2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Input-output decoupling with asymptotic stability of linear mechanical systems through connection with another mechanical system

Laura Menini and Antonio Tornambè

Abstract— **This paper deals with the input-output decoupling problem with asymptotic stability for a class of linear mechanical systems (with two inputs and two outputs), through parallel connection with another mechanical system, called the controller. The paper gives a procedure for the design of a controller, which solves the above problem under some mild sufficient conditions.**

I. INTRODUCTION

The control of mechanical systems is one of the main research topics in the systems and control field, in view of the great variety of related applications, e.g., in the robotics and aerospace areas. In recent years, the control of mechanical systems has received renewed attention, mainly due to the advances in the modelling and control of Hamiltonian systems (see, e.g., [1], [2]). In this paper, the classical problem of input-output decoupling is dealt with for a class of two-input two-output multi-body linear mechanical systems, with the requirement that the controller has to be another mechanical system to be physically connected to the given one (two terminal points of the controller are to be glued to the actuated bodies). A similar approach (for a control view point see [3]) is quite classical in vibration control, where the possibility of reducing the vibrations of a mechanical structure by connecting to it either mechanical dampers or electric RLC circuits is called passive control.

With respect to the standard way of designing a controller, constituted by a generic dynamical system taking as input the available outputs of the system (often the whole state), and giving as output the forces or torques to be applied to the actuated bodies, the approach taken here has two main differences: 1) the proposed controller has to be a very special dynamical system, with a strong structure (this limits severely the possible choices), 2) it is possible to use non-causal controllers.

Many of the concepts used in this paper are inspired by the classical tradition of studying mechanical systems through the analogy with suitable electric circuits (see, e.g., [4]) and by the use of passivity concepts [5].

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a linear mechanical system constituted by ideal point bodies, linear springs and linear dampers, moving on a line. Let $q_i(t)$ be the position at time $t \in \mathbb{R}$ with respect to an inertial reference frame of the i-th body, $i = 1, 2, ..., n$, where n is the number of the bodies and let $\mathbf{q}(t) := \begin{bmatrix} q_1(t) & q_2(t) & \cdots & q_n(t) \end{bmatrix}^T$; let $M_i \in \mathbb{R}, M_i > 0$, be the mass of the i-th body, $i = 1, 2, ..., n$. When present, let $K_{i,j} \in \mathbb{R}$ ($D_{i,j} \in \mathbb{R}$, $D_{i,j} \geq 0$) be the coefficient of elasticity (the damping factor) of the spring (the damper) possibly connecting the i-th body with the j-th one, $i = 1, 2, ..., n, j = i + 1, ..., n$; when present, let $K_{0,i} \in \mathbb{R}$ ($D_{0,i} \in \mathbb{R}$, $D_{0,i} \geq 0$) be the coefficient of elasticity (the damping coefficient) of the spring (the damper) possibly connecting the i-th body ($i = 1, 2, ..., n$) with the ground, constituted by an infinitely massive body (numbered with the index 0). Without loss of generality, in all the paper the length at rest of all the springs will be considered null.

Notation 1: $\mathbf{A} > 0$ (respectively, $\mathbf{A} \geq 0$) means that matrix $\mathbf{A}$ is real, symmetric and positive definite (respectively, semi-definite).

Let the system be described by the following kinetic and potential energies and by the following dissipation function, respectively: $\mathcal{T} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B}\, \dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n} M_i \dot{q}_i^2$, $\mathcal{U} = \frac{1}{2}\mathbf{q}^T \mathbf{H}\, \mathbf{q} = \frac{1}{2}\sum_{i=1}^{n} K_{0,i} q_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} K_{i,j}\,(q_i - q_j)^2$, $\mathcal{F} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{D}\, \dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n} D_{0,i}\dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} D_{i,j}\,(\dot{q}_i - \dot{q}_j)^2$, where $\mathbf{B}$ is the generalized inertia matrix which is diagonal and positive definite (all the bodies have non-null mass), $\mathbf{D}$ is symmetric and positive semi-definite, and $\mathbf{H}$ is symmetric and positive semi-definite if all the springs have non-negative coefficients of elasticity. Assume that the first 2 bodies are actuated by external forces $u_i(t), i = 1, 2$, and let $\mathbf{u}(t) := \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ be the input of the system. The relevant outputs of the system are both the positions $\mathbf{y}_q(t) = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}^T$ of the first two bodies and their velocities $\mathbf{y}_v(t) = \begin{bmatrix} \dot{q}_1(t) & \dot{q}_2(t) \end{bmatrix}^T$. Letting $\mathbf{E} \in \mathbb{R}^{n\times 2}, \mathbf{E} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \end{bmatrix}^T$, the mechanical system is then described by:

$$\mathbf{B}\,\ddot{\mathbf{q}}(t) + \mathbf{D}\,\dot{\mathbf{q}}(t) + \mathbf{H}\,\mathbf{q}(t) = \mathbf{E}\,\mathbf{u}(t), \quad (1)$$
$$\mathbf{y}_q(t) = \mathbf{E}^T\,\mathbf{q}(t), \quad (2)$$
$$\mathbf{y}_v(t) = \mathbf{E}^T\,\dot{\mathbf{q}}(t). \quad (3)$$

Notice that $\det\left(\mathbf{B}\,s^2 + \mathbf{D}\,s + \mathbf{H}\right)$ is not the null function since $\mathbf{B}$ is non-singular. By Laplace transformation:

$$\mathbf{y}_q(s) = \mathbf{E}^T\,(\mathbf{B}\,s^2 + \mathbf{D}\,s + \mathbf{H})^{-1}\mathbf{E}\,\mathbf{u}(s),$$

