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# Frequency Weighted Model Reduction Technique with Error Bounds

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***Abstract*— In this paper, we present a new frequency weighted balanced related technique which is based on a parametrized combination of the unweighted balanced technique [10], [9] and the partial fraction expansion technique [12]. The reduced order models which are guaranteed to be stable in the case of double-sided weighting, are obtained by either direct truncation or singular perturbation approximation. Simple, elegant and easily computable *a priori* error bounds are also derived. Numerical examples and comparison with other well-known techniques show the effectiveness of the proposed technique.**

## I. Introduction

Enns [2] has presented a scheme for reducing a stable high order model with frequency weighting, based on a modification of balanced truncation [10]. The method, known as frequency weighted balanced truncation, may use input weighting, output weighting, or both. With only one weighting present, stability of the reduced order model is guaranteed. With both weightings present, the method may yield unstable models. To overcome the potential drawback of instability, Lin and Chiu [8] proposed a new technique which yields stable models in case of double-sided weighting. Their technique was later generalized to include proper weights in [13]. However, as pointed out recently by Varga and Anderson [14] that this method can not be used in controller reduction applications due no pole-zero cancellation assumption required in the method. A modified method was proposed by Varga and Anderson [14] to rectify this problem however, this method suffers from the disadvantage that it yields proper reduced order models even when the original system is strictly proper. Another modification to Enns technique was proposed by Wang et al [15] which not only guarantees stability in case of double-sided weighting but also yields simple and elegant error bounds. As pointed out by Sreeram in [11], this method is realization dependent and hence yields different models for different realizations of the same original system. Some interesting results without explicitly predefining the frequency weights are presented in [3], [4].

Another group of methods which is based on partial fraction expansion was originally proposed by Latham and Anderson [7]. A number of frequency weighted model reduction methods have been proposed based on partial-fraction-expansion idea (see [1], [5], [16], [17], [12]). Error bounds exists for some special type of weighting functions [1], [12]. However the approximation error obtained using these methods are generally larger compared to Enns method with the exception of the method by Zhou [17] where in optimization is used to improve the approximation error.

In this paper, we propose a parametrized method which combines the advantages of the unweighted balancing [10], [9] with the frequency weighted partial fraction expansion technique [12]. This method has the following advantages: (i) stability of models in case of double sided weighting, (ii) simple, elegant and easily computable error bounds, (iii) applicability to both continuous and discrete systems, (iv) easily extendable to frequency weighted optimal Hankel norm approximation, (v) choice of free parameters to reduce the weighted error and error bounds, (vi) easily applicable to controller reduction problems (unlike the technique of Lin and Chiu [8], [12], [11]), and (vii) reduced order models and the error bounds are invariant under similarity transformation (unlike the technique of [15]).

## II. Preliminaries

In this section we review some of the well-known frequency weighted balanced model reduction techniques.

### A. Some Results on Frequency-weighted Balanced Truncation

Consider the transfer function of a linear time invariant system:

$$G(s) = C(sI-A)^{-1}B + D \tag{1}$$

where $\{A, B, C, D\}$ is its $n^{th}$ order minimal realization. Let the transfer functions of the stable input and output weights be given by equations (2) and (3) respectively:

$$V(s) = C_V(sI-A_V)^{-1}B_V + D_V \tag{2}$$
$$W(s) = C_W(sI-A_W)^{-1}B_W + D_W \tag{3}$$

where $\{A_V, B_V, C_V, D_V\}$ and $\{A_W, B_W, C_W, D_W\}$ are their $p^{th}$ and $q^{th}$ order minimal realizations respectively. The augmented systems given by

$$G(s)V(s) = \bar{C}_i(sI-\bar{A}_i)^{-1}\bar{B}_i + \bar{D}_i$$
$$W(s)G(s) = \bar{C}_o(sI-\bar{A}_o)^{-1}\bar{B}_o + \bar{D}_o$$

have the following minimal realizations:

$$\bar{A}_i = \begin{bmatrix} A & BC_V \\ 0 & A_V \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} BD_V \\ B_V \end{bmatrix}$$
$$\bar{C}_i = \begin{bmatrix} C & DC_V \end{bmatrix}, \quad \bar{D}_i = DD_V$$

