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Force/Position Output Feedback Tracking Control of Holonomically Constrained Rigid Bodies

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***Abstract*— In this note a new concept of force/position control approach for holonomically constrained rigid body systems is introduced. With this force control approach, the forces of the mechanical constraints between the rigid bodies as well as the forces of the constraints when the system's end effector interacting with the environment are to be directly controlled to *desired trajectories*. Our force control strategy is based on the use of a new dynamic model for constrained rigid body systems that determines the equations of the mechanical constraints between the elements of the constrained rigid body system in *closed form*. Our controller for the rigid body system during constrained motion ensures exponential position, end effector contact force and mechanical force tracking, and requires only position measurements.**

I. INTRODUCTION

Many robotic tasks involve intentional interaction between the manipulator and the environment. Usually, the end effector is required to follow in a stable way the edge or the surface of a workpiece while applying prescribed forces and torques. The need to handle complex contact situations of robotic tasks requires control also of the exchanged forces at the contact. An example is that of a surface finishing task where the tool motion is specified in the direction tangent to the piece, while along the normal direction it is desired to exert a force of given value.

During interaction, the environment imposes constraints on the geometric paths that can be followed by the manipulator's end effector. This situation is generally referred to as constrained motion. Another kind of contact that involves constrained motion is that between the links of the manipulator where the links are mechanically hinged. Indeed, the manipulator can be regarded as a set of rigid bodies (i.e., the links) that are subject to holonomically mechanical constraints. These constraints act on the Cartesian behavior of the manipulator and the resulting situation can be referred to as *Cartesian constrained motion*.

The mechanical forces resulting from the Cartesian constrained motion of the manipulator could be indirectly controlled by acting on the reference value of the position of manipulator motion control system and (if present) on that of the contact force control system. If it is desired to accurately control the mechanical forces, it is necessary to devise control schemes that allow directly specifying the desired mechanical forces. This is the objective we are trying to achieve by this work.

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In this note a new force/position control approach is formulated. With this force control approach, the mechanical force variables as well as the contact force variables are to be directly controlled to desired trajectories. Direct control of the mechanical forces essentially relates to the security of the structure against the deformation and the damage, and is motivated by the following: 1) Avoidance of high values of these force variables because they may deform or damage the articulations of the structure. For instance, a manipulator task of driving a screw in a hole when a hard end effector rotational motion can provoke reaction forces whose effects lead to an increase of mechanical forces that may damage the articulations of the structure. In practice, this is the consequence of the fact that the natural limit set by saturation of manipulator actuators is not well defined. 2) A desired trajectory of the mechanical forces can be considered as a nominal value via given optimal behavior of the articulations. This is useful, for example, to minimize the risque of damage in the structure during fugitive problems of contact with the environment (hard contact during the landing of an airplane).

Our presented force control strategy relies on the use of a new dynamic model for holonomically constrained rigid body systems (e.g., robot manipulators) introduced in [5]. This model is described in the Cartesian coordinate space and gives a redundant dynamics of the system. With the model formulation, we can for the first time explicitly determine the equations of the mechanical constraints between the elements of the rigid body system in *closed form*. In [5], a useful discussion showed how the dimension of the state of this model can be reduced by eliminating the redundancy in the equations of motion, thus obtaining the reduced order model of the Cartesian dynamics system.

In this note, we consider that the rigid body system's end effector is interacting with an infinitely stiff environment. Such interaction is then modelled by holonomic (algebraic) constraints imposed to the system's generalized motion. Based on the projection of the system dynamic equations on a submanifold described by the algebraic equation of constraints, it will be showed that the constrained motion of the generalized dynamics system can be represented by a reduced order model that facilitates the separation design of position and force control strategies. Various techniques for deriving the reduced order model have been proposed in the literature. Here, we will use the reduced order model introduced in [1]. Following the new concept of force/position control approach discussed here, we next design a position/force controller for the holonomically con-

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strained rigid body system during constrained motion that does not require measurement of link velocity, end effector contact force and mechanical force. Our control approach is based on the high-pass filtering of link position information to generate a velocity related signal. We prove exponential position, end effector contact force and mechanical force tracking using standard Lyapunov stability theory.

II. RIGID BODY DYNAMIC MODEL

We briefly introduce here our dynamic model for constrained rigid bodies. A detailed development of this latter can be found in [5]. Consider a system $\mathcal{B}_r$ of r interconnected rigid bodies. Let $q = \text{col}[q_1, \cdots, q_n]$ be a possible choice of generalized coordinates and n the number of degrees of freedom. Let $p_{C_i} \in \mathbb{R}^3$ be the vector of linear position of the i-th rigid body expressed in the base frame. Let $\phi_{C_i} \in \mathbb{R}^3$ be the vector of three rotational Euler angles of the i-th rigid body with respect to the base frame. Then, the Cartesian kinematic motion of the system $\mathcal{B}_r$ can be described in terms of the vector of Cartesian coordinates

$$\pi = \text{col}[p_{C_1}, \ldots, p_{C_r}, \phi_{C_1}, \ldots, \phi_{C_r}] \in \mathbb{R}^{6r}\,. \qquad (1)$$

