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Complexity-Reduced Guaranteed Cost Control Design for Delayed Uncertain Symmetrically Connected Systems

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Abstract— **Complexity reduced guaranteed cost control design problem is considered for a class of nonlinear symmetrically connected uncertain state-delayed systems. Nonlinear systems with identical nominal linear both subsystems and symmetric connections are supposed. Point time-delays and norm bounded parametric uncertainties are considered. A reduced-order guaranteed cost control problem is constructed first. The dimension of its state equals to the dimension of a subsystem in the original system. Then, a memoryless guaranteed cost dynamic output feedback controller is designed for the reduced problem using the LMI approach. It is proved that when this controller is implemented into each subsystem of the original system then such decentralized output controller is a guaranteed cost controller for the overall system.**

I. Introduction

Guaranteed cost control approach has the advantage of providing an upper bound on a given performance index. Recent advances in the theory of linear matrix inequality have allowed a revising of the guaranteed cost control approach for various classes of dynamic systems. Guaranteed cost control approach offers to employ this advantage also for large-scale systems. Symmetrically coupled systems belong to an important class of large-scale systems. Generally, they are characterized as coupled systems with identical subsystems and symmetric connections. Recently, guaranteed cost control approach has been extended on various classes of time-delayed uncertain systems to reflect real world phenomena in a more realistic way.

A. Relevant references

Guaranteed cost control design for delayed uncertain systems has been recently considered usually within the LMI approach. The state delay independent feedback considers [1]. The output delay independent feedback design presents [2]. Delay dependent output feedback includes [3]. The decentralized delay-less control design for power systems is studied in [4].

Motivation for studying a class of symmetrically coupled systems arises in very different application areas. Real world system examples can be found in parallel systems such as flow splitting parallel reactors with combined precooling [5], electric power systems operating in parallel [6], industrial manipulators with several degrees of freedom [7], flexible structures [8], space crystal furnace [9] or homogeneous interconnected systems such as seismic cables [10]. Recently, the problem of formations of vehicles in cyclic pursuit has been solved using circulant matrices in [11]. More exhaustive survey of other applications outside mainstream control interests is given in [5], [12].

Complexity-reduced control design for delay-less uncertain parameter symmetric composite system state space models considered [5], [13]-[19]. The problem of robust stabilization for a class of uncertain time delay systems using output feedback solved [20], [21]. This approach has been extended on a class of symmetric composite systems in [22], [23].

One of new open research issues within the guaranteed cost control approach for the class of symmetric composite systems is the extension to both delay–less and delayed systems. The present paper extends the results in [1], [2], [3] as well as [13], [14], [18], [22], [23] on the case of complexity-reduced dynamic output feedback controller design within the guaranteed cost control approach when considering a class of continuous-time symmetrically connected uncertain systems with state delays. Moreover, the results presented in [13], [14], [18], [22], [23] consider models with rank-one uncertainties which are overbounded by [24] to solve the problem of complexity reduced control design. This paper considers directly norm bounded uncertainties. It essentially simplifies the solution. To the author's knowledge, such extension has not been considered up to now.

B. Outline of the paper

This paper presents complexity–reduced decentralized guaranteed cost controller design for a class of nonlinear continuous-time uncertain time-delayed symmetrically coupled systems. The original guaranteed cost control problem is reduced to a low order control design guaranteed cost control problem preserving structural properties of the overall problem. The order of the reduced systems equals to the order of a subsystem's order of the overall original system. Guaranteed cost memoryless dynamic controller is designed for this reduced problem using the well known LMI approach. It is proved that when such controller is implemented into each subsystem, then the resulting decentralized controller is a guaranteed cost controller for the overall system. The presented procedure leads to an essential reduction of control design complexity.

