathrm{T}} \Lambda_2 H_2^{\mathrm{T}} (H_2 \Lambda_2 H_2^{\mathrm{T}})^{-1} + \Pi_2^{-1} \Gamma_2^{\frac{1}{2}} W_2 (H_2 \Lambda_2 H_2^{\mathrm{T}})^{-\frac{1}{2}}$$

Here, all matrices have the same forms as in Theorem 2, but with different matrices given by

$$\widetilde{\Theta}_{3,3} = -(1-\tau)\xi P + (1-\alpha)\beta P + (\varepsilon_2 + \varepsilon_4) G_2^{\mathrm{T}} G_2$$

$$U = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The theorem can be proved with similar lines as the proof of Theorem 2, and then omitted here.

VI. Numerical Example

In this section, we present an illustrative example to demonstrate the applicability of the proposed approach. Consider an uncertain stochastic time-delay system with

$$\boldsymbol{A} = \begin{bmatrix} -11 & 1 & 1 & 1 & 0 & 0 \\ 1 & -10 & -2 & 1 & 0.3 & 0 \\ -1 & 0 & -7 & 1 & 0 & 1 \\ 0 & 1 & 0 & -4 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 & 0 & -10 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 0.5 & 0 & 1 & 0 & -1 \\ 0 & -1 & 2 & 0 & 0.1 & 0 \\ 0.5 & -1 & 1 & 0.2 & 0 & 0.2 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ -0.5 & 0 & 0.2 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 1 \\ 0 & 1 & 0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0.5 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 0.1 & 0 & 2 & 0.4 & 0 \\ 0.3 & -1 & -0.2 & 0.1 & 0.3 & 0 \\ -0.2 & 0 & -0.5 & 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 & -0.4 & 0 & 0.2 \\ 0 & 0.2 & 0 & 0 & -0.3 & 0 \\ 0.1 & 0 & -0.1 & 0 & 0 & -0.5 \end{bmatrix}$$

$$M_d = \begin{bmatrix} 0.1 & 0.1 & 0.2 & 0.3 & 0 & -0.3 \\ 0 & -0.2 & 0.2 & 0 & 0.1 & 0 \\ 0.3 & -0.2 & 0.2 & 0.1 & 0 & 0 \\ 0.2 & 0 & 0 & 0.1 & 0 & -0.1 \\ -0.1 & 0 & 0.2 & 0 & -0.5 & 0 \\ 0.1 & 0 & -0.1 & 0 & 0 & -0.5 \end{bmatrix}, \quad N = \begin{bmatrix} 0.1 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.1 \\ 0 & 0.1 & 0.4 \\ -0.3 & 0 & 0.3 \\ 0.2 & 0 & 0.2 \\ 0.1 & 0 & 0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 & 0.3 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 & 5 \\ 0 & 5 & 5 \end{bmatrix}$$

where $U_1 = [0.1 \ \ 0 \ \ 0.1 \ \ 0.1 \ \ 0.1 \ \ 0.1]^{\mathrm{T}}$, $U_2 = [0 \ \ 0.1 \ \ 0 \ \ 0.1 \ \ 0 \ \ 0.1]^{\mathrm{T}}$, $U_3 = [0 \ \ 0.1]^{\mathrm{T}}$, $V_1 = [0.1 \ \ 0 \ \ 0.1 \ \ 0.1 \ \ 0 \ \ 0.1]$, $V_2 = [0.1 \ \ 0.1 \ \ 0 \ \ 0 \ \ 0.1 \ \ 0.2]$, $F = \sin\theta$.

It can be shown that this stochastic time-delay system is mean-square asymptotically stable. The time delay $h(t)$ is supposed to be equal to 1.6, $\tau = 0$. Here we are interested in constructing a first-order system (3) to approximate the above system. By solving the nonconvex feasibility problem in Theorem 2 with the help Algorithm 1, the subminimum L_1 performance for the error system is found to be $\gamma^* = 3.7298$ $(\alpha = 0.5, \beta = 0.91, \xi = 3.30)$. Choose $\Pi_i = 0.02 I_{2\times 2}$, $W_i = [0.99 I_{2\times 2} \ \ 0_{2\times 3}]$, $i = 1, 2$. The matrices of an admissible first-order L_1 reduced-order model (3) can be given by

$$\Xi_1 = \left[\begin{array}{c|c|c} \hat{M} & \hat{M}_d & \hat{N} \\ \hline \hat{A} & \hat{A}_d & \hat{B} \end{array}\right] = \left[\begin{array}{c|c|ccc} 1.5687 & -0.5570 & -0.1450 & 0.0795 & 0.2711 \\ \hline -6.7790 & 4.1695 & -0.3386 & 0.1622 & 0.4475 \end{array}\right]$$

$$\Xi_2 = \left[\begin{array}{c|c|c} \hat{C} & \hat{C}_d & \hat{D} \end{array}\right] = \left[\begin{array}{c|c|ccc} 2.4458 & 0.3320 & 4.9994 & 0.0006 & 5.0000 \\ -0.2516 & 3.3331 & 0.0003 & 4.9993 & 4.9997 \end{array}\right]$$

It can be seen that the reduced-order system approximate the original system under admissible error.

VII. Concluding Remarks

The problem of L_1 model reduction for continuous uncertain stochastic systems with state delay is investigated in this paper. The peak-to-peak gain criterion is first established for stochastic systems with state delay. Sufficient conditions are obtained for the existence of desired reduced-order models. Since these obtained conditions are not expressed as strict LMIs, the CCL method is exploited to cast them into nonlinear minimization problems subject to LMI constraints, which can be readily solved in the Matlab environment. Numerical example demonstrates the validity of the theoretical results.

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