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Singularly Perturbed Unified Systems with Low Sensitivity to Model Reduction

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***Abstract*—A method of designing a state feedback gain achieving a specified insensitivity of the closed-loop trajectory by the singularly perturbed unified system using the delta operators is proposed. The order of system is reduced by the singular perturbation technique by ignoring the fast mode in it. The proposed method takes care of the actual trajectory variations over the range of the singular perturbation parameter. Necessary conditions for optimality are also derived. The previous study was done in the continuous time system. The present paper extends the previous study to the discrete system and the delta operating system that unifies the continuous and discrete systems. Advantages of the proposed method are shown in the numerical example.**

I. Introduction

THE discrete algorithm well fits the computer. However, one of the major disadvantages of the q-operating discrete system is the truncation and round-off error owing to the finite word-length preassigned. For the continuous-time and the discrete-time systems, one uses the differential operator, d/dt, and the forwarding shift (q) operator respectively. The δ-operator is an incremental difference operator that unifies both the continuous and the discrete systems together. However, the first disadvantage of using the q-operator is an inconvenience as it is not like the differential operator, d/dt. The second weakness of the normal q-operator systems is that there are located the poles near the boundary of stability circle at small sampling interval. Middleton and Goodwin [13] showed that the delta operator approach numerically had better finite word-length characteristics compared with the q-operator approach when the q-operator poles are closer to 1+j0 than to the origin. This may cause both the instability and the pole/zero cancellation problem due to the low resolution of the stability circle [3]. If the discrete system is converted into the delta operator system, a numbers of the terms of discrete system are reduced without losing any generality. The finite word-length characteristics are improved by reducing the round-off and truncation errors. Salgado et al. [19] illustrated that the delta operator approach had less relative error than that of the q-operator for rapid sampling for a Kalman filter design. Middleton and Goodwin [14] provided an analytical foundation of the unified approach using the delta operators. Li and Gevers [8] presented some advantages of the delta operator state-space realization of the transfer function over that of the q-operator on the minimization of the round-off noise gain of the realization. They showed that the delta operator implementation is a special case of residue feedback. So, the minimum gain of the delta operator system is smaller than that of the q-operator if and only if the sum of the residue modes is smaller than the sum of the Hankel singular values. Li and Gevers [7] compared the q-operator and the delta operator state-space realizations in terms of the effects of finite word-length errors on the actual transfer function. They showed the parameterizations in the delta operator yielded a superior sensitivity performance over those in the q-operator. The sensitivity of the transfer function is concerned on coefficient errors in the q- or in the delta operator implementations respectively. Shim and Sawan [20] studied the linear quadratic regulator (LQR) design and the state feedback design with an aircraft example [21] in the singularly perturbed systems by the delta operator approach.

In engineering problems, there are two-time-scale models, whose eigenvalues are located into two groups of the fast and slow subsystems. Where the real parts of eigenvalues of the two-time-scale systems are grouped in the distance. It is the weakly coupled system, which is said the singularly perturbed systems. This system can be decoupled into the fast and slow subsystems by the singular perturbation technique. Chang [1], Kokotovic [4] and Chow and Kokotovic [2] used a matrix block diagonalization method to decouple the system matrices of the weakly decoupled systems. Kokotovic et al. [5] and Naidu [17] made contributions in the singularly perturbed continuous and discrete systems respectively. Those contributions are made in the time domain. Naidu and Price [15, 16, 18] presented some papers regarding the singularly perturbed discrete systems with detailed illustrations. Mahmoud et al. [9, 10, 11,

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12] made intensive studies for the singularly perturbed discrete systems.

Tran and Sawan [22] proposed a method of designing an optimal composite feedback controller whose actual trajectory variation over the range of the singular perturbation parameter in the continuous system is limited to a specific extent. The present paper extends the previous study to the three kinds of the systems, i.e., the discrete system, the discrete-like δ-operating system and the continuous-like δ-operating system. Finally, including the result in the continuous system, all the four kinds of results are compared with one another.