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II. PROBLEM FORMULATION

The uncertain linear symmetric composite system under consideration consists of N subsystems, where the i–th subsystem is described as follows

$$\begin{aligned}\dot{x}_i &= \Big[A + \Delta A_i(t)\Big]x_i + \Big[A_{di} + \Delta A_{di}(t)\Big]x_{di} + \\ &\quad \Big[B + \Delta B_i(t)\Big]u_i + s_{zi} \\ y_i &= (C + \Delta C_i(t))x_i \\ &\quad x_i(t_o) = \Phi_i(t_o) \quad \forall t_o \in [-d, 0] \quad i = 1, \ldots, N\end{aligned} \tag{1}$$

where x_i, u_i, s_{zi}, y_i are $n-, m-, \ p_s-, p_y-$dimensional vectors of the subsystem states, control inputs, interconnection inputs and measured outputs, respectively. $\Phi_i(t_o)$ is a given initial function. Interconnections are described in the form

$$s_{zi} = \sum_{j=1}^{N}(L_{ij}y_{zj} + L_{dij}y_{dzj}) \tag{2}$$

where y_{zj} is the p_z-dimensional vector of the interconnection output from the subsystem j which is related to the state vector in the form

$$y_{zj} = C_z x_j \qquad y_{dzj} = C_{dz}x_{dj} \tag{3}$$

$x_{dj} = x_j(t - d)$, d denotes a point time delay. The interconnection matrices L_{ij}, L_{dij} have the structure as follows:

$$\begin{aligned} L_{ii} &= 0 \qquad L_{ij} = L_q + \Delta L_{qij}(t) \\ L_{dii} &= 0 \qquad L_{dij} = L_{dq} + \Delta L_{dqij}(t), \quad (i \neq j)\end{aligned} \tag{4}$$

A, A_d, B, C, C_z and L_q, L_{dq} are constant nominal matrices, while $\Delta A_i(t), \Delta A_{di}(t), \Delta B_i(t), \Delta C_i(t)$, and $\Delta L_{qij}(t), \Delta L_{dqij}(t)$ are norm bounded uncertainties which admit the following structure:

$$\begin{aligned}\Delta A_i(t) &= D_A F_{Ai}(t) E_A \\ \Delta A_{di}(t) &= D_{dA} F_{dAi}(t) E_{dA} \\ \Delta B_i(t) &= D_B F_{Bi}(t) E_B \\ \Delta C_i(t) &= D_C F_{Ci}(t) E_C \\ \Delta L_{qij}(t) &= D_L F_{Lij}(t) E_L \\ \Delta L_{dqij}(t) &= D_{dL} F_{dLij}(t) E_{dL}\end{aligned} \tag{5}$$

$D_A, ..., E_{dL}$ are constant matrices. Uncertainties are lumped in unknown Lebesgue measurable functions $F_{(*)}$ satisfying the bounds. $F_{(*)}^T F_{(*)} \leq I$ for all $t \geq 0$. I denotes a unit matrix of appropriate dimensions.

Consider the performance index which is associated with the system (1)-(5) as follows

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} \int_0^{\infty} [x_i^T R_1 x_i(t) + u_i^T R_2 u_i(t)]dt \tag{6}$$

where R_1, R_2 are given positive definite matrices.

The goal is to find global decentralized dynamic controller quadratically stabilizing the system (1)-(5) and which guarantees the upper bound of the cost (6) for any uncertainty (5). We propose the decentralized dynamic full order memoryless controller to be composed of N local feedback controllers of the form

$$\begin{aligned}\dot{\hat{x}}_i &= A_c \hat{x}_i + B_c y_i \\ u_i &= C_c \hat{x}_i \qquad i = 1, \ldots, N\end{aligned} \tag{7}$$

where $\hat{x}_i$ is the $n-$dimensional controller state of the subsystem i. A_c, B_c, C_c are the controller matrices to be determined. Notice that these matrices are identical for all subsystems, thus taking advantage of the symmetric structure of the large scale system to reduce the control design complexity.

Definition 1. Suppose there exist a controller (7) and a positive scalar γ satisfying for any point time–delay and any realization of uncertainties (5) that the closed–loop system (1)–(5), (7) is quadratically stable and the closed–loop system value of the cost function (6) satisfies $J \leqq \gamma$. Then γ is said to be a guaranteed cost and (7) is said to be a guaranteed cost controller.

The problem

The goal is to derive a complexity-reduced procedure for designing a guaranteed cost decentralized memory–less dynamic output feedback controller (7) for the delayed symmetrically connected system (1)–(5) with the performance index (6).