Work supported by MIUR-COFIN and FIRB: TIGER. Corresponding author L. Menini. Tel. +39-06-7259 7431. Fax +39-06-7259 7460.
The authors are with Dipartimento di Informatica, Sistemi e Produzione, Università di Roma Tor Vergata, via del Politecnico, 1, 00133 Roma, Italy [menini,tornambe]@disp.uniroma2.it

0-7803-9098-9/05/$25.00 ©2005 AACC

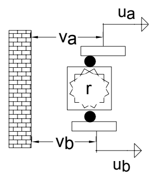

Fig. 1. The pictorial representation of the speed reducer.

$$\mathbf{y}_v(s) = \mathbf{E}^T\ (\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \frac{1}{s})^{-1}\mathbf{E}\ \mathbf{u}(s),$$

where $\mathbf{y}_q(s) = \mathcal{L}\left\{\mathbf{E}^T\ \mathbf{q}(t)\right\}$, $\mathbf{y}_v(s) = \mathcal{L}\left\{\mathbf{E}^T\ \dot{\mathbf{q}}(t)\right\}$, $\mathbf{u}(s) = \mathcal{L}\left\{\mathbf{u}(t)\right\}$. In the following, the **impedance matrix** $\mathbf{Z}(s) = \mathbf{E}^T\ (\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \frac{1}{s})^{-1}\mathbf{E}$ and the **admittance matrix** $\mathbf{Y}(s) = \mathbf{Z}^{-1}(s)$ will be used repeatedly.

It is well known that a square rational matrix function $\mathbf{Z}(s)$ is **positive real** (PR) if $\mathrm{Re}(\mathbf{Z}(s))$ is positive semi-definite for all s having $\mathrm{Re}(s) \geq 0$; $\mathbf{Z}(s)$ is **BIBO stable** if each entry $Z_{i,j}(s) = \dfrac{N_{i,j}(s)}{D_{i,j}(s)}$, with $N_{i,j}(s)$ and $D_{i,j}(s)$ being co-prime polynomials, is proper and its denominator $D_{i,j}(s)$ has all the roots with negative real part; the system (1), (2), (3) described by the impedance $\mathbf{Z}(s)$ is asymptotically stable if all the roots of $\det(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}) = 0$ have negative real part.

Lemma 1: Since $\det(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H})$ is not the null function, if $\mathbf{H}$ is positive semi-definite, in addition to $\mathbf{B}$ and $\mathbf{D}$ which are positive semi-definite as well, then the square rational matrix function $\mathbf{E}^T(\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \dfrac{1}{s})^{-1}\mathbf{E}$ is PR.

Taking into account that, when it exists, the inverse of a PR matrix is PR, under the hypotheses of Lemma 1, both the impedance and the admittance of system (1), (2), (3) are PR.

In this paper, the controller will not be a generic dynamical system taking as input $\mathbf{y}_q(t)$ and/or $\mathbf{y}_v(t)$ and giving as output $\mathbf{u}(t)$, but, rather, the controller will be another mechanical system having two terminal points to be physically connected to the first two bodies of the system. The connection can be either a direct one (i.e., the terminal point is glued to the mass of the body) or through an (ideal) speed reducer (e.g., an ideal gear reduction unit). The speed reducer, represented schematically in Figure 1, is a two terminal points object, without mass, friction and elasticity, characterized by the transmission ratio r. Denoting by v_i and u_i, $i \in \{a, b\}$, respectively, the velocity and the force applied to the i-th terminal point of the speed reducer, the equations describing its behaviour are $v_b = r\ v_a$, $u_b = \frac{1}{r} u_a$. If q_i, $i \in \{a, b\}$, denotes the position of the i-th terminal point of the speed reducer, we have $q_b = r\ q_a + c$, with c being an arbitrary constant that in this paper is taken equal to 0, without loss of generality. In the special case where $r = 1$, the speed reducer is equivalent to the direct connection, whereas when $r = -1$, it corresponds to inverting the velocity. In particular, if r_1 and r_2 are the transmission ratios of the reducers used for the connection (possibly, equal to 1), the controller is described by:

$$\mathbf{B}_c\ddot{\mathbf{q}}_c(t) + \mathbf{D}_c\dot{\mathbf{q}}_c(t) + \mathbf{H}_c\mathbf{q}_c(t) = \mathbf{0}, \quad (4)$$

$$\mathbf{y}_{c,q}(t) = \mathbf{E}_c^T\mathbf{q}_c(t), \quad (5)$$

with $\mathbf{q}_c(t) \in \mathbb{R}^{n_c}, \mathbf{E}_c \in \mathbb{R}^{n_c \times 2}, \mathbf{E}_c = \begin{bmatrix} \mathbf{R} & \mathbf{0} \end{bmatrix}^T$, $\mathbf{R} = \mathrm{diag}(r_1, r_2)$, $\mathbf{B}_c$ diagonal and positive semi-definite, $\mathbf{D}_c$ symmetric and positive semi-definite, $\mathbf{H}_c$ symmetric and $\det(\mathbf{B}_c\ s^2 + \mathbf{D}_c\ s + \mathbf{H}_c)$ being not the null function. The overall system is then described by the following equations:

$$\mathbf{B}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{H}\mathbf{q}(t) = \mathbf{E}\mathbf{u}(t) + \mathbf{E}\lambda(t), \quad (6)$$

$$\mathbf{B}_c\ddot{\mathbf{q}}_c(t) + \mathbf{D}_c\dot{\mathbf{q}}_c(t) + \mathbf{H}_c\mathbf{q}_c(t) = -\mathbf{E}_c\lambda(t), \quad (7)$$

$$\mathbf{y}_q(t) = \mathbf{y}_{c,q}(t), \quad (8)$$

where $\lambda(t)$ is the vector of the Lagrange multipliers used in order to take into account the equality constraint (8), which represents the forces exchanged between the system and the controller. Notice that, by eliminating the Lagrange multipliers and using the equality constraint (8), the overall system can be rewritten in the form (1), i.e., as an unconstrained mechanical system having $n + n_c - 2$ degrees of freedom. The input of the overall system (6), (7), (8) is still $\mathbf{u}(t)$ and the relevant outputs are still $\mathbf{y}_q(t)$ and $\mathbf{y}_v(t)$. The control problem studied in this paper is stated formally as follows.

Problem 1: Find, if any, a controller of the form (4), (5) such that the overall system (6), (7), (8) is asymptotically stable and input-output decoupled (the latter being equivalent to be have a non-singular and diagonal impedance/admittance matrix).

The overall system (6), (7), (8) will be called the (**mechanical**) **parallel connection** of the system and the controller: if $\mathbf{Y}(s)$ and $\mathbf{Y}_c(s)$ are the admittances of the mechanical system and of the controller, respectively, then the admittance of the parallel connection is $\mathbf{Y}_p(s) = \mathbf{Y}(s) + \mathbf{Y}_c(s)$. As for the impedance $\mathbf{Z}_p(s)$ of the parallel connection, it can be easily seen that $\mathbf{Z}_p(s) = \mathbf{Z}(s)\left(\mathbf{I} + \mathbf{Z}_c^{-1}(s)\mathbf{Z}(s)\right)^{-1}$, i.e. the parallel connection can be seen as a feedback system from the output $\mathbf{y}_v(t)$, in which the transfer matrix of the controller is $\mathbf{Z}_c^{-1}(s)$. Notice that $\mathbf{Z}_c^{-1}(s)$ is not necessarily proper; moreover, we are interested in a controller whose inverse be the impedance of a mechanical system, whence the classical tools for designing a controller that guarantees input-output decoupling with stability cannot be used.

If two systems having PR impedance matrices $\mathbf{Z}_1(t)$ and $\mathbf{Z}_2(t)$ are connected in parallel, the impedance matrix of the parallel connection is still PR. However, special care is to be used when the property of interest is the asymptotic stability of the system, which is stronger than the real positivity.

The following simple example shows that the parallel connection of two asymptotically stable mechanical systems needs not be asymptotically stable.

Example 1: Consider the mechanical system depicted in Figure 2-(a), which is constituted by three bodies moving on an horizontal line and connected by the springs having

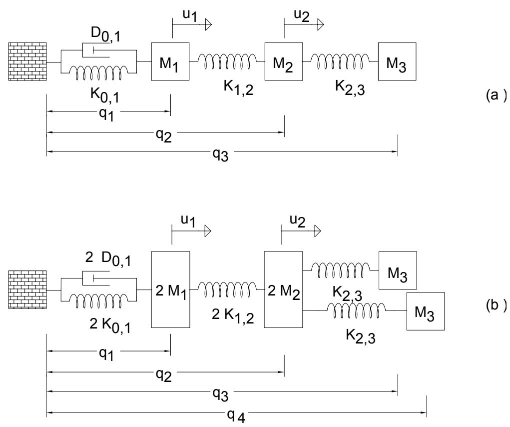

Fig. 2. The mechanical systems considered in Example1

positive coefficients of elasticity $K_{0,1}, K_{1,2}$ and $K_{2,3}$, and one damper having damping coefficient $D_{0,1} > 0$ as shown in Figure 2-(a). With the proposed notations, the system can be rewritten in the form (1), (2), (3), with

$$\mathbf{B} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} D_{0,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{H} = \begin{bmatrix} K_{0,1} + K_{1,2} & -K_{1,2} & 0 \\ -K_{1,2} & K_{1,2} + K_{2,3} & -K_{2,3} \\ 0 & -K_{2,3} & K_{2,3} \end{bmatrix},$$

from which it is easy to compute the polynomial $\det\left(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}\right)$ and to check that it has all the roots with negative real part for all positive $D_{0,1}$.

Taking two identical systems as the one in Figure 2-(a) and connecting them in mechanical parallel, the mechanical system depicted in Figure 2-(b) is obtained. Its description as unconstrained mechanical system has $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T$ as position vector, and it can be rewritten in the form (1), (2), (3), with

$$\mathbf{B} = \begin{bmatrix} 2M_1 & 0 & 0 & 0 \\ 0 & 2M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_3 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 2D_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{H} = \begin{bmatrix} 2K_{0,1}+2K_{1,2} & -2K_{1,2} & 0 & 0 \\ -2K_{1,2} & 2K_{1,2}+2K_{2,3} & -K_{2,3} & -K_{2,3} \\ 0 & -K_{2,3} & K_{2,3} & 0 \\ 0 & -K_{2,3} & 0 & K_{2,3} \end{bmatrix},$$

from which it is easy to compute

$$\det\left(\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}\right) = 2\left(M_3 s^2 + K_{2,3}\right)\hat{p}(s),$$

where $\hat{p}(s)$ is a polynomial having all the roots with negative real part. The polynomial (1) has $\pm j\sqrt{\frac{K_{2,3}}{M_3}}$ as roots for any value of $D_{0,1}$, which shows that the parallel connection of two asymptotically stable mechanical systems can be not asymptotically stable.