II. DELTA OPERATORS

Consider a linear and time-invariant continuous system

$$\frac{dx}{dt} = Ax(t) + Bu(t). \tag{1}$$

where x is a n x 1 state vector and u is a r x 1 control vector. A is a n x n matrix and B is n x r matrix. The corresponding sampled-data system with the zero-order hold (ZOH) and sampling interval Δ is given by

$$x(k+1) = A_q x(k) + B_q u(k),\ y(k) = C_q x(k). \tag{2}$$

where $A_q = e^{A\Delta},\ B_q = \int_0^{\Delta} e^{A(\Delta-\tau)} B d\tau.$

According to Middleton and Goodwin [14], the delta operator is defined as follow:

$$\delta = \frac{(q-1)}{\Delta}. \tag{3}$$

Now we have parameters identities between discrete and delta systems.

$$qx = A_q x(k) + B_q u(k),\ y(k) = C_q x(k). \tag{4}$$

$$\delta x(\tau) = A_\delta x(\tau) + B_\delta u(\tau),\ y(k) = C_\delta x(k). \tag{5}$$

The parameter identities of the q- and the δ-operators in (4) and (5) as

$$A_\delta = \frac{(A_q - I)}{\Delta},\ B_\delta = \frac{B_q}{\Delta},\ C_q = C_\delta. \tag{6}$$

The parameters between the continuous system and the delta system are identified as

$$A_\delta = \Omega A,\ B_\delta = \Omega B. \tag{7}$$

$$\Omega = \frac{1}{\Delta}\int_0^{\Delta} e^{A\tau}\, d\tau = I + \frac{A\Delta}{2!} + \frac{A^2\Delta^2}{3!} + \cdots. \tag{8}$$

Therefore, as Δ goes to zero, Ω becomes the identity matrix, thus, A_δ becomes A.

The continuous and the discrete equations are written as a comprehensive form using the δ operator as:

$$\rho x(\tau) = A_\rho x(\tau) + B_\rho u(\tau),\ y(\tau) = C_\rho x(\tau). \tag{9}$$

$$A_\rho = \begin{Bmatrix} A \\ A_q \\ A_\delta \end{Bmatrix},\ B_\rho = \begin{Bmatrix} B \\ B_q \\ B_\delta \end{Bmatrix},\ \rho = \begin{Bmatrix} d/dt \\ q \\ \delta \end{Bmatrix},\ \tau = \begin{Bmatrix} t \\ k \\ \tau_\delta \end{Bmatrix}.$$

It is noted that the first rows, the second rows and the third rows of A_ρ, B_ρ, ρ and τ denote the continuous system, the discrete system and the delta system respectively. When $\Delta \to 0$, then $A_\delta \to$A, $B_\delta \to$B. This means that, when the sampling time goes to zero, the discrete-like δ system becomes identical with the continuous system. Now the stability regions for various operators are introduced. For continuous systems, the operator is d/dt and the transform variable is s. For discrete systems, the operator is q and the transform variable is z. For unified systems, the operator is δ and the transform variable is γ. The stability regions are as follows:

$$\begin{aligned} &\text{Continuous case: } \operatorname{Re}\{s\} < 0, \\ &\text{Discrete case: } |z| < 1, \\ &\text{Unified case: } \frac{\Delta}{2}|\gamma|^2 + \operatorname{Re}\{\gamma\} < 0. \end{aligned} \tag{10}$$

As Δ approaches zero, the stability inequality of the unified case equals that of the continuous case.

III. TWO TIME SCALE UNIFIED SYSTEM

A. Open-loop system

Consider the linear system (9) and assume that the system satisfies the condition (10), and then we can write the *two-time-scale* system as:

$$\begin{bmatrix} \rho x(\tau) \\ \varepsilon \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{\delta 11} & A_{\delta 12} \\ A_{\delta 21} & A_{\delta 22} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta 1} \\ B_{\delta 2} \end{bmatrix} u(\tau), \tag{11}$$

where x and z are n and m dimensional state vectors, u is an r dimensional control vector, and $A_{\delta ij}$ are matrices of appropriate dimensionality. Also, it is required that $A_{\delta 22}$ be non-singular. System (11) has a two-time-scale property, if