Let $\dot{p}_{C_i} \in \mathbb{R}^3$ and $\omega_{C_i} \in \mathbb{R}^3$ be the vectors of linear and angular velocities of the center of mass of the i-th rigid body expressed in the base frame. Let the constant m_{C_i} denote the mass of the i-th rigid body. Let $I_{C_i}^{C_i} \in \mathbb{R}^{3\times3}$ correspond to the *constant* inertia tensor of the i-th rigid body relative to its center of mass expressed in a frame attached to the rigid body by the center of mass. Let $R_{C_i} = R_{C_i}(\phi_{C_i}) \in \mathbb{R}^{3\times3}$ be the rotation matrix expressing the orientation of the i-th rigid body frame with respect to the base frame. Let $T(\phi_{C_i}) \in \mathbb{R}^{3\times3}$ be the transformation matrix relating the angular velocity of the i-th rigid body to the time derivative of Euler angles, that is $\omega_{C_i} = T(\phi_{C_i})\,\dot{\phi}_{C_i}$. We will collect all the vectors $\dot{p}_{C_i}$ and $\dot{\phi}_{C_i}$, for $i = 1, \ldots, r$, in $\nu := \dot{\pi} = \text{col}[\dot{p}_{C_1}, \ldots, \dot{p}_{C_r}, \dot{\phi}_{C_1}, \ldots, \dot{\phi}_{C_r}]$ to obtain the vector of Cartesian velocities of $\mathcal{B}_r$.

Based on the definitions above, we introduce the "Cartesian inertia matrix"

$$\mathcal{M}(\pi) = \mathcal{F}(\pi)^\top M\, \mathcal{F}(\pi) \in \mathbb{R}^{6r\times6r} \qquad (2)$$

with

$$M = \text{block-diag}\{m_{C_1}I_3, \cdots, m_{C_r}I_3, I_{C_1}^{C_1}, \cdots, I_{C_r}^{C_r}\} \qquad (3)$$

$$R(\pi) = \text{block-diag}\left\{R_{C_1}(\phi_{C_1}), \cdots, R_{C_r}(\phi_{C_r})\right\} \qquad (4)$$

$$T_\phi(\pi) = \text{block-diag}\left\{T(\phi_{C_1}), \cdots, T(\phi_{C_r})\right\} \qquad (5)$$

$$\mathcal{F}(\pi) = \text{block-diag}\left\{I_{3r},\ R(\pi)^\top T_\phi(\pi)\right\} \qquad (6)$$

where $M \in \mathbb{R}^{6r\times6r}$ is constant symmetric positive definite, $R(\pi) \in \mathbb{R}^{3r\times3r}$ has full rank for all π, $T_\phi(\pi) \in \mathbb{R}^{3r\times3r}$ has full rank for all π through the fact that $T(\phi_{C_i}) \in \mathbb{R}^{3\times3}$, for $i = 1, \ldots, r$, has full rank for certain sequences of elementary rotations of Euler angles ϕ_{C_i}. Hence, $\mathcal{M}(\pi)$ is symmetric and positive definite for all π.

Then, our dynamic model for the system B_r is

$$\mathcal{M}(\pi)\,\dot{\nu} + \dot{\mathcal{M}}(\pi,\nu)\,\nu - N(\pi,\nu) + v = \tau + \tau_c \qquad (7)$$

with

$$N(\pi,\nu) = \frac{1}{2}\sum_{i=1}^{6r}\sum_{j=1}^{6r}\frac{\partial m_{ij}(\pi)}{\partial \pi}\,\nu_i\,\nu_j \qquad (8)$$

$$v = \text{col}\left[-m_{C_1}g_o, \cdots, -m_{C_r}g_o,\, 0_{3r\times1}\right] \qquad (9)$$

where m_{ij} is the generic element of $\mathcal{M}(\pi)$, $\tau \in \mathbb{R}^{6r}$ is the vector of Cartesian forces and torques, $\tau_c \in \mathbb{R}^{6r}$ is the vector of forces and torques corresponding to the holonomically mechanical constraints between the different rigid bodies of $\mathcal{B}_r$, and $g_o \in \mathbb{R}^3$ is the gravity acceleration vector in the base frame.