We recall the well known fact (see [6]) that, for mechanical systems of the form (1), (2), (3), the stabilizability from the input $\mathbf{u}(t)$ and the detectability from the output $\mathbf{y}_v(t)$ can be tested, respectively, by means of the following two necessary and sufficient conditions:

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H} & \mathbf{E} \end{bmatrix}\right) = n, \quad \forall s \in \mathbb{C}, \operatorname{Re}(s) \geq 0, \tag{9}$$
$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H} \\ s\ \mathbf{E}^T \end{bmatrix}\right) = n, \quad \forall s \in \mathbb{C}, \operatorname{Re}(s) \geq 0. \tag{10}$$

Remark 1: If $\det(\mathbf{H}) \neq 0$, then the stabilizability and detectability conditions (9), (10) are equivalent. As a matter of fact, if $\det(\mathbf{H}) \neq 0$, then $\operatorname{rank}\left(\left[\mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H}\right]\right)$ is maximum for $s = 0$, and therefore we can assume without loss of generality that $s \neq 0$; under such an assumption,

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H} \\ s\ \mathbf{E}^T \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H} \\ \mathbf{E}^T \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ s^2 + \mathbf{D}\ s + \mathbf{H} & \mathbf{E} \end{bmatrix}\right).$$

Remark 2: Considering again the parallel connection described in Example 1, it can be easily verified that also the properties of stabilizability and detectability have been lost. As a matter of fact, for $\hat{s} = \sqrt{\frac{K_{2,3}}{M_3}}$, we have $\operatorname{rank}\left(\begin{bmatrix} \mathbf{B}\ \hat{s}^2 + \mathbf{D}\ \hat{s} + \mathbf{H} & \mathbf{E} \end{bmatrix}\right) = 3 < 4$. On the other hand, the simple system depicted in Figure 2-(a) is reachable and observable, whence stabilizable and detectable; therefore, it is clear that also the structural properties of stabilizability and detectability can be lost by the mechanical parallel connection.

The goal of this paper is to find a controller having admittance matrix $\mathbf{Y}_c(s)$ such that the overall system is input-output decoupled and asymptotically stable. The next three lemmas recall important facts that will be useful in the proof of the main result. The first one (see also [7]) is concerned with the possibility of stabilizing a mechanical system — having PR impedance $\mathbf{Z}(s)$ — by connecting the two actuated bodies with the ground by means of two identical dampers having damping coefficient equal to $D > 0$. Such a connection (which is practically equivalent to using a derivative control law) can be seen as the parallel connection of the given mechanical system and of the controller with singular $\mathbf{B}_c$ constituted by just the two dampers, having admittance matrix $\mathbf{Y}_c(s) = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} = \mathbf{Z}_c^{-1}(s)$. Therefore, such a parallel connection has the following impedance matrix $\mathbf{Z}_p(s) = \mathbf{Z}(s)(\mathbf{I} + D\ \mathbf{Z}(s))^{-1}$.

Lemma 2: If $D > 0$ and $\mathbf{Z}(s)$ is PR, then $\mathbf{Z}_p(s) = \mathbf{Z}(s)(\mathbf{I} + D\ \mathbf{Z}(s))^{-1}$ is BIBO stable. If the stabilizability and detectability conditions (9) and (10) hold, then the parallel connection having $\mathbf{Z}_p(s)$ as impedance matrix is asymptotically stable.

The second intermediate result is concerned with the possibility of rendering PR the impedance matrix of a mechanical system by connecting the two actuated bodies with the ground by means of two identical springs having a positive and sufficiently high coefficient of elasticity K. Such a connection (which is practically equivalent to using a proportional control law) can be seen as the parallel connection of the given mechanical system and of the controller with singular $\mathbf{B}_c$ constituted by just the two springs, having admittance matrix $\mathbf{Y}_c(s) = \begin{bmatrix} \frac{K}{s} & 0 \\ 0 & \frac{K}{s} \end{bmatrix} = \mathbf{Z}_c^{-1}(s)$. Moreover, the description of the parallel connection in the form (1), (2), (3) has the same $\mathbf{B}$ and $\mathbf{D}$ matrices of the given mechanical system, whereas for its matrix $\mathbf{H}_p$ we have:

$$\mathbf{H}_p = \mathbf{H} + \mathrm{diag}\Big(K, K, \underbrace{0, ..., 0}_{n-2 \text{ times}}\Big). \qquad (11)$$

The following lemma, gives a necessary and sufficient condition for the impedance matrix of the mechanical system to be PR.

Lemma 3: If $\mathbf{B} > 0$, $\mathbf{D} \geq 0$ and the stabilizability and detectability conditions (9) and (10) hold, then $\mathbf{E}^T(\mathbf{B}\ s + \mathbf{D} + \mathbf{H}\ \frac{1}{s})^{-1}\mathbf{E}$ is PR if and only if $\mathbf{H} \geq 0$.

As for the possibility of rendering matrix $\mathbf{H}_p$ positive definite through an appropriate choice of K, a necessary and sufficient condition is given by the following lemma (see [8]).