$$0 < |E_{s1}| < |E_{s2}| \cdots < |E_{sn}| < |E_{f1}| < |E_{f2}| \cdots < |E_{fm}| < |\frac{2}{\Delta}|, \tag{12}$$

$$\varepsilon = |E_{sn}| / |E_{f1}| << 1, \tag{13}$$

where E denotes eigenvalues of the system. So, we have

$$|E_{\max}(A_{\delta s})| << |E_{\min}(A_{\delta f})|. \tag{14}$$

If the norm properties of the invertible matrices are used, this can be equivalent to

$$|A_{\delta f}|^{-1} << |A_{\delta s}|^{-1}. \tag{15}$$

Now, we need to de-couple the system (11) into the slow and fast subsystems as

$$x_s(\tau) = (I_s - M_\delta L_\delta)x(\tau) - M_\delta z(\tau),\ z_f(\tau) = L_\delta x(\tau) + I_{\delta f} z(\tau). \tag{16}$$

From (16) the slow and fast subsystems are obtained,

$$\begin{bmatrix} \rho x(\tau) \\ \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{\delta s} & 0 \\ 0 & A_{\delta f} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta s} \\ B_{\delta f} \end{bmatrix} u(\tau), \tag{17}$$

where $A_{\delta s} = A_{\delta 11} - A_{\delta 12} L_\delta, \quad A_{\delta f} = A_{\delta 22} + L_\delta A_{\delta 22},$
$B_{\delta s} = B_{\delta 1} - M_\delta B_{\delta 2} - M_\delta L_\delta B_{\delta 1}, \quad B_{\delta f} = B_{\delta 2} + L_\delta B_{\delta 1}.$

Here, L and M are the solutions of the nonlinear algebraic Riccati-type equations as

$$LA_{\delta 11} + A_{\delta 21} - LA_{\delta 12}L - A_{\delta 22}L = 0, \tag{18}$$
$$A_{\delta 11}M - A_{\delta 12}LM - MA_{\delta 22} - MLA_{\delta 12} + A_{\delta 12} = 0, \tag{19}$$

with initial conditions

$$L_0 = A_{\delta 22}^{-1} A_{\delta 21},\ M_0 = A_{\delta 12} A_{\delta 22}^{-1}. \tag{20}$$
$$A_{\delta 0} = A_{\delta 11} - A_{\delta 12} L_0,\ B_{\delta 0} = B_{\delta 1} - M_0 B_{\delta 2}. \tag{21}$$

The sequences L_K and M_K are defined by

$$L_{k+1} = A_{\delta 22}^{-1}(A_{\delta 21} + L_k A_{\delta 11} - L_k A_{\delta 12} L_k), \tag{22}$$
$$M_{k+1} = ((A_{\delta 11} + A_{\delta 12} - A_{\delta 12} L_k M_k - M_k L_k A_{\delta 12}) A_{\delta 22}^{-1}. \tag{23}$$

and L_0 as in (20) converges to a real bounded root of (19). Moreover

$$\| L_{k+1} - L \| \le \| L_k - L \|,\ k = 0,1,2,3 \cdots. \tag{24}$$

Quasi-steady state approximation [1,2,4] is summarized as

$$\begin{bmatrix} \rho \bar{x}(\tau) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{\delta 11} & A_{\delta 12} \\ A_{\delta 21} & A_{\delta 22} \end{bmatrix} \begin{bmatrix} \bar{x}(\tau) \\ \bar{z}(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta 1} \\ B_{\delta 2} \end{bmatrix} \bar{u}(\tau), \tag{25}$$

where a bar denotes *quasi-steady-state*. The system (25) is reduced to

$$\rho \bar{x}(\tau) = A_{\delta 0} \bar{x}(\tau),\ \bar{z}(\tau) = A_{\delta 22}^{-1}(A_{\delta 21}\bar{x}(\tau) + B_{\delta 2}\bar{u}(\tau)) \tag{26}$$