III. SOLUTION

In this section, we propose a constructive procedure to design the matrices A_c, B_c, and C_c. This procedure is divided into two steps: first, a reduced-order guaranteed cost control problem is constructed. The state dimension of the reduced system equals to the dimension of any subsystem in (1). The objective of such construction is to obtain a low order guaranteed cost control problem with dynamic properties which are equivalent to the overall guaranteed cost control problem (1)–(6). It is proved in Appendix that the particular structural features of (1)–(7) enable such construction. Second, a dynamic output memoryless feedback controller is considered to be designed using the standard well known LMI procedure for the reduced-order guaranteed cost control problem. If such controller exists, then it quadratically stabilizes the reduced-order control design system with guaranteed cost. When this output controller is applied to each subsystem of the original overall system (1)–(5) as a global decentralized controller with identical local controllers, it renders the overall closed–loop system (1)–(5), (7) quadratically stable with guaranteed cost equal to the guaranteed cost of the control design system multiplied by a constant depending on N in a nonlinear way.

First, introduce

$$\begin{aligned}\Delta A_a(t) &= D_a F_a(t) E_a \\ \Delta A_{da}(t) &= D_{da} F_{da} E_{da}\end{aligned} \tag{8}$$

where $D_a, ..., E_{da}$ are constant block matrices given by decomposing the matrices $\frac{N}{2}L_qC_z, \frac{N}{2}L_{dq}C_{dz}$ into the form

$$\begin{aligned}\frac{N}{2}L_qC_z &= D_aE_a \\ \frac{N}{2}L_{dq}C_{dz} &= D_{da}E_{da}\end{aligned} \tag{9}$$

$F_{(.)}(t)$ in (8) are unknown Lebesgue measurable functions satisfying $F_{(*)}(t)^TF_{(*)}(t) \leq I$ for all $t \geq 0$.

Define the n-dimensional system using the uncertainties (8) as follows

$$\begin{aligned}\dot{x}_m &= (A_m + \Delta A_m(t))x_m + (A_{dm} + \Delta A_{dm}(t))x_{dm} \\ &\quad + (B + \Delta B_m(t))u_m \\ y_m &= (C + \Delta C_m(t))x_m\end{aligned} \tag{10}$$

where the nominal matrices are defined by the expressions

$$\begin{aligned}A_m &= A + (\frac{N}{2} - 1)L_qC_z \\ A_{dm} &= A_d + (\frac{N}{2} - 1)L_{dq}C_{dz}\end{aligned} \tag{11}$$

with the uncertainties given as follows

$$\begin{aligned}\Delta A_m(t) &= D_AF_A(t)E_A + (\frac{N}{2} - 1)D_LF_L(t)E_L \\ &\quad + D_aF_a(t)E_a \\ &= \Delta A(t) + \Delta A_a(t) \\ &= D_{Am}F_{Am}(t)E_{Am} \\ \Delta A_{dm}(t) &= (\frac{N}{2} - 1)D_{dA}F_{(dA)}E_{dA} \\ &\quad + D_{da}F_{da}E_{da} \\ &= \Delta A_d(t) + \Delta A_{da}(t) \\ &= D_{Adm}F_{Adm}(t)E_{Adm} \\ \Delta B_m(t) &= D_BF_{Bm}(t)E_B = D_{Bm}F_{Bm}(t)E_{Bm} \\ \Delta C_m(t) &= D_CF_{Cm}(t)E_C = D_{Cm}F_{Cm}(t)E_{Cm}\end{aligned} \tag{12}$$

$F_{(.)}(t)$ in (12) are unknown Lebesgue measurable functions satisfying standard norm bounded conditions.

Consider the performance index which is associated with the system (10)-(12) as follows

$$J_m = \int_0^\infty [x_m^TR_1x_m(t) + u_m^TR_2u_m(t)]dt \tag{13}$$

where R_1, R_2 are positive definite matrices given in (6).

Consider a full order memoryless dynamic feedback controller for the guaranteed cost control problem (10)-(13) in the form

$$\begin{aligned}\dot{\hat{x}}_m &= A_c\hat{x}_m + B_cy_m \\ u_m &= C_c\hat{x}_m\end{aligned} \tag{14}$$