Lemma 4: The matrix $\mathbf{H}_p$ in (11) can be rendered positive definite with a suitable choice of K if and only if the matrix $\mathbf{H}_{22} \in R^{(n-2)\times(n-2)}$ obtained by removing the first two rows and columns of $\mathbf{H}$ is positive definite. Moreover, if $\mathbf{H}_{22} > 0$, then there exists $\bar{K} \geq 0$ such that $\mathbf{H}_p > 0$ for all $K > \bar{K}$.

III. Main result

Now, in order to design a controller solving Problem 1, consider the pictorial representation of the given mechanical system as a non-directed graph having $n+1$ vertices, one for each body-mass and one for the ground, and one edge for each spring and damper. The following assumption can be made without loss of generality.

Assumption 1: Assume that the graph associated with the given mechanical system is connected.

Denote by $\mathcal{X}_1$ the set of the vertices (masses) that are connected by a path of the graph with the vertex corresponding to M_1 after removing the vertices corresponding to M_2 and the ground, and all the edges connecting such vertices. Symmetrically, define $\mathcal{X}_2$ by removing M_2, the ground and the relevant edges. Let $S_{12} = \mathcal{X}_1 \cap \mathcal{X}_2$, $S_1 = \mathcal{X}_1 \setminus \{\mathcal{X}_1 \cap S_{12}\}$ and $S_2 = \mathcal{X}_2 \setminus \{\mathcal{X}_2 \cap S_{12}\}$. Let n_1, n_2 and n_3 be the cardinalities of S_1, S_2 and S_{12}, respectively.

In this way, the n degrees of freedom of the given system can be partitioned into 5 sets, represented pictorially in Figure 3, with $n = n_1+n_2+n_3+2$. In Figure 3, the spring labeled by $\mathcal{K}_1$ represents a set of springs with possibly different coefficients of elasticity, each one connecting a different mass of the set S_{12} with M_1 ($\mathcal{K}_1$ can be understood as the vector of such coefficients of elasticity); the same happens for the springs labeled by $\mathcal{K}_2, ..., \mathcal{K}_7$ and the dampers labeled by $\mathcal{D}_1, ..., \mathcal{D}_7$. Furthermore, not all such springs and dampers need to be actually present, since the case when a spring is missing can be considered by letting its coefficient of elasticity be equal to zero, and similarly for the dampers (but for each $i \in \{1, 2, 4, 6\}$ either $\mathcal{D}_i$ or $\mathcal{K}_i \neq 0$).

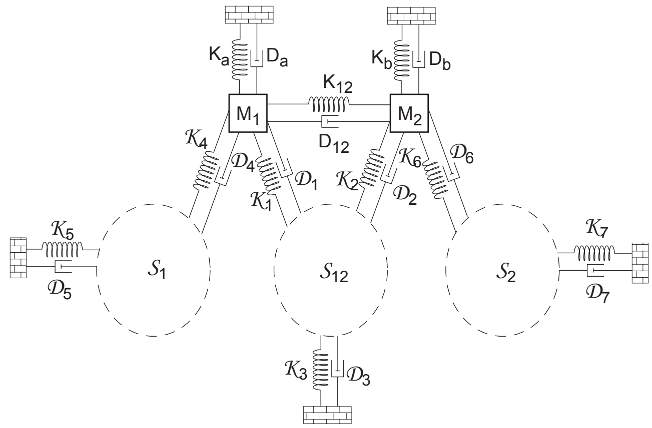

Fig. 3. Decomposition of the given system. For space reasons springs and dampers are depicted in different directions, but the reader should imagine all the motions as horizontal.

The proposed controller is a n_c-degrees of freedom mechanical system, with $n_c = n_3+2$, constituted by a copy of the masses M_1 and M_2, whose coordinates will be denoted by $q_{c,1}$ and $q_{c,2}$, respectively, and all the masses contained in the set S_{12}, with (i) all the springs and dampers that in the given system connect such masses with each other and with the ground, (ii) two additional dampers having damping coefficient $D > 0$ connecting the bodies with coordinates $q_{c,1}$ and $q_{c,2}$ with the ground and (iii) two additional springs with sufficiently high coefficient of elasticity K connecting the same two bodies with the ground. Such a coefficient of elasticity is to be chosen sufficiently high as detailed in the subsequent proof of Theorem 1. Moreover, a speed reducer characterized by $r = -1$ is to be used to connect the body having coordinate $q_{c,2}$ with the second body of the given system, whereas the body having coordinate $q_{c,1}$ is to be glued with the first body of the system. In this way, the matrix $\mathbf{R}$ used in the description of the controller is $\mathbf{R} = \mathrm{diag}\,(1, -1)$. In order to give the main result of the paper, let Σ_1 denote the SISO mechanical system obtained from the given one by fixing to the ground the mass M_2, and removing the masses in S_2 and all the springs and dampers directly connected with the removed masses so to obtain a system with $n_1 + n_3 + 1$ degrees of freedom, having input u_1 and output y_1. Symmetrically, define Σ_2 by fixing M_1 and removing all the masses in S_1, with the relevant springs and dampers, so to obtain a system with n_2+n_3+1 degrees of freedom, having input u_2 and output y_2.

Theorem 1: Under Assumption 1, if (i) the matrix $\mathbf{H}_{22}$ defined as in Lemma 4 is positive definite, (ii) Σ_1 and Σ_2

are reachable, then there exists $\bar{K} \geq 0$ such that for each $K > \bar{K}$ the mechanical parallel connection of the given system with the proposed controller is asymptotically stable and input-output decoupled.