In words, (7) without the term of constraint τ_c gives the dynamics of a set of r *free* rigid bodies whose elements can reach any position in space. By taking into account the term of constraint τ_c we obtain a nonreduced-order (i.e., redundant) dynamics for the holonomically constrained rigid body system $\mathcal{B}_r$. On the other hand, the holonomic constraints between the elements of the system $\mathcal{B}_r$ allow eliminating s out of $6r =: m$ coordinates of the redundant dynamics (7). With the remaining $n = m - s$ coordinates, it is possible to determine the minimal configuration of $\mathcal{B}_r$. Those coordinates which have already been defined as the vector q are the *nonredundant* generalized coordinates and n is the number of degrees of freedom of the holonomically constrained system $\mathcal{B}_r$. Consequently, the Cartesian kinematic motion of the system $\mathcal{B}_r$ with n degrees of freedom and s holonomic constraints can be described by equation of the form

$$\pi = \pi(q(t))\,. \qquad (10)$$

By differentiating the equation above with respect to time, we obtain the Cartesian kinematics equation of the system

$$\nu = \left(\frac{\partial \pi(q)}{\partial q}\right)^\top \dot{q} =: \mathcal{J}(q)\,\dot{q} \qquad (11)$$

where the "Jacobian matrix" $\mathcal{J}(q)$ of dimension $(m \times n)$ has full-column rank, globally with respect to q. Note that, from the system kinematics equation (11), we also have

$$\dot{q} = \mathcal{J}^\dagger(q)\,\nu \qquad (12)$$

where $\mathcal{J}^\dagger(q)$ is any left pseudo-inverse of the Jacobian matrix $\mathcal{J}(q)$.

A. Reduced Order Model

By substituting (12) in the system Cartesian kinematics equation (11), the holonomic constraints between the elements of $\mathcal{B}_r$ can be *explicitly* defined by the following equations

$$P(q)\,\nu := \left(I_m - \mathcal{J}(q)\,\mathcal{J}^\dagger(q)\right)\,\nu = 0_{m\times1}\,. \qquad (13)$$

Among the m constraint equations (13), only $s = m - n$ equations are independent [5]. Moreover, there may exist several sets of these $(m - n)$ independent equations, for a given left pseudo-inverse matrix $\mathcal{J}^\dagger(q)$. Let $F(q) \in \mathbb{R}^{(m-n)\times m}$ be the matrix given by a possible choice of

$(m-n)$ linearly independent rows of the constraint matrix $P(q)$ and suppose that $F(q)$ has full-row rank, globally for q or at least locally in a neighbourhood of the operating point. The matrix $F(q)$ can then be expressed in terms of the constraint matrix $P(q)$ as

$$F(q) = S\,P(q) \tag{14}$$

where $S \in \mathbb{R}^{(m-n)\times m}$ is a selecting matrix of the constraint vector $P(q)\,\nu$. For example, if the $(m-n)$ first rows of $P(q)$ are linearly independent we can select these rows to construct $F(q)$ and then the selecting matrix S becomes

$$S = [I_{m-n}\;\; 0_{(m-n)\times n}]\,. \tag{15}$$

Hence, the equations of the constraints between the elements of the constrained rigid body system $\mathcal{B}_r$ can be defined by the reduced form

$$F(q)\,\nu = 0_{(m-n)\times 1}\,. \tag{16}$$

Following the principle of virtual work, the vector of Cartesian constraint forces τ_c can be expressed in terms of a $((m-n)\times 1)$-vector of Lagrangian multipliers λ_c as

$$\tau_c = F(q)^\top\,\lambda_c\,. \tag{17}$$

Differentiating (16) with respect to time and solving (7) for $\dot{\nu}$, the force multipliers λ_c are

$$\lambda_c = Y(q)(\mathcal{M}(\pi)\dot{\mathcal{J}}(q,\dot{q})\dot{q} + \dot{\mathcal{M}}(\pi,\nu)\,\nu - N(\pi,\nu) + v - \tau) \tag{18}$$

with

$$Y(q) = \left(F(q)\mathcal{M}^{-1}(\pi)F(q)^\top\right)^{-1} F(q)\mathcal{M}^{-1}(\pi) \tag{19}$$

where (11) and the fact that $F(q)\mathcal{J}(q) = 0$ have been used.

From (7) and (17)-(19), the "weighting matrix" $\mathcal{K}(q) := (I - F(q)^\top Y(q))$ of the Cartesian forces vector τ in the dynamics (7) is so that $\mathcal{K}(q)F(q)^\top = 0$. That is $\mathcal{K}$ is a projection operator that filters out all Cartesian forces lying in the range of the transpose of the reduced constraint matrix $F(q)$. These correspond to Cartesian forces that tend to violate the imposed Cartesian space constraints. On the other hand, even though all generalized positions q can be expressed by means of a suitable Cartesian position vector π, there exist Cartesian positions π in $\mathbb{R}^m$ which cannot be expressed by means of q. Then, the obtained Cartesian positions π are supposed to be in a subset Ω_π of $\mathbb{R}^m$.