Therefore, the decoupled systems are as

$$\rho x_s = A_{\delta 0} x_s(\tau) + B_{\delta 0} u_s(\tau),\ x_s(0) = x_{s0},$$
$$\rho z_f(\tau) = A_{\delta 22} z_f(\tau) + B_{\delta 2} u_f(\tau), z_f(0) = z_0 - \bar{z}(0). \tag{27}$$

B. Closed-loop system

The optimal feedback gain in the infinite horizon is used to design a near optimal controller for the closed-loop system. So, we use the linear equation (11) and the quadratic cost index as

$$J = \tfrac{1}{2} S_{\tau=0}^{\infty} \{[x^T(\tau) z^T(\tau)]^T Q [x^T(\tau) z^T(\tau)] + u(\tau)^T R u(\tau)\} d\tau, \tag{28}$$
$$u(\tau)_{opt} = -G(\varepsilon)\bar{x}(\tau), \tag{29}$$
$$G(\varepsilon) = (\frac{R}{\Delta} + \Delta B^T K(\varepsilon) B)^{-1} B^T K (I + A\Delta), \tag{30}$$
$$\bar{x}(\tau) = \begin{bmatrix} x^T(\tau) & z^T(\tau) \end{bmatrix}^T, \tag{31}$$

K is the solution of the algebraic Riccati equation (ARE) as follows:

$$0 = KA + A^T K + \frac{Q}{\Delta} + \Delta A^T K A - G^T (\frac{R}{\Delta} + \Delta B^T K B) G. \tag{32}$$

Therefore, parameter A of the closed-loop system is formulated as

$$A_C = A + BG = A - BR^{-1}B^T K. \tag{33}$$

One can separate the cost equation (28) into the slow and fast formulae as

$$J_s = \tfrac{1}{2} S_{\tau=0}^{\infty} \{x^T(\tau) Q_s x(\tau) + u_s^T(\tau) R_s u_s(\tau)\} d\tau,$$
$$J_f = \tfrac{1}{2} S_{\tau=0}^{\infty} \{z^T(\tau) Q_f z(\tau) + u_f^T(\tau) R_f u_f(\tau)\} d\tau. \tag{34}$$

Accordingly, one computes the optimal feedback gains from the ARE

$$G_s = -(\frac{R_s}{\Delta} + \Delta B_s^T K_s B_s)^{-1} B_s^T K_s (I + A_s \Delta),$$
$$G_f = -(\frac{R_f}{\Delta} + \Delta B_f^T K_f B_f)^{-1} B_f^T K_f (I + A_f \Delta), \tag{35}$$
$$0 = K_s A_s + A_s^T K_s + \frac{Q_s}{\Delta} + \Delta A_s^T K_s A_s - G_s^T (\frac{R_s}{\Delta} + \Delta B_s^T K_s B_s) G_s,$$
$$0 = K_f A_f + A_f^T K_f + \frac{Q_f}{\Delta} + \Delta A_f^T K_f A_f - G_f^T (\frac{R_f}{\Delta} + \Delta B_f^T K_f B_f) G_f, \tag{36}$$

$$A_{Csub} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} \begin{bmatrix} G_s & G_f \end{bmatrix} \quad (37)$$

C. Insensitive design

We multiply the output-related term by the Lagrange coefficient in the cost functional and consider it at the reduced value. Thus, one can rewrite (34) (29), (35), (36), (33) as

$$J_s(\lambda) = \tfrac{1}{2} S_{\tau=0}^{\infty} \{\lambda[x^T(\tau) Q_s x(\tau)] + u_s^T(\tau) R_s u_s(\tau)\} d\tau, \quad (38)$$

$$u_s(\tau)_\lambda = -G_s(\lambda) x_s(\tau), \quad (39)$$

$$G_s(\lambda) = -(\frac{R_s}{\Delta} + \Delta B_s^T K_s(\lambda) B_s)^{-1} B_s^T K_s(\lambda)(I + A_s \Delta), \quad (40)$$