Proof: Consider the admittance matrices of the system and of the controller

$$\mathbf{Y}(s)=\begin{bmatrix} Y_{11}(s) & Y_{12}(s) \\ Y_{12}(s) & Y_{22}(s) \end{bmatrix}, \ \mathbf{Y}_c(s)=\begin{bmatrix} Y_{c,11}(s) & Y_{c,12}(s) \\ Y_{c,12}(s) & Y_{c,22}(s) \end{bmatrix}.$$

Since $u_1(s) = Y_{11}(s)v_1(s) + Y_{12}(s)v_2(s)$, it follows that $Y_{12}(s)$ is the transfer function from the velocity $v_2(t)$ of the second body to the force $u_1(s)$ acting on the first body as a consequence of the imposed $v_2(s)$, when $v_1(s) = 0$, i.e., when the first body is rigidly fixed to the ground. Hence, looking at the pictorial representation in Figure 3, it is clear that $Y_{12}(s)$ is due only to the spring and damper possibly connecting M_1 with M_2 and to the masses belonging to S_{12}, their interconnections, and the springs and dampers connecting them with M_1, M_2 and the ground. Such a subsystem is exactly replicated in the controller, and, due to the speed reducer with $r = -1$ used to connect the second mass of the controller, we have $Y_{c,12}(s) = -Y_{12}(s)$. This shows that the mechanical parallel connection of the system and the controller has a diagonal admittance matrix, since $\mathbf{Y}_p(s) = \mathbf{Y}(s) + \mathbf{Y}_c(s)$. In order to show that the overall system is asymptotically stable and has a non-singular admittance matrix, consider the following slight modification of the overall system that is totally equivalent to the proposed one. Rather than connecting the two bodies having coordinates $q_{c,1}$ and $q_{c,2}$ with the ground by means of two springs having coefficient of elasticity K, it is possible to use four springs having coefficient of elasticity $\frac{K}{2}$, two of them connecting the bodies with coordinates $q_{c,1}$ and $q_{c,2}$ with the ground. This amounts to have a modified system with admittance matrix

$$\bar{\mathbf{Y}}(s) \ = \ \mathbf{Y}(s) + \begin{bmatrix} \frac{K}{2} & 0 \\ 0 & \frac{K}{2} \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11}(s) & Y_{12}(s) \\ Y_{12}(s) & \bar{Y}_{22}(s) \end{bmatrix}$$

and a modified controller with admittance matrix

$$\bar{\mathbf{Y}}_c(s) = \mathbf{Y}_c(s) - \begin{bmatrix} \frac{K}{2} & 0 \\ 0 & \frac{K}{2} \end{bmatrix} = \begin{bmatrix} \bar{Y}_{c,11}(s) & Y_{c,12}(s) \\ Y_{c,12}(s) & \bar{Y}_{c,22}(s) \end{bmatrix}.$$

The asymptotic stability and the non-singularity of $\mathbf{Y}_p(s)$ will be proven for the modified system. Now, assume that the overall system is written in the form (1), with matrices $\mathbf{B}_p$, $\mathbf{D}_p$ and $\mathbf{H}_p$ replacing matrices $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$, and with the overall state vector given by $\mathbf{q}_p = \begin{bmatrix} q_1 & q_2 & \mathbf{q}_3^T & \mathbf{q}_{3c}^T & \mathbf{q}_4^T & \mathbf{q}_5^T \end{bmatrix}^T \in \mathbb{R}^{n+n_3}$, $\mathbf{q}_3 \in \mathbb{R}^{n_3}, \mathbf{q}_{3c} \in \mathbb{R}^{n_3}, \mathbf{q}_4 \in \mathbb{R}^{n_1}$, and $\mathbf{q}_5 \in \mathbb{R}^{n_2}$, with $\mathbf{q}_3$, $\mathbf{q}_4$ and $\mathbf{q}_5$ being the coordinates of the bodies in S_{12}, S_1 and S_2, respectively, and $\mathbf{q}_{3c}$ the coordinates of the bodies that constitute the controller, apart from the two having coordinates $q_{c,1}$ and $q_{c,2}$ (such coordinates disappear from the overall description since $q_{c,1} = q_1$ and $q_{c,2} = -q_2$). Matrix $\mathbf{H}_p$ has the following form:

$$\mathbf{H}_p = \begin{bmatrix} 2H_1+K & 0 & \mathbf{H}_{13} & \mathbf{H}_{13} & \mathbf{H}_{14} & \mathbf{0} \\ 0 & 2H_2+K & \mathbf{H}_{23} & -\mathbf{H}_{23} & \mathbf{0} & \mathbf{H}_{25} \\ \mathbf{H}_{13}^T & \mathbf{H}_{23}^T & \mathbf{H}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_{13}^T & -\mathbf{H}_{23}^T & \mathbf{0} & \mathbf{H}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_{14}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{25}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_5 \end{bmatrix}, \qquad (12)$$

where H_1 and H_2 are scalar, $\mathbf{H}_3 \in \mathbb{R}^{n_3 \times n_3}$, $\mathbf{H}_4 \in \mathbb{R}^{n_1 \times n_1}$, and $\mathbf{H}_5 \in \mathbb{R}^{n_2 \times n_2}$. It is clear that if the matrix $\mathbf{H}_{22}$ of the given system, $\mathbf{H}_{22} = \text{blockdiag}(\mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5)$, is positive definite as guaranteed by condition (i), then the corresponding matrix for the parallel connection, given by $\mathbf{H}_{p,22} = \text{blockdiag}(\mathbf{H}_3, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5)$ is definite positive too; hence, by Lemma 4, the coefficient K can be chosen as $K = K_0$ sufficiently high so to guarantee that $\mathbf{H}_p > 0$. By choosing