To eliminate the Cartesian constraint forces τ_c and therefore reduce the dimension of the constrained system (7), it suffices to use the system Cartesian kinematics equation (11) in the dynamics (7) and premultiply on both sides of (7) by $\mathcal{J}(q)^\top$. Hence, the Cartesian constraint forces τ_c are eliminated owing to the fact that $\mathcal{J}(q)^\top\tau_c = 0$, and thus the reduced order model of the constrained system $\mathcal{B}_r$ is given by the following equation

$$\begin{aligned} &\underbrace{\mathcal{J}(q)^\top\mathcal{M}(\pi)\,\mathcal{J}(q)}_{D(q)}\,\ddot{q} \;\; + \\ &\underbrace{\mathcal{J}(q)^\top\left(\mathcal{M}(\pi)\dot{\mathcal{J}}(q,\dot{q})\,\dot{q} + \dot{\mathcal{M}}(\pi,\nu)\,\mathcal{J}(q)\,\dot{q} - N(\pi,\nu)\right)}_{C(q,\dot{q})\dot{q}} \\ &\quad + \underbrace{\mathcal{J}(q)^\top v}_{g(q)} \;=\; \underbrace{\mathcal{J}(q)^\top\tau}_{e} \end{aligned} \tag{20}$$

where the generalized matrices and vectors $D(q)$, $C(q,\dot{q})$ and $g(q)$ are now given by jacobian-type expressions. Also, $\mathcal{J}(q)^\top\tau = e$ gives the relationship between the Cartesian forces vector τ and the generalized forces vector e. It is worth noting that $N(\pi,\nu)$ of (8) can be written as $O(\pi,\nu)\,\nu$ with several choices of the matrix $O(\pi,\nu)$ among them there exists one which renders the matrix $\dot{D}(q,\dot{q}) - 2C(q,\dot{q})$ skew-symmetric.

B. Contact with the Environment

We assume that the end effector of the rigid body system $\mathcal{B}_r$ is now under interaction with an infinitely stiff environment with no friction. Then, the motion is constrained to a smooth $(n-k)$-dimensional submanifold defined by

$$\psi(q) = 0_{k\times 1} \tag{21}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}^k$ is at least twice continuously differentiable, and k is the number of the holonomic constraints. By taking into account the effects of these holonomic constraints into the generalized dynamics (20), the motion equations of the system $\mathcal{B}_r$ become

$$D(q)\,\ddot{q} + C(q,\dot{q})\,\dot{q} + g(q) = \mathcal{J}(q)^\top\tau + \tau_q \tag{22}$$

where $\tau_q \in \mathbb{R}^n$ is the vector of generalized constraint forces.

It is worth noting that the presence of the term of constraint τ_q in the generalized dynamics (20) leads to another term $\tau_q^\star$ that will be added to the dynamics (7). This term $\tau_q^\star$ results from the generalized constraints (21) on the Cartesian variables π and can be defined such that $\mathcal{J}(q)^\top\tau_q^\star := \tau_q$. Interestingly enough, the presence of this term in the dynamics (7) does not affect the expression of the Cartesian constraint forces τ_c of (17).

For solvability of the constraints equation (21) and therefore the determination of the *overall* reduced order model of the system $\mathcal{B}_r$, the following assumptions are required:

Assumption 1 *Given $q := col[q^{r_1}, q^{r_2}]$ with $q^{r_1} \in \mathbb{R}^{n-k}$ and $q^{r_2} \in \mathbb{R}^k$. Assume that there exists an operating region $\Omega_q \subseteq \mathbb{R}^n$ defined as $\Omega_q := \Omega_{q^{r_1}} \times \Omega_{q^{r_2}}$, where $\Omega_{q^{r_1}}$ is a convex subset of $\mathbb{R}^{n-k}$ and $\Omega_{q^{r_2}}$ is an open subset of $\mathbb{R}^k$. We also assume the existence of a function $s : \Omega_{q^{r_1}} \to \mathbb{R}^k$ twice continuously differentiable such that $\psi(q^{r_1}, s(q^{r_1})) = 0$, for all $q^{r_1} \in \Omega_{q^{r_1}}$. Under these conditions, the vector q^{r_2} can be uniquely defined by the vector q^{r_1} such that $q^{r_2} = s(q^{r_1})$, for all $q^{r_1} \in \Omega_{q^{r_1}}$.* □

Assumption 2 *We assume that the system's end effector is always on the constraint surface during closed-loop operation. This means that, under Assumption 1, the state trajectory $q(t) \in \Omega_q$, for all $t \geq 0$.* $\square$

Note that under Assumption 1, we can partition the matrix $\mathcal{J}_q(q)^\top := \partial\psi(q)/\partial q$ as $\mathcal{J}_q(q) = [J_{q^{r_1}}(q)\ \ J_{q^{r_2}}(q)]$, where $J_{q^{r_1}}(q)^\top := \partial\psi(q)/\partial q^{r_1}$, $J_{q^{r_2}}(q)^\top := \partial\psi(q)/\partial q^{r_2}$, and $J_{q^{r_2}}(q) \in \mathbb{R}^{k\times k}$ never degenerates in the set Ω_q.

By choosing q^{r1} as the independent variables, we can express the generalized velocities $\dot{q}$ in terms of the independent velocities $\dot{q}^{r_1}$ as

$$\dot{q} = H(q)\,\dot{q}^{r_1} \tag{23}$$

where

$$H(q) = \begin{pmatrix} I_{n-k} \\ -J_{q^{r_2}}^{-1}(q)J_{q^{r_1}}(q) \end{pmatrix} \tag{24}$$

which has full rank for all $q \in \Omega_q$.