The matrices A_c, B_c, C_c can be determined by the procedure as follows. Denote $\bar{x}_m = [x_m^T, \hat{x}_m^T]^T$. The closed–loop system (10)-(12), (14) results in

$$\begin{aligned}\dot{\bar{x}}_m(t) &= (\bar{A}_m + \bar{D}_{1m}\bar{F}_{1m}(t)\bar{E}_{1m})\bar{x}_m(t) \\ &\quad + (\bar{A}_{dm} + \bar{D}_{2m}\bar{F}_{2m}(t)\bar{E}_{2m})x(t-d)\end{aligned} \tag{15}$$

where

$$\begin{aligned}&\bar{A}_m = \begin{pmatrix} A_m & BC_c \\ B_cC & A_c\end{pmatrix} \qquad \bar{A}_{dm} = \begin{pmatrix} A_{dm} \\ 0 \end{pmatrix} \\ &\bar{D}_{1m} = \begin{pmatrix} D_{Am} & D_{Bm} & 0 \\ 0 & 0 & B_cD_{Cm}\end{pmatrix} \qquad \bar{D}_{2m} = D_{Adm} \\ &\bar{E}_{1m}^T = \begin{pmatrix} E_{Am}^T & 0 & E_{Cm}^T \\ 0 & E_{Bm}^T & 0\end{pmatrix} \qquad \bar{E}_{2m} = E_{Adm} \\ &\bar{F}_{1m} = diag(F_{Am}, F_{Bm}, F_{Cm}) \qquad \bar{F}_{2m} = F_{Adm}\end{aligned} \tag{16}$$

To establish the quadratic stability consider the Lyapunov functional for the system (15)-(16)

$$\begin{aligned}V_m(\bar{x}_m, t) &= \bar{x}_m(t)^TP_m\bar{x}_m(t) \\ &\quad + \int_0^d x_m(t-s)^TP_{1m}x_m(t-s)ds\end{aligned} \tag{17}$$

where $P_m \in \Re^{2n\times 2n}$ and $P_{1m} \in \Re^{n\times n}$ are positive definite matrices.

Definition 1. The controller (14) is a quadratically stabilizing controller for the system (15)–(16 if there exist matrices $P_m > 0$ and $P_{1m} > 0$ satisfying

$$\begin{pmatrix} Y_{11m} & Y_{12m} \\ Y_{12m}^T & Y_{22m}\end{pmatrix} < 0 \tag{18}$$

where $Y_{11m} = (\bar{A}_m + \bar{D}_{1m}\bar{F}_{1m}(t)\bar{E}_{1m})^TP_m + P_m(\bar{A}_m + \bar{D}_{1m}\bar{F}_{1m}(t)\bar{E}_{1m}) + \hat{P}_m$, $Y_{12m} = P_m(\bar{A}_{dm} + \bar{D}_{2m}\bar{F}_{2m}(t)\bar{E}_{2m})$, $Y_{22m} = P_{1m}$, and $\hat{P}_m = diag(P_{1m}, 0)$.

The closed–loop system cost value has the form

$$J_m = \int_0^\infty [x_m^TR_1x_m(t) + \hat{x}_m^TC_c^TR_2C_c\hat{x}_m(t)]dt \tag{19}$$

Denote $\bar{C}_m = diag(R_{1m}^{\frac{1}{2}}, R_{2m}^{\frac{1}{2}}C_c)$.

Lemma 1. Consider the system (10)–(12 and the cost function (19). Suppose there exist a controller (14) and given matrices $P_m > 0$, $P_{1m} > 0$, $V_{1m} \geq 0$, and $V_{2m} > 0$ such that

$$Y_m(A_m) = \begin{pmatrix} Y_{11m} + \bar{C}_m^T\bar{C}_m & Y_{12m} \\ Y_{12m}^T & Y_{22m}\end{pmatrix} < 0 \tag{20}$$

Then this controller quadratically stabilizes the system (10)–(12). The closed–loop system cost (19) satisfies the bound

$$J_m \leq tr\{P_mV_m\} + tr\{P_{1m}\int_{-d}^\infty x_m(s)x_m(s)^Tds\} = \gamma_m \tag{21}$$

where $V_m = diag(V_{1m}, B_cV_{2m}B_c^T)$. (20), (21) hold for all admissible uncertainties.

Lemma 1 is based on Theorem 3.1 in [2]. This result is not convenient for computations of controller parameters.

However, applying Schur complements to (20) and applying the method of changing variables, we get two known LMIs serving for the control design. They are given by Theorem 3.2. in [2]. This result enables also to solve the problem of finding an output-feedback controller guaranteeing the cost bound to be less than some specific value [2]. Because these results are well known, they are omitted here.