$$K > 2K_0 + 2\max\{H_1, \ H_2\}, \qquad (13)$$

the asymptotic stability of the parallel connection can be proven through Lemmas 2 and 3, by showing that the overall system is stabilizable and detectable. In the following, it will be shown that, under condition (ii), the overall system is actually reachable and observable. First, notice that the system $\bar{\Sigma}_1$ obtained by adding to Σ_1 a spring with coefficient of elasticity $\frac{K}{2}$ that connects its mass M_1 with the ground is reachable if Σ_1 is reachable. Notice also that, in view of Remark 1, system $\bar{\Sigma}_1$ is also observable if its matrix $\mathbf{H}$ is non-singular. By using the same symbols as above, the matrix $\mathbf{H}$ of system $\bar{\Sigma}_1$ is given by

$$\bar{\mathbf{H}}_1 = \begin{bmatrix} H_1 + \frac{K}{2} & \mathbf{H}_{13} & \mathbf{H}_{14} \\ \mathbf{H}_{13}^T & \mathbf{H}_3 & \mathbf{0} \\ \mathbf{H}_{14}^T & \mathbf{0} & \mathbf{H}_4 \end{bmatrix},$$

which, by Shur complements, is positive definite if

$$H_1 + \frac{K}{2} - \mathbf{H}_{13}\mathbf{H}_3^{-1}\mathbf{H}_{13}^T - \mathbf{H}_{14}\mathbf{H}_4^{-1}\mathbf{H}_{14}^T > 0. \qquad (14)$$

By using Shur complements on $\mathbf{H}_p$ with $K = K_0$, which is positive definite, one obtains

$$2H_1 + K_0 - 2\mathbf{H}_{13}\mathbf{H}_3^{-1}\mathbf{H}_{13}^T - \mathbf{H}_{14}\mathbf{H}_4^{-1}\mathbf{H}_{14}^T > 0, \quad (15)$$

$$2H_2 + K_0 - 2\mathbf{H}_{23}\mathbf{H}_3^{-1}\mathbf{H}_{23}^T - \mathbf{H}_{25}\mathbf{H}_5^{-1}\mathbf{H}_{25}^T > 0. \quad (16)$$

Taking into account that $\mathbf{H}_4 > 0$, condition (15) and (13) obviously imply condition (14), thus showing that $\bar{\Sigma}_1$ is observable. The same considerations hold for $\bar{\Sigma}_2$ obtained similarly from Σ_2, thus showing that also $\bar{\Sigma}_2$ is reachable and observable. Now, consider the pictorial representation of $\bar{\Sigma}_1$ reported in Figure 4.

It is clear that the admittance of $\bar{\Sigma}_1$, which coincides with $\bar{Y}_{11}(s)$ can be decomposed as $\bar{Y}_{11}(s) = \bar{Y}_{11,R}(s) + \bar{Y}_{11,L}(s)$, where $\bar{Y}_{11,R}(s)$ corresponds to the force that is due to the inertia of M_1 and all the masses in S_{12} (the part of the system on the right of point A in Figure 4), whereas $\bar{Y}_{11,L}(s)$ corresponds to the force due to the masses in S_1 (the part of the system on the left of point A in Figure 4). The term $\bar{Y}_{11,R}(s)$, which is the term that is duplicated by the proposed (modified) controller, is a non-proper rational

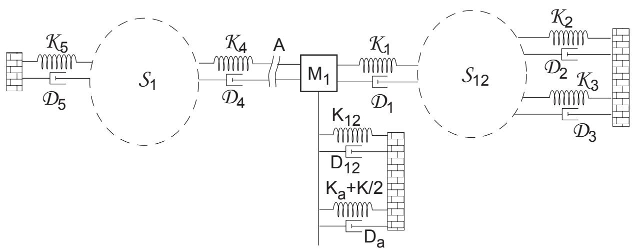

Fig. 4. The subsystem $\bar{\Sigma}_1$.

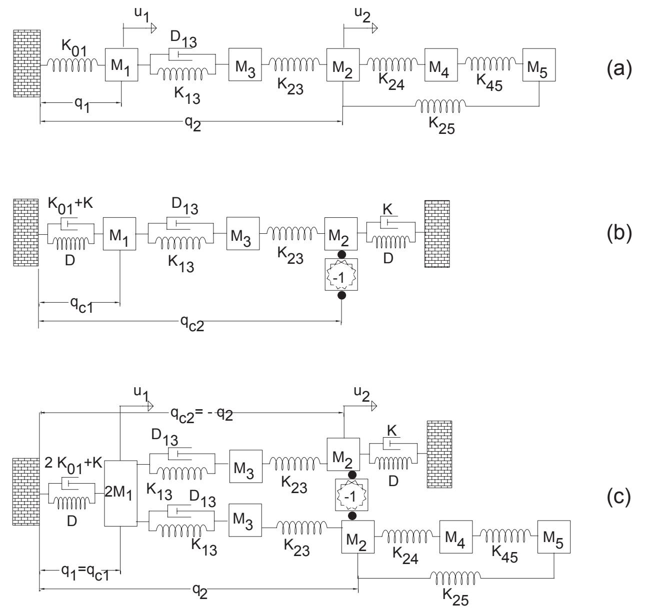

Fig. 5. The mechanical system, the controller and their parallel connection, considered in Example2.