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and

$$\bar{A}_o = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \quad \bar{B}_o = \begin{bmatrix} B \\ B_W D \end{bmatrix}$$
$$\bar{C}_o = \begin{bmatrix} D_W C & C_W \end{bmatrix}, \quad \bar{D}_o = D_W D$$

The frequency weighted Gramians

$$\bar{P}_i = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_V \end{bmatrix}, \quad \bar{Q}_o = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_W \end{bmatrix}$$

satisfy the following Lyapunov equations:

$$\bar{A}_i \bar{P}_i + \bar{P}_i \bar{A}_i^T + \bar{B}_i \bar{B}_i^T = 0 \quad (4)$$
$$\bar{A}^T \bar{Q}_o + \bar{Q}_o \bar{A}_o + \bar{C}_o^T \bar{C}_o = 0 \quad (5)$$

### B. Enns [2] Method

Expanding the (1,1) blocks of the above Lyapunov equations ((4) and (5)) yield the following:

$$AP_{11} + P_{11}A^T + P_E = 0 \quad (6)$$
$$A^T Q_{11} + Q_{11} A + Q_E = 0 \quad (7)$$

where

$$P_E = BC_V P_{12} + P_{12}^T C_V^T B^T + BD_V D_V^T B^T \quad (8)$$
$$Q_E = C^T B_W^T Q_{12}^T + Q_{12} B_W C + C^T D_W^T D_W C \quad (9)$$

Simultaneously diagonalizing the weighted Gramians, we get

$$T^{-T} Q_{11} T^{-1} = T P_{11} T^T = \begin{bmatrix} \Sigma_{E1} & 0 \\ 0 & \Sigma_{E2} \end{bmatrix}$$

where $\Sigma_{E1} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, $\Sigma_{E2} = diag\{\sigma_{r+1}, \ldots, \sigma_n\}$, $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$. Now transform and partition the original system as below:

$$T^{-1} A T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$
$$CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad D = D$$

where $A_{11} \in R^{r \times r}$. The reduced order model $G_r(s)$ is given by

$$G_r(s) = \left[ \begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array} \right]$$

Enns method is based on simultaneously diagonalizing the solution of Lyapunov equations (6) and (7). Since the matrices $P_E$ and $Q_E$ in these equations may not be positive semidefinite, the models obtained by Enns technique may not be stable. Several modifications to Enns technique are proposed in the literature to overcome the stability problem.

### C. Wang et al's Technique

The stability problem is solved in Wang et al's technique [15] by making the matrices $P_E$ (8) and $Q_E$ (9) positive semidefinite. In this technique, the new controllability and observability Gramians obtained as the solution to Lyapunov equations:

$$AP_W + P_W A^T + B_W B_W^T = 0$$
$$A^T Q_W + Q_W A + C_W^T C_W = 0$$

are diagonalized. The matrices $B_W$ and $C_W$ in the above Lyapunov equations are fictitious input and output matrices determined from:

$$B_W = U |S_X|^{1/2}$$
$$C_W = |R_Y|^{1/2} V$$

The terms on RHS of the above equations, $U$, $S_X$, $V$, and $R_Y$ are obtained from the following orthogonal eigen decomposition of symmetric matrices

$$P_E = U S_X U^T$$
$$Q_E = V R_Y V^T$$

Since

$$P_E \leq B_W B_W^T \geq 0$$
$$Q_E \leq C_W^T C_W \geq 0$$

and $(A, B_W, C_W)$ is minimal, the stability of the reduced order model in case of double-sided weighting is guaranteed.

**Remark 1:** *The following error bounds hold:*

$$\|W(s)(G(s) - G_r(s))V(s)\|_\infty = ktr\Sigma_2$$

*where $k = 2\|W(s)L\|_\infty \|KV(s)\|_\infty$ with $L$ and $K$ being constant matrices which depend on the weights and the original system.*

**Remark 2:** *When $P_E \geq 0$ and $Q_E \geq 0$,Wang et al's technique is equivalent to Enns Technique. However, in general, these matrices are indefinite, then the model reduction error and error bounds are not invariant under similarity transformation, and can be very large also[11].*

### D. Lin and Chiu Technique

In Lin and Chiu's technique [8], [13], instead of diagonalizing Gramians $P_{11}$ and $Q_{11}$, the following Gramians

$$P_{LC} = P_{11} - P_{12} P_V^{-1} P_{12}^T$$
$$Q_{LC} = Q_{11} - Q_{12}^T Q_W^{-1} Q_{12}$$

are simultaneously diagonalized:

$$T^{-T} Q_{LC} T^{-1} = T P_{LC} T^T = \begin{bmatrix} \Sigma_{LC1} & 0 \\ 0 & \Sigma_{LC2} \end{bmatrix}$$

where $\Sigma_{LC1} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, $\Sigma_{LC2} = diag\{\sigma_{r+1}, \ldots, \sigma_n\}$, $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$.