To give the expression on the bounds of costs, consider

$$T_J(1,1) = 1$$
$$T_J(1,s) = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ -1 & -1 & \dots & -1 & 1 \end{pmatrix} \quad s > 1 \tag{22}$$

Denote

$$\begin{aligned} \overline{T}_J(i) &= diag[T_J(1, N-i), ..., 1], i = 0, ..., N-1 \\ G_J &= \overline{T}_J(0)\overline{T}_J(1) \quad ... \quad \overline{T}_J(N-1) \\ G_J^T G_J &= diag(2 \quad 6 \quad ... \quad N(N-1) \quad N) \\ K_J &= tr\{G_J^T G_J\} \end{aligned} \tag{23}$$

The following theorem states the main result.

Theorem. Consider the guaranteed cost control problem defined by equations (1)–(6). Construct the reduced guaranteed cost control problem defined by equations (10)–(13). Suppose the matrices A_c, B_c, C_c are selected so that (14) is a quadratically stabilizing output feedback controller for the system (10)–(12) with a guaranteed cost γ_m. Then, using the matrices A_m, A_{dm} and the matrices A_c, B_c, C_c in (7), the global closed–loop overall system (1)–(5), (7) is quadratically stable with a guaranteed cost $\gamma = K_J \gamma_m$.

Proof: See the Appendix.

Remark. Lemma 1 serves not only as a base for the control design when it is rewritten into computationally convenient well known LMIs by [2], but mainly as an assertion which is required to prove the main result presented by Theorem. Finally, the resulting reduced-order output feedback controller is implemented to each subsystem of the overall original system. The result states that the original guaranteed cost control problem can be solved using a guaranteed cost dynamic controller design for the reduced-order guaranteed cost control problem. The dimension of the reduced-order control design problem equals to a subsystem dimension. The resulting global controller is a decentralized controller with identical local controllers. Moreover, the bound on the cost is a constant determined as a nonlinear function of N multiplied by the cost bound for the reduced guaranteed cost control problem.

IV. Conclusion

The paper contributes by the procedure for a complexity–reduced guaranteed cost control design when considering a class of nonlinear uncertain state-delayed continuous-time symmetrically connected systems. The construction of a reduced guaranteed cost control problem has been presented. It has been proved that the guaranteed cost controller designed for this reduced problem can be used as a decentralized guaranteed cost controller for the overall system when implemented into each subsystem of this system.

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APPENDIX

Necessary preliminaries are introduced only to prove Theorem. First, the overall system description is given. The systems has a specific structural properties which are used to transform this systems into low order equivalent system which is convenient for the proper control design. Denote the global system description of the system (1)–(5) as follows

$$\begin{aligned}\dot{x} &= \Big[\overline{A} + \Delta\overline{A}(t)\Big]x + \Big[\overline{A}_d + \Delta\overline{A}_d(t)\Big]x_d \\ &+ \Big[\overline{B} + \Delta\overline{B}(t)\Big]u \\ y &= (\overline{C} + \Delta\overline{C}(t))x \qquad x(t_o) = \Phi_o(t_o) \quad \forall t_o \in [-d, 0]\end{aligned} \tag{24}$$

where x, u, y are $nN-, mN-, pN-$dimensional vectors of the system states, control inputs and measured outputs, respectively. $\Phi_o(t_o)$ is a given initial function. Further $x_d(t) = x(t-d)$. The nominal matrices are defined as follows

$$\begin{aligned}\overline{A} &= (\overline{A}_{ij}) \qquad \overline{A}_{ii} = A \qquad \overline{A}_{ij} = L_{ij}C_z \\ \overline{A}_d &= (\overline{A}_{dij}) \qquad \overline{A}_{dii} = A_d, \qquad \overline{A}_{dij} = L_{dij}C_{dz} \\ \overline{B} &= diag(B, ..., B) \\ \overline{C} &= diag(C, ..., C)\end{aligned} \tag{25}$$

The uncertainty terms have the form

$$\begin{aligned}\Delta\overline{A}(t) &= \overline{D}_A\overline{F}_A(t)\overline{E}_A \\ \Delta\overline{A}_d(t) &= \overline{D}_{dA}\overline{F}_{dA}(t)\overline{E}_{dA} \\ \Delta\overline{B}(t) &= \overline{D}_B\overline{F}_B(t)\overline{E}_B \\ \Delta\overline{C}(t) &= \overline{D}_C\overline{F}_C(t)\overline{E}_C\end{aligned} \tag{26}$$