Following the principle of virtual work, the vector of generalized constraint forces τ_q can be expressed in terms of a $(k \times 1)$-vector of Lagrangian multipliers λ_q as

$$\tau_q = \mathcal{J}_q(q)^\top \lambda_q\,. \tag{25}$$

The force multipliers λ_q of dimension $(k \times 1)$ can be eliminated by solving (22) for $\ddot{q}$ and substituting it into the second time derivative of (21) to obtain

$$\lambda_q = Z(q)\left(C_\lambda(q,\dot{q})\dot{q}^{r_1} + g(q) - \mathcal{J}(q)^\top\tau\right) \tag{26}$$

with

$$Z(q) = \left(\mathcal{J}_q(q)D^{-1}(q)\mathcal{J}_q(q)^\top\right)^{-1}\mathcal{J}_q(q)D^{-1}(q) \tag{27}$$

$$C_\lambda(q,\dot{q}) = D(q)\dot{H}(q,\dot{q}) + C(q,\dot{q})H(q) \tag{28}$$

where (23) and the fact that $\mathcal{J}_q(q)\,H(q) = 0$ have been used.

C. Overall Reduced Order Model

We present here the reduced order model of the holonomically constrained rigid body system $\mathcal{B}_r$ during constrained motion with the environment. Both the Cartesian mechanical constraints beween the rigid bodies and the generalized constraints resulting from interaction of the end effector with the environment are to be considered to derive this model. To that end, differentiating (23) for $\ddot{q}$ with respect to time and substituting it into (22) and then premultiplying the resulting equation on both sides by $H(q)^\top$, we obtain the *overall* reduced order model of the *two* constrained motion of $\mathcal{B}_r$ as

$$D_\star(q)\,\ddot{q}^{r_1} + C_\star(q,\dot{q})\,\dot{q}^{r_1} + g_\star(q) = H(q)^\top\mathcal{J}(q)^\top\tau \tag{29}$$

with

$$\begin{aligned} D_\star(q) &= H(q)^\top D(q)\,H(q) && (30)\\ C_\star(q,\dot{q}) &= H(q)^\top C_\lambda(q,\dot{q}) && (31)\\ g_\star(q) &= H(q)^\top g(q) && (32) \end{aligned}$$

where the generalized constraint forces τ_q are eliminated by using the fact that $\mathcal{J}_q(q)\,H(q) = 0$.

The following properties will be useful in the subsequent control development and stability analysis.

Property 1 [2] *The matrix $D_\star(q)$ is symmetric and positive definite for all $q \in \Omega_q$. Moreover, it can be bounded as follows*

$$m_1\|x\|^2 \leq x^\top D_\star(q)x \leq m_2(\|q\|)\|x\|^2\ \ \forall x \in \mathbb{R}^{n-k} \tag{33}$$

where m_1 is a known positive scalar constant and $m_2(\|q\|)$ is a known positive nondecreasing scalar function. $\square$

Property 2 [4] *For a suitable choice of $C_\star(q,\dot{q})$, we have that for all $q \in \mathbb{R}^n$ the matrix $\dot{D}_\star(q,\dot{q}) - 2C_\star(q,\dot{q})$ is skew-symmetric.* $\square$

III. CONTROL DESIGN AND STABILITY

A. Control Problem

We define the control problem we solve in this note as follows. Suppose that only Cartesian/generalized positions are available for measurements. Defining the position and force tracking errors as $\tilde{\pi} := \pi - \pi_d$, $\tilde{\tau}_q := \tau_q - \tau_{qd}$ and $\tilde{\tau}_c := \tau_c - \tau_{cd}$, where $\pi_d(t), \tau_{qd}(t)$ and $\tau_{cd}(t)$ stand for the desired values of the positions, generalized and Cartesian constraint forces, respectively. Then, we seek for a dynamic control law $\tau = \tau(t,\pi,q,p), \dot{p} = \phi(t,p,q)$ such that $\tilde{\pi}(t), \tilde{\tau}_q(t)$ and $\tilde{\tau}_c(t)$ converge to zero exponentially fast. The desired values $\pi_d(t), \tau_{qd}(t)$ and $\tau_{cd}(t)$ are supposed to satisfy the following assumptions:

Assumption 3 *The desired trajectory $\pi_d(t)$ is $\mathcal{C}^2$ and with its time derivative up to order 2 are bounded.* $\square$