Note that the Gramians $P_{11} - P_{12} P_V^{-1} P_{12}^T$ and

$Q_{11} - Q_{12}^T Q_W^{-1} Q_{12}$ are the Schur compliments of the (1,1) blocks of the matrices $\bar{P}_i$ and $\bar{Q}_o$ and satisfy the following Lyapunov equations:

$$AP_{LC} + P_{LC}A^T + B_{LC}B_{LC}^T = 0$$
$$A^T Q_{LC} + Q_{LC}A + C_{LC}^T C_{LC} = 0$$

where

$$\begin{aligned} B_{LC} &= BD_V - P_{12}P_V^{-1}B_V \\ C_{LC} &= D_W C - C_W Q_W^{-1} Q_{12} \end{aligned}$$

Since the realization $\{A, B_{LC}, C_{LC}\}$ is minimal and the diagonalized Gramians satisfy the Lyapunov equation, Lin and Chiu's technique yields stable models in the case of double-sided weighting.

### *E. Varga and Anderson's Technique*

The main weakness of Lin and Chiu's technique, as pointed out in [14], is the requirement that no pole-zero cancellation occur when forming the augmented systems $K(s)V(s)$ and $W(s)K(s)$. This prevents the applicability of this method when solving controller reduction problems involving weights of the form:

$$\begin{aligned} W(s) &= (I + G(s)K(s))^{-1}G(s) \\ V(s) &= (I + G(s)K(s))^{-1} \end{aligned}$$

where $G(s)$ and $K(s)$ are the transfer functions of the system and its controller respectively.
To overcome this drawback, Varga and Anderson proposed simultaneously diagonalizing the following Gramians:

$$\begin{aligned} P_{VA} &= P_{11} - \alpha^2 P_{12}P_V^{-1}P_{12}^T \\ Q_{VA} &= Q_{11} - \alpha^2 Q_{12}^T Q_W^{-1} Q_{12} \end{aligned}$$

where $0 \leq \alpha \leq 1$. Note that when $\alpha = 0$, we get Enns method and when $\alpha = 1$, we have Lin and Chiu's technique.

**Remark 3:** *An obvious drawback of this technique is that the reduced order models are proper even though the original system is strictly proper. Moreover, there are no error bounds for this technique.*

### *F. Partial Fraction Expansion Technique*

In the partial fraction expansion technique [1], [12], [17], instead of diagonalizing Gramians $P$ and $Q$, the following Gramians $P_{PF}$ and $Q_{PF}$ are simultaneously diagonalized.

$$\begin{aligned} T^{-T}Q_{PF}T^{-1} &= TP_{PF}T^T \\ &= \text{diag}\,\{\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n\} \end{aligned}$$

where $\sigma_i \geq \sigma_{i+1},\ i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$ and

$$\begin{aligned} P_{PF} &= P_{11} - P_{12}X^T - XP_{12}^T + XP_VX^T \\ Q_{PF} &= Q_{11} - Q_{12}Y - Y^TQ_{12}^T + Y^TQ_WY \end{aligned}$$

The matrices $X$ and $Y$ satisfy the following matrix equations

$$AX - XA_V + BC_V = 0 \quad (10)$$
$$YA - A_WY + B_WC = 0 \quad (11)$$

**Remark 4:** *Note that the solutions $X$ and $Y$ are unique when $G(s)$ has no pole in common with $V(s)$ and $W(s)$ respectively.*

The Gramians $P_{PF}$ and $Q_{PF}$ satisfy following Lyapunov equations:

$$\begin{aligned} AP_{PF} + P_{PF}A^T + B_{PF}B_{PF}^T &= 0 \\ A^TQ_{PF} + Q_{PF}A + C_{PF}^TC_{PF} &= 0 \end{aligned}$$

where

$$B_{PF} = BD_V - XB_V \quad (12)$$
$$C_{PF} = D_WC - C_WY \quad (13)$$

Since the realization $\{A, B_{PF}, C_{PF}\}$ is minimal and the Gramians diagonalized satisfy the Lyapunov equations, the partial fraction expansion technique yields stable models in the case of double-sided weightings.