$$0 = K_s(\lambda) A_s + A_s^T K_s(\lambda) + \frac{\lambda Q_s}{\Delta} + \Delta A_s^T K_s(\lambda) A_s \quad (41)$$
$$- G_s^T(\frac{R_s}{\Delta} + \Delta B_s^T K_s(\lambda) B_s) G_s,$$

$$A_{sC}(\lambda) = A_s - B_s R_s^T B_s^T K_s(\lambda). \quad (42)$$

In the case of non-zero ε, (42) is rewritten as

$$A_{sC}(\lambda)_{\varepsilon \neq 0} = A - BR^T B^T K_s(\lambda)_{aug}. \quad (43)$$

where $K_s(\lambda)_{aug}$ is augmented by *0* to match the order of the system. As described in Krishnan and Brzezowski [6], the variation of the closed-loop response is limited to a prespecified extent if the following condition is satisfied

$$Max \| P \| \leq \rho, \quad (44)$$

where ρ is a positive scalar of the upper bound of the allowable trajectory variation, and $\| P \|$ is the maximum eigenvalue of the real, symmetric matrix *P* of the solution of the Lyapunov equation as

$$0 = PA_{sC}(\lambda) + A_{sC}^T(\lambda) P + Q + \Delta A_{sC}^T P A_{sC}. \quad (45)$$

The steps in the numerical algorithm are listed as follows:

Step 1: Obtain A_s, B_s.

Step 2: Select ρ and λ.

Step 3: Compute the Riccati equation (41) and find its solution, $K_s(\lambda)$.

Step 4: Find $A_{sC}(\lambda)$.

Step 5: Obtain the solution *P*, and the check for the sensitivity criterion as in (44). If not satisfied, increase the value of λ and repeat the process from step 3.

IV. TWO TIME SCALE DISCRETE SYSTEM

A. Open-loop system

Consider the linear system

$$x(k+1) = A_{d11} x(k) + A_{d12} z(k) + B_{d1} u(k),$$
$$\varepsilon z(k+1) = A_{d21} x(k) + A_{d22} z(k) + B_{d2} u(k). \quad (46)$$

We can arrange the eigenvalues of the system as

$$|1| > |p_{s1}| > |p_{s2}| > \cdots > |p_{sn}| > |p_{f1}| > |p_{f2}| \cdots > |p_{fm}|, \quad (47)$$

$$\varepsilon = | p_{f1} | / | p_{sn} | << 1. \quad (48)$$

The system (46) is decoupled as

$$\begin{bmatrix} x_s(k+1) \\ z_f(k+1) \end{bmatrix} = \begin{bmatrix} A_{ds} & 0 \\ 0 & A_{df} \end{bmatrix} \begin{bmatrix} x_s(k) \\ z_f(k) \end{bmatrix} + \begin{bmatrix} B_{ds} \\ B_{df} \end{bmatrix} u(k), \quad (49)$$

$$A_{d22} D - DA_{d11} + DA_{d12} D - A_{d21} = 0. \quad (50)$$

$$E(A_{d22} + DA_{d12}) - (A_{d11} - A_{d12} D) E + A_{d12} = 0. \quad (51)$$

The iterative solutions of (50) and (51) are obtained by (52) and (53).

$$D_{i+1} = (A_{d22} D_i + D_i A_{d12} D_i - A_{d21}) A_{d11}^{-1},\ D_0 = -A_{d21} A_{d11}^{-1}. \quad (52)$$

$$E_{i+1} = A_{d11}^{-1}(E_i A_{d22} + E_i D A_{d12} + A_{d12} D E_i + A_{d12}),\ E_0 = A_{d11}^{-1} A_{d12}. \quad (53)$$