Assumption 4 *The desired trajectory $\pi_d(t) \in \Omega_\pi$ for all $t \geq 0$. This means that $\pi_d(t) = \pi(q_d(t))$ and $\nu_d(t) = \mathcal{J}(q_d(t))\,\dot{q}_d(t)$ for a given consistent values of $q_d(t), \dot{q}_d(t)$. A consistent value $q_d(t)$ is chosen into Ω_q where the generalized constraint equation $\psi(q_d(t)) = 0$ is satisfied. Furthermore, the desired generalized and Cartesian constraint forces satisfy $\tau_{qd}(t) = \mathcal{J}_q(q_d(t))^\top\lambda_{qd}(t)$ and $\tau_{cd}(t) = F(q_d(t))^\top\lambda_{cd}(t)$ for all $t \geq 0$, where $\lambda_{qd}(t)$ and $\lambda_{cd}(t)$ are the desired values of the independent generalized and Cartesian constraint forces, respectively.* $\square$

Because only $n-k$ *independent* position variables, k *independent* generalized force variables and $m-n$ *independent* Cartesian force variables need to be controlled, the control objective above can be achieved by forcing to zero exponentially fast the independent position, generalized and Cartesian force tracking errors

$$e := q^{r_1} - q_d^{r_1} \tag{34}$$

$$\tilde{\lambda}_q := \lambda_q - \lambda_{qd} \tag{35}$$

$$\tilde{\lambda}_c := \lambda_c - \lambda_{cd} \tag{36}$$

where $q_d^{r_1}(t)$ is the desired value of the independent positions. From Assumption 3, $q_d^{r_1}(t)$ is $\mathcal{C}^2$ and with its time derivative up to order 2 are bounded. Then, there exists $\zeta_d > 0$ such that

$$\max\{\|q_d^{r_1}(t)\|, \|\dot{q}_d^{r_1}(t)\|, \|\ddot{q}_d^{r_1}(t)\|\} \leq \zeta_d \,. \qquad (37)$$

As for many works in the field, we will introduce a control law which allows to control the generalized positions and constraint forces separately. Thus, we consider the *decoupling* control law τ of the form

$$\tau = \mathcal{J}^\dagger(q)^\top H^\dagger(q)^\top e_1 + \mathcal{J}^\dagger(q)^\top \mathcal{J}_q(q)^\top e_2 + F(q)^\top e_3 \quad (38)$$

where $e_1 \in \mathbb{R}^{n-k}$, $e_2 \in \mathbb{R}^k$, $e_3 \in \mathbb{R}^{m-n}$ are to be defined in the subsequent development control, and $H^\dagger(q)$ is a left pseudo-inverse of $H(q)$.

Then, the system model (29), (26) and (18) in closed-loop with the control law (38) becomes

$$D_\star(q)\,\ddot{q}^{r_1} + C_\star(q,\dot{q})\,\dot{q}^{r_1} + g_\star(q) = e_1 \qquad (39)$$

$$\lambda_q = Z(q)\left(C_\lambda(q,\dot{q})\dot{q}^{r_1} + g(q) - H^\dagger(q)^\top e_1\right) - e_2 \quad (40)$$

$$\lambda_c = Y(q)\Big(\mathcal{M}(\pi)\dot{\mathcal{J}}(q,\dot{q})\,\dot{q} + \dot{\mathcal{M}}(\pi,\nu)\,\nu - N(\pi,\nu)$$
$$+ v - \mathcal{J}^\dagger(q)^\top H^\dagger(q)^\top e_1 - \mathcal{J}^\dagger(q)^\top \mathcal{J}_q(q)^\top e_2\Big) - e_3 \,. (41)$$

Assumption 5 *In order to facilitate the control development and stability analysis, we will assume that the elements of the equations (39), (40) and (41) are all bounded provided that* $q, \dot{q} \in L_\infty^n$. □

B. High-Pass Filter Approach

We will design a position controller based on the use of a filter that generates a velocity tracking error-related signal, to compensate for the need for velocity measurements.

1) Filter Formulation:: The filter is given by the following dynamic relationship [2]

$$\dot{p} = -k_\alpha(k+1)p + (-kk_\beta + k_\alpha + k_\alpha k + k_\alpha k^2)e \quad (42)$$
$$e_f = -ke + p \qquad (43)$$

with state $p \in \mathbb{R}^{n-k}$, input $e \in \mathbb{R}^{n-k}$, output $e_f \in \mathbb{R}^{n-k}$ and initial conditions $p(0) = ke(0)$. The filter gains k_α, k_β are scalar positive constant and k is to be defined later.

C. Position Controller

We will first develop the open-loop position tracking error dynamics. We start by obtaining the dynamics for the high-pass filter output e_f. To that end, we differentiate (43) with respect to time and substitute (42) for $\dot{p}$ to obtain

$$\dot{e}_f = -k\dot{e} - k_\alpha(k+1)p + (-kk_\beta + k_\alpha + k_\alpha k + k_\alpha k^2)e \,. \quad (44)$$