**Remark 5:** *As pointed out in [5], [16], [17], the approximation error $\|W(s)[G(s) - G_r(s)]V(s)\|_\infty$ using this method can be reduced significantly by using maximum phase weighting functions. In other words if the weighting functions, $V(s)$ and $W(s)$ are minimum phase, then by using the maximum phase weighting functions $V(-s)$ and $W(-s)$ in the above frequency weighted model reduction technique, we obtain lower order model, $G_r(s)$ with lower approximation error.*

## III. Main Results

The proposed frequency weighted balanced related technique is based on a combination of the unweighted balanced technique ([10] or [9]) and partial-fraction based frequency weighted balanced reduction technique [12].

Instead of simultaneously diagonalizing $P_{PF}$ and $Q_{PF}$ as in [12], we propose to simultaneously diagonalize

$$\begin{aligned} P_X &= \alpha^2 P + P_{PF} \\ Q_Y &= \beta^2 Q + Q_{PF} \end{aligned}$$

where $\alpha > 0,\ \beta > 0$ and $P$ and $Q$ are the unweighted Gramians satisfying:

$$\begin{aligned} AP + PA^T + BB^T &= 0 \\ A^TQ + QA + C^TC &= 0 \end{aligned}$$

and $P_{PF}$ and $Q_{PF}$ are the partial fraction expansion frequency weighted Gramians satisfying:

$$\begin{aligned} AP_{PF} + P_{PF}A^T + B_{PF}B_{PF}^T &= 0 \\ A^TQ_{PF} + Q_{PF}A + C_{PF}^TC_{PF} &= 0 \end{aligned}$$

where $B_{PF}$ and $C_{PF}$ are given by eqns. (12) and (13) respectively. It is straightforward to see that the Gramians $P_X$ and $Q_Y$ satisfy the following Lyapunov equations:

$$\begin{aligned} AP_X + P_XA^T + B_XB_X^T &= 0 \\ A^TQ_Y + Q_YA + C_Y^TC_Y &= 0 \end{aligned}$$

where

$$B_X = \begin{bmatrix} \alpha B & B_{PF} \end{bmatrix}$$
$$C_Y = \begin{bmatrix} \beta C \\ C_{PF} \end{bmatrix}$$

**Remark 6:** *Note that when $\alpha = 0$ and $\beta = 0$, the new fictitious input and output matrices are equal to $B_{PF}$ and $C_{PF}$ respectively, i.e.,*

$$B_X|_{\alpha=0} = B_{PF}$$
$$C_Y|_{\beta=0} = C_{PF}$$

To establish the relationship between the input and output matrices ($B$ and $C$), and the new fictitious input and output matrices ($B_X$ and $C_Y$), we define two constant matrices:

$$K = \begin{bmatrix} \frac{I}{\alpha} \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} \frac{I}{\beta} & 0 \end{bmatrix}$$

The following relationship holds:

$$B = B_X K$$
$$C = L C_Y$$

**Theorem 1:** *The realization $\{A, B_X, C_Y\}$ is stable and minimal.*

**Proof:** When $\alpha > 0$ and $\beta > 0$, the stability and minimality of $\{A, B_X, C_Y\}$ follows directly from the stability and the minimality of $\{A, B, C\}$.

Given the original system realization, $\{A, B, C, D\}$ and the weights $\{A_W, B_W, C_W, D_W\}$ and $\{A_V, B_V, C_V, D_V\}$, the proposed technique is based on balancing the realization $\{A, B_X, C_Y\}$. The reduced order models are obtained either by direct truncation or by singular perturbation approximation.