The constant matrices are defined as follows

$$\begin{aligned}\overline{D}_A &= diag(\overline{D}_1, ..., \overline{D}_N) \\ &\quad \overline{D}_i = (D_L ... D_L \quad D_A \quad C_L ... D_L) \\ \overline{E}_A &= diag(\overline{E}_1 ..., \overline{E}_N) \\ &\quad \overline{E}_i = (E_L ... E_L \quad E_A \quad E_L ... E_L) \\ \overline{D}_{dA} &= diag(\overline{D}_{d1}, ..., \overline{D}_{dN}) \\ &\quad \overline{D}_{di} = (D_{dL} ... D_{dL} \quad D_{dA} \quad D_{dL} ... D_{dL}) \\ \overline{E}_{dA} &= diag(\overline{E}_{d1} ..., \overline{E}_{dN}) \\ &\quad \overline{E}_{di} = (E_{dL} ... E_{dL} \quad E_{dL} \quad E_{dL} ... E_{dL}) \\ \overline{D}_B &= diag(D_B, ..., D_B) \\ \overline{E}_B &= diag(E_B, ..., E_B) \\ \overline{D}_C &= diag(D_C, ..., D_C) \\ \overline{E}_C &= diag(E_C, ..., E_C)\end{aligned} \tag{27}$$

D_A is located at the $i-th$ position in D_i. An analogous symbols are used for the locations in E_i, D_{di}, E_{di}. The uncertainty structure is lumped in uncertainty functions in the form

$$\begin{aligned}\overline{F}_A(t) &= diag(F_{A1}, ..., F_{AN}) \\ &\quad F_{Ai} = diag(F_{Li1}, ... \\ &\qquad ..., F_{Li\,i-1}, F_{Ai}, F_{Li\,i+1}, ..., F_{LiN}) \\ \overline{F}_{dA}(t) &= diag(F_{dA1}, ..., F_{dAN}) \\ &\quad F_{dAi} = diag(F_{dLi1}, ... \\ &\qquad ..., F_{dLi\,i-1}, F_{dAi}, F_{dLi\,i+1}, ..., F_{dLiN}) \\ \overline{F}_B(t) &= diag(F_{B1}, ..., F_{BN}) \\ \overline{F}_C(t) &= diag(F_C, ..., F_{CN})\end{aligned} \tag{28}$$

Uncertainties $\overline{F}_A(t), \overline{F}_{dA}(t), \overline{F}_B(t), \overline{F}_C(t)$ are unknown Lebesgue measurable functions satisfying standard norm bounded conditions.

Consider the overall guaranteed cost control problem (6), (24)–(28).

Denote the overall decentralized controller (7) rewritten into a compact form as follows

$$\begin{aligned}\dot{\hat{x}} &= \bar{A}_c\hat{x} + \bar{B}_c y \\ u &= \bar{C}_c\hat{x}\end{aligned} \tag{29}$$

where all matrices have a diagonal form of the corresponding dimensions.

Denote $\bar{x} = [x^T, \hat{x}^T]^T$. The closed–loop system (24)-(29) results in

$$\begin{aligned}\dot{\bar{x}}(t) &= (\bar{A}_c + \bar{D}_{1c}\bar{F}_{1c}(t)\bar{E}_{1c})\bar{x}(t) \\ &+ (\bar{A}_{dc} + \bar{D}_{2c}\bar{F}_{2c}(t)\bar{E}_{2c})x(t-d)\end{aligned} \tag{30}$$

where

$$\begin{aligned}\bar{A}_c &= \begin{pmatrix}\bar{A} & \bar{B}\bar{C}_c \\ \bar{B}_c\bar{C} & \bar{A}_c\end{pmatrix} \qquad \bar{A}_{dm} = \begin{pmatrix}\bar{A}_{dm} \\ 0\end{pmatrix} \\ \bar{D}_1 &= \begin{pmatrix}\bar{D}_A & \bar{D}_B & 0 \\ 0 & 0 & \bar{B}_c\bar{D}_C\end{pmatrix} \qquad \bar{D}_{2m} = \bar{D}_{dA} \\ \bar{E}_{1m}^T &= \begin{pmatrix}\bar{E}_A^T & 0 & \bar{E}_C^T \\ 0 & \bar{E}_B^T & 0\end{pmatrix} \qquad \bar{E}_2 = \bar{E}_{dA} \\ \bar{F}_1 &= diag(\bar{F}_A, \bar{F}_B, \bar{F}_C) \qquad \bar{F}_2 = \bar{F}_{dA}\end{aligned} \tag{31}$$