After solving for p from (43) and substituting the resulting expression into (44), we obtain the dynamics for e_f as

$$\dot{e}_f = -k_\alpha e_f - k\eta + k_\alpha e \qquad (45)$$

where the filtered tracking error term η is defined as

$$\eta = \dot{e} + k_\alpha e_f + k_\beta e \,. \qquad (46)$$

From (46), it can be shown that the dynamics of e is

$$\dot{e} = -k_\beta e + \eta - k_\alpha e_f \,. \qquad (47)$$

To develop the open-loop dynamics for η, we take the time derivative of (46) and premultiply the resulting expression by $D_\star(q)$ to yield

$$D_\star\dot{\eta} = D_\star(\ddot{q}^{r_1} - \ddot{q}_d^{r_1}) + k_\alpha D_\star\dot{e}_f + k_\beta D_\star\dot{e} \qquad (48)$$

where the argument of $D_\star(q)$ has been dropped for simplicity and (34) has been used. Using (39) for $D_\star\ddot{q}^{r_1}$, (45) for $\dot{e}_f$ and (47) for $\dot{e}$ into equation above we obtain

$$D_\star\dot{\eta} = -D_\star\ddot{q}_d^{r_1} - C_\star\,\dot{q}_d^{r_1} - g_\star(q) - C_\star\dot{e} + e_1 + k_\alpha D_\star$$
$$.(-k_\alpha e_f - k\eta + k_\alpha e) + k_\beta D_\star(-k_\beta e + \eta - k_\alpha e_f) \; (49)$$

where $\dot{q}^{r_1}$ has been replaced by $\dot{q}_d^{r_1} + \dot{e}$ in accordance with the time derivative of (34). Now, adding and subtracting the feedforward term $X_d(q_d^{r_1}, \dot{q}_d^{r_1}, \ddot{q}_d^{r_1})$ defined as

$$X_d = -D_\star(q_d)\,\ddot{q}_d^{r_1} - C_\star(q_d,\dot{q}_d)\,\dot{q}_d^{r_1} - g_\star(q_d) \qquad (50)$$

to the right-hand side of (49), substituting (47) for $\dot{e}$, and then rearranging the resulting expression in an advantageous manner, we have the final representation for the open-loop dynamics of η as follows

$$D_\star\dot{\eta} = \mathcal{X} + e_1 + X_d - kk_\alpha D_\star\eta - C_\star\,\eta \qquad (51)$$

where the state disturbance variable $\mathcal{X}$ is defined as

$$\mathcal{X} = -D_\star\ddot{q}_d^{r_1} - C_\star\dot{q}_d^{r_1} + k_\beta D_\star(-k_\beta e + \eta - k_\alpha e_f)$$
$$-g_\star(q) - X_d + k_\alpha^2 D_\star(e - e_f) + C_\star(k_\beta e + k_\alpha e_f) \,. (52)$$

Using [3, Lemma B.1], Assumption 5, and (37), it is straightforward to show that $\|\mathcal{X}\|$ can be upper bounded as

$$\|\mathcal{X}\| \leq \rho_o(\zeta_d, \|z\|)\|z\| =: \rho(\|z\|)\|z\| \qquad (53)$$

where $\rho(\|z\|)$ is some positive, non-decreasing, scalar function, and z is defined as

$$z = \mathrm{col}[e_f,\, e,\, \eta] \,. \qquad (54)$$

Based on the structure of (51) and the subsequent stability analysis, we propose the following position controller

$$e_1 = -X_d + ke_f - e \,. \qquad (55)$$

After substituting (55) into (51), we have the closed-loop dynamics for η as

$$D_\star\dot{\eta} = -C_\star\eta + \mathcal{X} - kk_\alpha D_\star\eta + ke_f - e \,. \qquad (56)$$

Theorem 1 *The state trajectories of the dynamics (45), (47) and (51) in closed-loop with the controller (55) converge to zero exponentially fast. More precisely, let* $k = (1/m_1 k_\alpha)(1 + k_{n_2})$ *such that* $k_{n_2} > (1/\lambda_3)\rho^2((\lambda_2(\|q(0)\|)/\lambda_1)^{1/2}\|z(0)\|)$*, we then have for all* $t \geq 0$

$$\|z(t)\| \leq \left(\frac{\lambda_2(\|q(0)\|)}{\lambda_1}\right)^{1/2} \|z(0)\|\, e^{(-\lambda_4 t)} \qquad (57)$$

where $\lambda_1 := \min\{1, m_1\}, \lambda_2(\|q\|) := \max\{1, m_2(\|q\|)\}$, $\lambda_4 := \beta/\lambda_2(\zeta_d + (\lambda_2(\|q(0)\|)/\lambda_1)^{1/2}\|z(0)\|), \lambda_3 := \min\{k_\alpha, k_\beta, 1\}$ *and* β *is some positive constant.* □

For the sake of space constraints, the proof of this theorem is not presented here and is given in [6].