**Algorithm I:** In this Algorithm, the reduced order models are obtained by direct truncation.

1) Given a stable minimal realization $\{A, B, C, D\}$, and minimal realizations $\{A_V, B_V, C_V, D_V\}$, and $\{A_W, B_W, C_W, D_W\}$, compute $X$ and $Y$ by solving the equations:

$$AX - XA_V + BC_V = 0$$
$$YA - A_W Y + B_W C = 0$$

2) Compute the matrices $B_{PF}$ and $C_{PF}$ as follows:

$$B_{PF} = BD_V - XB_V$$
$$C_{PF} = D_W C - C_W Y$$

3) Compute the fictitious input and output matrices:

$$B_X = \begin{bmatrix} \alpha B & B_{PF} \end{bmatrix}$$
$$C_Y = \begin{bmatrix} \beta C \\ C_{PF} \end{bmatrix}$$

4) Solve the Lyapunov equations:

$$AP_X + P_X A^T + B_X B_X^T = 0$$
$$A^T Q_Y + Q_Y A + C_Y^T C_Y = 0$$

5) Calculate the transformation $T$, which simultaneously diagonalizes the positive definite matrices $P_X$ and $Q_Y$, such that

$$T^{-T} Q_Y T^{-1} = T P_X T^T$$
$$= \text{diag } \{\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n\}$$

where $\sigma_i \geq \sigma_{i+1},\ i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$.

6) Compute the frequency weighted balanced realization.

$$\hat{A} = T^{-1} A T, \quad \hat{B} = T^{-1} B, \quad \hat{C} = CT$$

7) Partition $\{\hat{A}, \hat{B}, \hat{C}\}$ as follows:

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$\hat{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where $A_{11} \in R^{r \times r},\ B_1 \in R^{r \times p},\ C_1 \in R^{q \times r}$ and $r < n$.

8) The reduced order model is $G_r(s) = C_1(sI - A_{11})^{-1} B_1 + D$.

**Algorithm II:** In this Algorithm , the reduced order models are obtained via singular perturbation approximation.

1) - 7) Same as Algorithm I.

8) The reduced order model is $G_{SPA}(s) = C_{SPA}(sI - A_{SPA})^{-1} B_{SPA} + D_{SPA}$ where

$$A_{SPA} = A_{11} - A_{12} A_{22}^{-1} A_{21}$$
$$B_{SPA} = B_1 - A_{12} A_{22}^{-1} B_2$$
$$C_{SPA} = C_1 - C_2 A_{22}^{-1} A_{21}$$
$$D_{SPA} = D - C_2 A_{22}^{-1} B_2$$

**Theorem 2:** *The reduced order models obtained via Algorithm I and Algorithm II are stable.*

The proof follows immediately from the proof of stability of the unweighted cases [10], [9], and therefore omitted here.

**Remark 7:** *The proposed Algorithm I produces proper models for proper original system, and strictly proper models for strictly proper original systems. However, the proposed Algorithm II produces proper models even for strictly proper original systems.*

## IV. Error Bounds

In this section we derive the error bounds for the reduced order models obtained using the two algorithms proposed.

**Theorem 3:** *Let $G(s)$ be a proper stable transfer function of order $n$ , and $V(s)$ and $W(s)$ be proper weighting functions. If $G_r(s)$ is a proper stable, reduced order model obtained using Algorithm I, then the following error bound holds:*

$$\|W(s)(G(s) - G_r(s))V(s)\|_\infty \leq \gamma \sum_{i=r+1}^{n} \sigma_i$$

*where* $\gamma = \frac{2}{\alpha\beta}\|W(s)\|_\infty \|V(s)\|_\infty$, $\alpha$ *and* $\beta$ *are real numbers greater than zero to be chosen.*

**Proof:** Partitioning

$$B_X = \begin{bmatrix} B_{X1} \\ B_{X2} \end{bmatrix}, \qquad C_Y = \begin{bmatrix} C_{Y1} & C_{Y2} \end{bmatrix}$$