To establish the quadratic stability using the specific structure of the system (24)–(28), consider the Lyapunov functional

$$\begin{aligned}\bar{V}(\bar{x}, t) &= \bar{x}(t)^T P\bar{x}(t) \\ &+ \int_0^d x(t-s)^T P_1 x(t-s) ds\end{aligned} \tag{32}$$

where $P \in \Re^{2N\times 2N}$ and $P_1 \in \Re^{N\times N}$ are positive definite diagonal matrices with diagonal blocks of dimensions $2n \times 2n$ and $n \times n$, respectively.

Definition 2. The controller (29) is a d-quadratically stabilizing controller for the system (24)–(28) if there exist diagonal matrices $P > 0$ and $P_1 > 0$ with diagonal blocks of dimensions $2n \times 2n$ and $n \times n$, respectively, satisfying

$$\begin{pmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ \bar{Y}_{12}^T & \bar{Y}_{22} \end{pmatrix} < 0 \tag{33}$$

where $\bar{Y}_{11} = (\bar{A}+\bar{D}_1\bar{F}_1(t)\bar{E}_1)^T P + P(\bar{A}+\bar{D}_1\bar{F}_1(t)\bar{E}_1) + \hat{P}$, $\bar{Y}_{12} = P(\bar{A}_d + \bar{D}_2\bar{F}_2(t)\bar{E}_2)$, $\bar{Y}_{22} = P_1$, and $\hat{P} = diag(P_1, 0)$.

The closed–loop system cost value has the form

$$J = \int_0^\infty [x^T \bar{R}_1 x(t) + \hat{x}^T \bar{C}_c^T \bar{R}_2 \bar{C}_c \hat{x}(t)]dt \tag{34}$$

where $\bar{R}_1 = diag(R_1, ..., R_1)$ and $\bar{R}_2 = diag(R_2, ..., R_2)$. Denote $\bar{C} = diag(\bar{R}_1^{\frac{1}{2}}, \bar{R}_2^{\frac{1}{2}} C_c)$.

Lemma 2. Consider the system (24)–(28) and the cost function (6). Suppose there exist a controller (29) and given positive definite matrices $P = diag(P_m, ..., P_m)$, $P_1 = diag(P_{1m}, ..., P_{1m}$ and $V_1 = diag(V_{1m}, ..., V_{1m}) \geq 0$, $V_2 = diag(P_{2m}, ..., P_{2m}) > 0$ such that

$$\bar{Y}(\bar{A}) = \begin{pmatrix} \bar{Y}_{11} + \bar{C}^T\bar{C} & \bar{Y}_{12} \\ \bar{J}Y_{12}^T & \bar{Y}_{22} \end{pmatrix} < 0 \tag{35}$$

Then this controller d-quadratically stabilizes the system (30). The closed–loop system cost (34) satisfies the bound

$$J \leq tr\{PV\} + tr\{P_1 \int_{-d}^\infty x(s)x(s)^T ds\} = \gamma \tag{36}$$

where $V = diag(V_1, \bar{B}_c V_2 \bar{B}_c^T)$. (35), (36) hold for all admissible uncertainties.

Notice only a d-quadratic stability means a quadratic stability with the corresponding diagonal matrices.

To get a deeper insight into the structural properties of the considered system (24)-(28), introduce a useful transformation as follows.