B. Closed-loop system

One uses the linear equation (46) and the quadratic cost index as

$$J = \tfrac{1}{2} \sum_{k=0}^{\infty} \{ x^T(k) z^T(k)]^T Q [x^T(k) z^T(k)] + u^T(k) R u(k) \}, \quad (54)$$

$$u(k) = -FX(k), \quad (55)$$

$$F = (R + B^T PB)^{-1} B^T PA. \quad (56)$$

where $X(k) = [x(k)\, z(k)]^T$, and *P* is the solution of the ARE in the discrete system as

$$P = A_C^T P A_C + F^T R F + Q, \quad (57)$$

$$A_C = A - BF. \quad (58)$$

As the same manner for the continuous time case, one has two kinds of the cost functions as

$$J_s = \tfrac{1}{2} \sum_{k=0}^{\infty} \{ x(k)^T Q_s x(k) + u_s^T(k) R_s u_s(k) \},$$
$$J_f = \tfrac{1}{2} \sum_{k=0}^{\infty} \{ z(k)^T Q_f z(k) + u_f^T(k) R_f u_f(k) \}. \quad (59)$$

Also one has two sorts of the optimal gains and the corresponding the AREs as

$$F_s = (R_s + B_s^T P_s B_s)^{-1} B_s^T P_s A_s,$$
$$F_f = (R_f + B_f^T P_f B_f)^{-1} B_f^T P_f A_f, \quad (60)$$
$$P_s = A_{sC}^T P_s A_{sC} + F_s^T R_s F_s + Q_s,$$
$$P_f = A_{fC}^T P_f A_{fC} + F_f^T R_f F_f + Q_f, \quad (61)$$
$$A_{Csub} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} - \begin{bmatrix} B_s \\ B_f \end{bmatrix} \begin{bmatrix} F_s & F_f \end{bmatrix} \quad (62)$$

C. Insensitive design

We multiply the output-related term by the Lagrange coefficient in the cost functional and consider it at the reduced value. Thus, one can rewrite (59) (55), (60), (61), (58) as

$$J_s(\lambda) = \tfrac{1}{2}\sum_{k=0}^{\infty}\{\lambda[x_s(k)^T Q_s x_s(k)] + u_s^T(k) R_s u_s(k)\}, \quad (63)$$
$$u_s(k)_\lambda = -F_s(\lambda) x_s(k), \quad (64)$$
$$F_s(\lambda) = (R_s + B_s^T P_s(\lambda) B_s)^{-1} B_s^T P_s(\lambda) A_s, \quad (65)$$
$$P_s(\lambda) = A_{sC}^T P_s(\lambda) A_{sC} + F_s^T(\lambda) R_s F_s(\lambda) + Q_s, \quad (66)$$
$$A_{sC} = A_s - B_s F_s(\lambda). \quad (67)$$

In the case of non-zero ε, (67) is rewritten as

$$A_{sC}(\lambda)_{\varepsilon \neq 0} = A - BR^T B^T F_s(\lambda)_{aug}. \quad (68)$$

where $K_s(\lambda)_{aug}$ is augmented by *0* to match the order of the system. Similarly as in the unified case, the variation of the closed-loop response is limited to a prespecified extent if the following condition [6] is satisfied

$$Max \| P_s \| \le \rho, \quad (69)$$

where ρ is a positive scalar of the upper bound of the allowable trajectory variation, and $\| P \|$ is the maximum eigenvalue of the real, symmetric matrix P_s of the solution of the Lyapunov equation as (66). Note that the discrete algebraic Riccati equation is the same as the discrete algebraic Lyapunov equation. Of course, equations of the fast variable $z(k)$ are not used for the proposed insensitive design.

The steps in the numerical algorithm are listed as follows:

Step 1: Obtain A_s, B_s.

Step 2: Select ρ and λ.

Step 3: Using (65),(66), and (68), find its solution, $P_s(\lambda)$.

Step 4: Find $A_{sC}(\lambda)$.

Step 5: Check for the sensitivity criterion as in (69). If not satisfied, increase the value of λ and repeat the process from step 3.