function, since it contains the inertia of mass M_1, whereas $\bar{Y}_{11,L}(s)$ is a proper transfer function (possibly not strictly proper); the proofs of these facts can be made easily through the electric circuit analog to the mechanical system $\bar{\Sigma}_1$ (see [4]) or by simple algebraic manipulations in the Laplace domain. Obviously, a wholly similar decomposition $\bar{Y}_{22}(s) = \bar{Y}_{22,R}(s) + \bar{Y}_{22,L}(s)$ holds for the admittance of $\bar{\Sigma}_2$, where $\bar{Y}_{22,R}(s)$ corresponds to S_{12} and $\bar{Y}_{22,L}(s)$ to S_2. The admittance matrix of the parallel connection is

$$\mathbf{Y}_p(s) = \begin{bmatrix} 2\ \bar{Y}_{11,R}(s)+\bar{Y}_{11,L}(s) & 0 \\ 0 & 2\ \bar{Y}_{22,R}(s)+\bar{Y}_{22,L}(s) \end{bmatrix} =: \begin{bmatrix} Y_{p,1}(s) & 0 \\ 0 & Y_{p,2}(s) \end{bmatrix}.$$

In view of the diagonal structure of $\mathbf{Y}_p(s)$, the overall system is reachable and observable if the sum of the degrees of the numerators of the diagonal elements of $\mathbf{Y}_p(s)$ (the denominators of the diagonal elements of $\mathbf{Z}_p(s) = \mathbf{Y}_p^{-1}(s)$) is equal to $2(n+n_c-2) = 2(2n_3+n_1+n_2+2)$ (i.e., twice the number of the degrees of freedom of the overall system). Now, since $\bar{\Sigma}_1$ is reachable and observable, the degree of the numerator of $\bar{Y}_{11}(s)$ is $2(n_1+n_3+1)$. Moreover, since $\bar{Y}_{11}(s)$ and $Y_{p,1}(s)$ are linear combination of the same two rational functions $\bar{Y}_{11,R}(s)$ and $\bar{Y}_{11,L}(s)$, with different non-null coefficients, and $\bar{Y}_{11,R}(s)$ and $\bar{Y}_{11,L}(s)$ have different relative degree, it follows that also the degree of the numerator of $Y_{p,1}(s)$ is $2(n_1+n_3+1)$. In the same way, it can be seen that the degree of the numerator of $Y_{p,2}(s)$ is $2(n_2+n_3+1)$, thus completing the proof of the asymptotic stability. Finally, it is clear that both $Y_{p,1}(s)$ and $Y_{p,2}(s)$ are not zero, whence $\mathbf{Y}_p(s)$ and $\mathbf{Z}_p(s)$ are non-singular. ■

In the following example, the procedure for the design of the controller is illustrated for a non-trivial mechanical system.

Example 2: Consider the mechanical system depicted in Figure 5-(a), where $n = 5$, $M_i > 0$, $D_{1,3} > 0$, and $K_{i,j} > 0$. The relevant sets for the design of the controller are $\mathcal{S}_{12} = \{M_3\}$, $\mathcal{S}_1 = \{\emptyset\}$ and $\mathcal{S}_2 = \{M_4,\ M_5\}$; the resulting controller, with $n_c = 3$, is depicted in Figure 5-(b), with $K, D > 0$. The overall system, i.e., the mechanical parallel connection of the system and the controller, is depicted in Figure 5-(c). As for the values of K that guarantee that $H_p > 0$, it can be seen that $\bar{K} = 0$, whence, any positive value of K can be chosen. With the method proposed in Example 1, it can be verified that the overall system is asymptotically stable for any $K, D > 0$.

IV. Conclusions

In this paper the problem of input-output decoupling has been dealt with for linear mechanical systems under the requirement that the controller is another mechanical system to be physically connected to the given one. The problem has been solved for two-input two-output systems, under some weak conditions on the structural properties of the system.

Further work will be devoted to the case of m-inputs and m-outputs, and to nonlinear mechanical systems.

References

[1] R. Ortega, M. W. Spong, F. Gmez-Estern, and G. Blankenstein, “Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment,” *IEEE Trans. on Automat. Contr.*, vol. 47, no. 8, pp. 1218–1233, 2002.

[2] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, “Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems,” *Automatica*, vol. 38, no. 4, pp. 585–596, 2002.

[3] J. C. Willems, “On Interconnections, Control, and Feedback,” *IEEE Trans. on Automat. Contr.*, vol. 42, no. 3, pp. 326–339, 1997.

[4] A. MacFarlane, *Dynamical system models*. London: Harrap, 1970.

[5] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez, *Passivity-based Control of Euler-Lagrange Systems: Mechanical, Electrical and Electromechanical Applications.* London: Springer Verlag (Series Comunications and Control Engineering), 1998.

[6] T. Kailath, *Linear systems*. Englewood Cliffs, N.J.: Prentice-Hall Information and System Sciences Series, 1980.

[7] C. Byrnes, A. Isidori and J. C. Willems , “Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems,” *IEEE Trans. on Automat. Contr.*, vol. 36, no. 11, pp. 1228-1240, 1991.

[8] H. Nijmeijer and A. van der Schaft, *Nonlinear dynamical control systems*. New York: Springer-Verlag, 1990.