and substituting

$$B_1 = B_{X1}K, \quad C_1 = LC_{Y1}$$

We have

$$\begin{aligned}
& \|W(s)(G(s)-G_r(s))V(s)\|_\infty \\
= & \left\|W(s)(C(sI-A)^{-1}B \right. \\
& \left. -C_1(sI-A_{11})^{-1}B_1)V(s)\right\|_\infty \\
= & \left\|W(s)(LC_Y(sI-A)^{-1}B_XK \right. \\
& \left. -LC_{Y1}(sI-A_{11})^{-1}B_{X1}K)V(s)\right\|_\infty \\
= & \left\|W(s)L(C_Y(sI-A)^{-1}B_X \right. \\
& \left. -C_{Y1}(sI-A_{11})^{-1}B_{X1})KV(s)\right\|_\infty \\
\leq & \|W(s)L\|_\infty \left\|(C_Y(sI-A)^{-1}B_X \right. \\
& \left. -C_{Y1}(sI-A_{11})^{-1}B_{X1})\right\|_\infty \|KV(s)\|_\infty \\
\leq & \frac{1}{\beta}\|W(s)\|_\infty \left\|(C_Y(sI-A)^{-1}B_X \right. \\
& \left. -C_{Y1}(sI-A_{11})^{-1}B_{X1})\right\|_\infty \frac{1}{\alpha}\|V(s)\|_\infty
\end{aligned}$$

Since $\left[\begin{array}{c|c} A & B_X \\ \hline C_Y & D \end{array}\right]$ is a balanced realization and $\left[\begin{array}{c|c} A_{11} & B_{X1} \\ \hline C_{Y1} & D \end{array}\right]$ is its reduced order model, we have from [2]

$$\left\|(C_Y(sI-A)^{-1}B_X - C_{Y1}(sI-A_{11})^{-1}B_{X1})\right\|_\infty \leq 2\sum_{i=r+1}^{n}\sigma_i$$

Let $\gamma = \frac{2}{\alpha\beta}\|W(s)\|_\infty \|V(s)\|_\infty$, then

$$\|W(s)(G(s)-G_r(s))V(s)\|_\infty \quad \leq \quad \gamma \sum_{i=r+1}^{n}\sigma_i$$

where $\sigma_i$ is obtained from step 5 of the Algorithm.

**Remark 8:** *The error bound formula for the reduced order models obtained using Algorithm II is identical to the one shown in Theorem 3. The proof of this formula is slightly different, due to a direct component term, but is very similar to the proof shown above, and is therefore omitted.*

**Corollary 1:** *In the case of input weighting only, the error bound is given by*

$$\|(G(s)-G_r(s))V(s)\|_\infty \leq \frac{2}{\alpha}\|V(s)\|_\infty \sum_{i=r+1}^{n}\sigma_i$$

*Similarly, in the case of output weighting only, we have*

$$\|W(s)(G(s)-G_r(s))\|_\infty \leq \frac{2}{\beta}\|W(s)\|_\infty \sum_{i=r+1}^{n}\sigma_i$$

**Remark 9:** *Note that the proposed error bound formula is not only much simpler but also can be made much tighter (by varying* $\alpha$ *and* $\beta$*) than the Wang et al's technique [15]. In Wang et al's technique, the matrices* $L$ *and* $K$ *have to be computed before evaluating the infinity norms* $\|W(s)L\|_\infty$ *and* $\|KV(s)\|_\infty$*. Here, because of the special structure of the* $L$ *and* $K$ *matrices, we have*

$$\begin{aligned}
\|KV(s)\|_\infty &= \frac{1}{\alpha}\|V(s)\|_\infty \\
\|W(s)L\|_\infty &= \frac{1}{\beta}\|W(s)\|_\infty
\end{aligned}$$

## V. Examples

In the both the examples considered below, to reduce the approximation error ($\|W(s)[G(s)-G_r(s)]V(s)\|_\infty$), we have used maximum phase weighting functions $V(-s)$ and $W(-s)$ in our proposed algorithms (please see Remark 5).

### A. Example I

Consider the fourth order system used in [8], [13], [15], [14]

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -5 \\ 1/2 & -3/2 \\ 1 & -5 \\ -1/2 & 1/6 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4/15 & 1 & 0 & 1 \end{bmatrix}$$

with the following input and output weights:

$$V(s) = W(s) = \{-4.5I_2, \; 3I_2, \; 1.5I_2, I_2\}$$

where $I_2$ denotes an identity matrix of second order.