V. NUMERICAL EXAMPLE

Parameters of a linear time-invariant singularly perturbed contiuous model (11) are given as [22].

$$A_{11} = \begin{bmatrix} -3 & -1 \\ 0 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} -34 \\ 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix}, A_{22} = -18.$$
$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_2 = 1.$$

Q is a *3x3* dimensional identity matrix. *R=1*. The simulation results are obtained as in the following tables,

TABLE I

$Max\|P\|$ $\varepsilon = 0, \Delta_d = 0.01, \Delta_{ud} = 0.01, \Delta_{uc} = 0.01$

	λ=0.1	λ=1	λ=5
ρ	0.3	0.25	0.24
$Max\|P\|_{cont}$	0.2508	0.2033	0.1430
$Max\|P\|_{del-disc}$	0.2507	0.2027	0.1423
$Max\|P\|_{disc}$	0.2598	0.2137	0.1379
$Max\|P\|_{del-cont}$	0.2532	0.2053	0.1449

TABLE II

$Max\|P\|$ $\varepsilon = 0.25, \Delta_d = 0.01, \Delta_{ud} = 0.01, \Delta_{uc} = 0.01$

	λ=0.1	λ=1	λ=5
ρ	0.3	0.25	0.24
$Max\|P\|_{cont}$	0.2544	0.2131	0.1581
$Max\|P\|_{del-disc}$	0.2543	0.2126	0.1586
$Max\|P\|_{disc}$	0.2589	0.2149	0.1431
$Max\|P\|_{del-cont}$	0.2565	0.2140	0.1540

TABLE III

$Max\|P\|$ $\varepsilon = 0.5, \Delta_d = 0.01, \Delta_{ud} = 0.01, \Delta_{uc} = 0.01$

	λ=0.1	λ=1	λ=5
ρ	0.3	0.25	0.24
$Max\|P\|_{cont}$	0.2634	0.2285	0.2329
$Max\|P\|_{del-disc}$	0.2634	0.2283	0.2340
$Max\|P\|_{disc}$	0.2682	0.2310	0.2221
$Max\|P\|_{del-cont}$	0.2648	0.2272	0.2248

TABLE IV

$Max\|P\|$ $\varepsilon = 0.25,\ \lambda = 0.1$ with different Δ

	$\Delta_d = 0.1$, $\Delta_{ud} = 0.1$, $\Delta_{uc} = 0.1$.	$\Delta_d = 0.01$, $\Delta_{ud} = 0.01$, $\Delta_{uc} = 0.01$.	$\Delta_d = 0.001$, $\Delta_{ud} = 0.001$, $\Delta_{uc} = 0.001$.

$Max \| P \|_{cont}$	0.2544	0.2544	0.2544
$Max \| P \|_{del-disc}$	0.2537	0.2543	0.2544
$Max \| P \|_{disc}$	0.3055	0.2589	0.2552
$Max \| P \|_{del-cont}$	0.2770	0.2565	0.2546

where subscripts '*cont*', '*del-disc*', '*disc*', and '*del-con* ' denote '*continuous system*', '*discrete-like delta system*', '*discrete system*', and '*continuous-like delta system*' respectively. Δ_d is a sampling interval when converting the continuous system into the discrete system. Δ_{ud} is a sampling interval to obtain the discrete-like unified system as shown in (32), (35), (36), (40), (41), and (45). Δ_{uc} is a sampling interval to obtain the continuous-like unified system as shown in (6).

The results obtained as in the table 1, table 2, and table 3 shows that, as λ increases, the maximum eigenvalue of *P* in each case decreases. This means that the sensitivity is improved as λ increases. Note that increasing λ value means increasing the cost.

In the table 4, it is proved that, as sampling interval approaches zero, the maximum eigenvalue of the discrete-like unified system is identified with that of the continuous system. For $\Delta_d = 0.001$ and $\Delta_{uc} = 0.001$, both the maximum eigenvalues of the discrete system and the continuous-like unified system are nearly equal to each other. Also, it is shown that all the eigenvalues of *P* in the four kinds of systems have nearly close values when Δ_d, Δ_{ud}, and Δ_{uc} goes to zero. When a coarse sampling interval is used, the discrete system shows the worst results among those four systems. However, the δ operating unified approach shows very fine results even if a coarse sampling interval is used.

VI. CONCLUSION

It is shown that the finite word-length characteristics are improved when using the δ operating unified approach when making a model reduction by ignoring the fast mode for the singularly perturbed closed-loop systems. The δ systems have less error and better robustness over the discrete systems.

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