Remark 1 *As a direct implication of Theorem 1 that velocity tracking error goes to zero exponentially fast. Indeed, from the fact that* $e(t), e_f(t)$ *and* $\eta(t)$ *go all to zero exponentially fast and by applying the triangular inequality to (47) we have that* $\dot{e}(t)$ *will also go to zero exponentially fast.*

D. Forces Controller

1) Generalized Force Controller:: We start by rewriting the equation of the generalized constraint forces (40) as

$$\lambda_q = W(q,\dot{q}) - Z(q)\, H^{\dagger}(q)^{\top} e_1 - e_2 \tag{58}$$

where the term $W(q,\dot{q})$ is defined as

$$W = Z(q)\, C_\lambda(q,\dot{q})\, \dot{q}^{r_1} + Z(q)\, g(q)\,. \tag{59}$$

Based on the structure of (58), we propose the generalized force controller as

$$e_2 = -Z(q)\, H^{\dagger}(q)^{\top} e_1 - \lambda_{qd} + W_d \tag{60}$$

where the feedforward term $W_d(q_d, \dot{q}_d)$ is defined as

$$W_d = Z(q_d)\, C_\lambda(q_d,\dot{q}_d)\, \dot{q}_d^{r_1} + Z(q_d)\, g(q_d)\,. \tag{61}$$

Then, the generalized constraint forces model (58) in closed-loop with the controller (60) becomes

$$\tilde{\lambda}_q = W - W_d\,. \tag{62}$$

Based on the observations used to prove (53), the term on the right-hand side of (62) can be upper bounded as

$$\|W - W_d\| \le \rho_1(\zeta_d, \|z\|)\, \|z\| + \rho_2(\zeta_d, \|z\|)\, \|\dot{e}\| \tag{63}$$

where $\rho_1(\zeta_d, \|z\|)$ and $\rho_2(\zeta_d, \|z\|)$ are some positive, non-decreasing, scalar functions. Using the stability result of Theorem 1 and Remark 1, exponential convergence to zero of the generalized force tracking error $\tilde{\lambda}_q$ follows.

2) Cartesian Force Controller:: The equation of the Cartesian constraint forces (41) can be rewritten as

$$\begin{aligned}\lambda_c = &-Y(q)\, \mathcal{J}^{\dagger}(q)^{\top} H^{\dagger}(q)^{\top} e_1 - Y(q)\, \mathcal{J}^{\dagger}(q)^{\top} \mathcal{J}_q(q)^{\top} e_2 \\ &+ Q(q,\dot{q}) - e_3\end{aligned} \tag{64}$$

where the term $Q(q,\dot{q})$ is defined as

$$Q = Y(q)(\mathcal{M}(\pi)\dot{\mathcal{J}}(q,\dot{q})\, \dot{q} + \dot{\mathcal{M}}(\pi,\nu)\, \nu - N(\pi,\nu) + v)\,. \tag{65}$$

Based on the structure of (64), we propose the Cartesian force controller as

$$\begin{aligned}e_3 = &-Y(q)\, \mathcal{J}^{\dagger}(q)^{\top} H^{\dagger}(q)^{\top} e_1 - Y(q)\, \mathcal{J}^{\dagger}(q)^{\top} \mathcal{J}_q(q)^{\top} e_2 \\ &+ Q_d - \lambda_{cd}\end{aligned} \tag{66}$$

when the feedforward term $Q_d(q_d, \dot{q}_d)$ is defined as

$$\begin{aligned}Q_d = Y(q_d)\Big(&\mathcal{M}(\pi_d)\dot{\mathcal{J}}(q_d,\dot{q}_d)\, \dot{q}_d \\ &+ \dot{\mathcal{M}}(\pi_d,\nu_d)\, \nu_d - N(\pi_d,\nu_d) + v\Big)\end{aligned} \tag{67}$$

where we recall that the desired values π_d and ν_d are given as $\pi_d = \pi(q_d)$ and $\nu_d = \mathcal{J}(q_d)\, \dot{q}_d$.

Then, the Cartesian constraint forces model (64) in closed-loop with the controller (66) becomes

$$\tilde{\lambda}_c = Q - Q_d\,. \tag{68}$$

Based on the same observations above, the term on the right-hand side of (68) can be upper bounded as

$$\|Q - Q_d\| \le \rho_3(\zeta_d, \|z\|)\, \|z\| + \rho_4(\zeta_d, \|z\|)\, \|\dot{e}\| \tag{69}$$

where $\rho_3(\zeta_d, \|z\|)$ and $\rho_4(\zeta_d, \|z\|)$ are some positive, non-decreasing, scalar functions. Using the stability result of Theorem 1 and Remark 1, exponential convergence to zero of the Cartesian force tracking error $\tilde{\lambda}_c$ follows.

E. Simulation Results

Due to space constraints, the results of simulation are omitted here and given in [6].

IV. CONCLUSION

We have introduced a new concept of force/position control approach for holonomically constrained rigid body systems. With this control approach the forces of the mechanical constraints between the elements of the mechanical constrained rigid body system as well as the forces of the constraints when the system's end effector interacting with the environment are directly controlled to desired trajectories. We proposed a controller for the constrained rigid body system during constrained motion with the environment that exhibits exponential position, end effector contact force and mechanical force tracking. The controller was based on exact knowledge of the system model and required only position measurements.

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