In the Table I, the data in the 2nd, 3rd, 5th, and 6th columns represents the errors obtained using Enns' technique [2], Varga and Anderson's technique [14], and Algorithm I and II respectively. The data in the last column represents the error bounds obtained via the proposed techniques. It is well known that the Enns' technique [2] yields strictly proper models when the original system is strictly proper, whereas the Varga and Anderson's technique [14] yields proper models. Algorithm I produces strictly proper models when the original system is strictly proper, whereas Algorithm II produces proper models. To make a fair comparison, we compare the results of Algorithm I with the Enns' technique [2] and Algorithm II with Varga and Anderson's technique [14]. It is clear that our method produces lower approximation error in the respective comparisons. Moreover, Enns' technique [2] and the Varga and Anderson's technique [14] do not have any *a priori* error bounds.

Observe that for large values of $\alpha$ and $\beta$, the results obtained using the proposed Algorithms I and II approach the results of unweighted balanced truncation [10] and

TABLE I

THE ERRORS AND ERROR BOUNDS FOR THE MODELS.

| $r$ | Enns [2] | VA [14] | $\alpha$=$\beta$ | A-I | A-II | Error Bound |
|---|---|---|---|---|---|---|
| 1 | 2.1291 | 1.405 | 1 | 2.1269 | 1.4089 | 10.2672 |
| | | | 2 | 2.1662 | 1.3630 | 4.7588 |
| | | | 5 | 2.2682 | 1.3182 | 3.2024 |
| 2 | 0.2660 | 0.250 | 0.35 | 0.2655 | 0.2509 | 11.7471 |
| | | | 1 | 0.2754 | 0.2469 | 1.8835 |
| | | | 3 | 0.2990 | 0.2478 | 0.6516 |
| | | | 5 | 0.3064 | 0.2482 | 0.5516 |
| 3 | 0.1131 | 0.069 | 1 | 0.1125 | 0.0653 | 0.4390 |
| | | | 3 | 0.1205 | 0.0621 | 0.1714 |
| | | | 5 | 0.1274 | 0.0613 | 0.1494 |

TABLE II

THE ERRORS AND ERROR BOUNDS FOR THE MODELS.

| $r$ | Enns [2] | $\alpha$ | A-I | Error Bound |
|---|---|---|---|---|
| 1 | 320.752 | 1 | 20.1362 | 8226 |
| | | 3 | 20.1362 | 8226 |
| 2 | 0.1313 | 1 | 0.1165 | 1.5490 |
| | | 3 | 0.1167 | 1.5283 |
| | | 5 | 0.1167 | 1.5266 |
| 3 | 0.0669 | 1 | 0.1271 | 0.4589 |
| | | 3 | 0.1265 | 0.4424 |
| | | 5 | 0.1265 | 0.4411 |
| 4 | $9.1745 \times 10^{-4}$ | .3 | $9.5211 \times 10^{-4}$ | 0.0071 |
| | | 1 | $10.7282 \times 10^{-4}$ | 0.0046 |
| | | 3 | $11.5303 \times 10^{-4}$ | 0.0042 |
| | | 5 | $11.5953 \times 10^{-4}$ | 0.0042 |

unweighted balanced singular perturbation approximation [9], respectively. This is because increasing the scaling factors ($\alpha$ and $\beta$) increases the weighting of the unweighted Gramians when compared to the weighting of the partial fraction frequency weighted Gramians. It was observed that the weighted error may not necessarily decrease with increase in $\alpha$ and $\beta$, or vice versa. Further work is necessary to determine the optimal values of $\alpha$ and $\beta$.

### *B. Example II*

Consider the controller reduction example of Kim et al [6] where the stable controller transfer function $K(s)$ has poles at $s = \{-1.5,\ -0.7 \pm j0.71414,\ -0.01,\ -0.001\}$ and zeros at $s = \{-2,\ -0.8\}$. The stable input weighting is given in the form:

$$V(s) = \frac{K(s)^{-1}}{(s+1)^2(s+2)}$$

In the Table II, the approximation errors and the error bounds obtained via Algorithm I are compared with Enns' method [2]. It can be seen that Algorithm I produces lower approximation errors in most cases.

## VI. CONCLUSION

A frequency weighted balanced related technique based on a combination of unweighted balancing and frequency weighted balanced partial fraction expansion [12] is proposed. By varying the free parameters, it is possible to obtain lower approximation errors and tighter error bounds than other well known techniques. The models obtained are guaranteed to be stable in the case of double-sided weighting. The proposed method can be applied to both continuous and discrete systems and can also be easily extended to optimal Hankel norm approximation.

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