Consider a real snxsn matrix $T(n, s)$ in the form

$$\begin{aligned} T(n,1) &= I \\ T(n,s) &= \begin{pmatrix} I & 0 & \dots & 0 & I \\ 0 & I & \dots & 0 & I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & I \\ -I & -I & \dots & -I & I \end{pmatrix} \qquad s > 1 \end{aligned} \tag{37}$$

where I denotes here nxn identical matrix. Denote

$$\begin{aligned} \overline{T}(i) &= diag[T(n, N-i)I, ..., I] \quad i = 0, ..., N-1 \\ G &= \overline{T}(0)\overline{T}(1) \quad ... \quad \overline{T}(N-1) \end{aligned} \tag{38}$$

Lemma 3 [18]. Consider the matrix $\overline{A}$ in the system (24)–(28) and any given $J = diag[J_o, ..., J_o]$, where J, J_o are nNxnN, nxn matrices. Then the following equalities

$$\begin{aligned} G^{-1}AG &= diag(A_s...A_s \quad A_c) \\ G^T AG &= diag(2A_s \quad 6A_s...N(N-1)A_s \quad NA_c) \\ G^{-1}J(G^{-1})^T &= diag(\frac{1}{2}J_o \quad \frac{1}{6}J_o...\frac{1}{N(N-1)}J_o) \\ G^T JG &= diag(2J_o...N(N-1)J_o \quad NJ_o) \end{aligned} \tag{39}$$

hold.

Proof of Theorem

Consider the system (1)–(5). Denote $A_s = A - L_q C_z$, $A_c = A_s + NL_q C_z$, $A_{ds} = A_d - L_{dq} C_{dz}$, $A_{dc} = A_{ds} + NL_{dq} C_{dz}$. Consider two particular cases of the uncertainties (8) such as $\Delta A_a(t) = \frac{N}{2} L_q C_z, \Delta A_{da}(t) = \frac{N}{2} L_{dq} C_{dz}$ and $\Delta A_a(t) = -\frac{N}{2} L_q C_z, \Delta A_{da}(t) = -\frac{N}{2} L_{dq} C_{dz}$. It evident that using (12)

$$\begin{aligned} A_m + \Delta A_m(t) &= A_s + \Delta A_a(t) + \Delta A(t) \\ &= A_c - \Delta A(t) \\ A_{dm} + \Delta A_{dm}(t) &= A_{ds} + \Delta A_{da}(t) + \Delta A_d(t) \\ &= A_{dc} - \Delta A_d(t) \end{aligned} \tag{40}$$

Suppose given matrices $P_m, ..., V_{2m}$ by Lemma 1 and the matrices A_c, B_c, C_c satisfying (20) with the guaranteed closed–loop cost γ_m (21). Then by Theorem it is sufficient to show that these matrices satisfy the relation (35) with the guaranteed cost (36).

Denote $\bar{G}_Y = diag(\bar{G}, \bar{G})$, where $\bar{G} = diag(G, G)$ and G defined by (38). We get using Lemma 3 when applying only standard operations on all terms

$$\begin{aligned} &\bar{G}_Y^T \bar{Y}(\bar{A})\bar{G}_Y = \bar{Y}(\bar{A}) \\ &\quad = \begin{pmatrix} \bar{G}^T(Y_{11} + \bar{C}^T\bar{C})\bar{G} & \bar{G}^T\bar{Y}_{12}\bar{G} \\ \bar{G}^T\bar{J}Y_{12}^T\bar{G} & \bar{G}^T\bar{Y}_{22}\bar{G} \end{pmatrix} \\ &\quad = diag[2Y_m(A_s) \ ... \ N(N-1)Y_m(A_s) \quad NY_m(A_c)] \end{aligned} \tag{41}$$

However, if $Y_m(A_m) < 0$, then also $Y_m(A_s) < 0$ and $Y_m(A_c) < 0$ hold because the uncertainties of the system (10), (11) includes both systems with the matrices A_s, A_c as special cases. The matrix $\bar{G}_Y$ is nonsingular. Therefore, the relation $\bar{G}_Y^T \bar{Y}(\bar{A})\bar{G}_Y < 0$ holds.

Consider (36) and suppose given J_m. Now, applying the above transformation using $\overline{G}$

$$J \leq tr\{\bar{G}^T PV\bar{G}\} + tr\{\bar{G}^T P_1 \int_{-d}^\infty x(s)x(s)^T ds\bar{G}\} \tag{42}$$

and Lemma 3 for $J_o = 1$ in (22), (23), we get directly the relation $\gamma = K_J\gamma_m$.

Thereby, the closed loop system (1)-(7) is quadratically stabilized using the guaranteed cost controller (7) with the guaranteed cost bound (36). Q.E.D.