$\mathcal{H}^\infty$ control of a piezo-actuated flexible beam using an analytical bound approach

Robert Sweeney‡ Michael A. Demetriou‡ Karolos M. Grigoriadis§

***Abstract*— The paper considers the $\mathcal{H}^\infty$ control of collocated structural systems in vector second order form using an analytical bound approach. The model of a cantilevered beam with a collocated piezoelectric sensor/actuator pair is used to validate explicit analytical expressions that have been derived for the $\mathcal{H}^\infty$ norm bounds and the corresponding static output feedback control gains that achieve these bounds. The analytical bound results are compared with the standard $\mathcal{H}^\infty$ analysis and control results in terms of accuracy and computational efficiency to demonstrate the great computational benefits of the analytical bound approach.**

***Index Terms*— Flexible structures, $\mathcal{H}^\infty$ control, piezoceramic actuator/sensor**

I. Introduction

The control of structural systems with collocated sensors and actuators has been shown to provide great advantages from a stability, passivity, robustness and an implementation viewpoint. For example, collocated control can easily be achieved in a space structure when an attitude rate sensor is placed at the same point as a torque actuator. Collocation of sensors and actuators leads to symmetric transfer functions. Several other classes of engineering systems, such as circuit systems, chemical reactors and power networks, can be modelled as systems with symmetric transfer functions. Stabilization, robustness, model reduction and control of such systems has been examined recently.

State space $\mathcal{H}^\infty$ control based on the standard Riccati equation approach or the recent linear matrix inequality (LMI) formulation is now a well developed control synthesis tool. The optimal static state feedback and full-order dynamic output feedback $\mathcal{H}^\infty$ control synthesis problems can be solved using iterations on the corresponding Riccati solutions or via the computational solution of a convex LMI optimization problem. On the other hand, the static output feedback and the fixed-order dynamic output feedback $\mathcal{H}^\infty$ control synthesis problems are difficult computational problems since they require the solution of (nonconvex) bilinear matrix inequalities (BMIs) or LMIs with coupling rank constraints [4].

Recently, analytical expressions for an $\mathcal{H}^\infty$ norm bound of collocated structural systems have been obtained using a particular solution of the Bounded Real Lemma (BRL). In addition, explicit expressions for the static output feedback gains that achieve a desired closed-loop $\mathcal{H}^\infty$ norm bound have been derived [6]. In the present work we utilize these bounds and analytical formulas to design $\mathcal{H}^\infty$ controllers for a cantilevered beam with a collocated piezoelectric sensor/actuator pair. The results provide a validation of the developed bounds and control gain expressions for a realistic structural model and demonstrate the tightness of the bounds, as well as, the computational advantages of the analytical approach.

‡Department of Mechanical Engineering, Worcester Polytechnic Institute, Worcester, MA 01609, USA, {sweenerj,mdemetri}@wpi.edu

§Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA, karolos@uh.edu

The notation to be used in this paper is as follows: Given a real matrix N, the orthogonal complement $N^\perp$ is defined as the (possibly non-unique) matrix with maximum row rank that satisfies $N^\perp N = 0$ and $N^\perp N^{\perp T} > 0$. Hence, $N^\perp$ can be computed from the singular value decomposition of N as follows: $N^\perp = TU_2^T$ where T is an arbitrary nonsingular matrix and U_2 is defined from the singular value decomposition of N

$$N = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

The standard notation $>$ ($<$) is used to denote the positive (negative) definite ordering of symmetric matrices. The ith eigenvalue of a real symmetric matrix N will be denoted by $\lambda_i(N)$ where the ordering of the eigenvalues is defined as $\lambda_{\max}(N) = \lambda_1(N) \geq \lambda_2(N) \geq \ldots \geq \lambda_n(N)$. The maximum singular value of a (not necessarily square) matrix N will be denoted by $\sigma_{\max}(N)$, which is also its spectral norm $||N||$. N^+ will denote the Moore-Penrose generalized inverse of a matrix N.

II. $\mathcal{H}^\infty$ Control Analysis

The class of dynamical systems under consideration is assumed to be described by a vector second order form with collocated sensors and actuators

$$\begin{aligned} M\ddot{q} + D\dot{q} + Kq &= B_0 u \\ y &= B_0^T \dot{q} \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^n$ is the generalized coordinate vector, $u \in \mathbb{R}^m$ is the input vector and $y(t) \in \mathbb{R}^m$is the measured output vector ($m < n$). The matrices M, D and K are symmetric positive definite matrices that represent the structural *mass, damping* and *stiffness* distribution, respectively, i.e. we consider non-gyroscopic systems with $D = D^T$ [1]. The above formulation is often encountered in the dynamics of structural systems resulting from a finite element approximation. For control design of such systems, one usually brings (1) to a first order form in order to take advantage of the plethora of control design schemes for linear time invariant systems.

The state-space realization of (1) is given via

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx \end{aligned} \tag{2}$$

where the (A,B,C) triple is given by

$$A=\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix},\ B=\begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix},$$

$$C=\begin{bmatrix} 0 & B_0^T \end{bmatrix},$$

where $x(t)=\begin{bmatrix} q(t)^T & \dot{q}(t)^T \end{bmatrix}^T$. The associated transfer function from u to y is then given by

$$T(s)=\begin{bmatrix} 0 & B_0^T \end{bmatrix}\begin{bmatrix} sI & -I \\ M^{-1}K & sI+M^{-1}D \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix}.$$

The above is certainly a valid approach to obtain such a transfer function, but it does not take advantage of (i) the algebraic structure of the second order form and (ii) the symmetry of the system. Without utilizing the first order formulation (2), one may calculate the transfer function directly from (1), which is given via the quadratic pencil

$$T(s)=B_0^T s\left(Ms^2+Ds+K\right)^{-1}B_0. \tag{3}$$

Notice that by direct inspection, the above transfer function is symmetric, i.e. $T(s)=T^T(s)$. The system (2) is an *externally symmetric* state space realization, that is, there exists a nonsingular matrix L such that

$$A^TL=LA, \qquad C^T=TB.$$

For the specific structural system considered here, the associated matrix L is given by

$$L=\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}.$$

This class of systems is more general than the class of *internally* or *state symmetric* systems that satisfy the symmetry conditions with a positive definite transformation matrix L. It can observed that state-space symmetry implies external symmetry, *but* the converse is not true, that is, there exist symmetric transfer matrices for which there is no internally symmetric realization.

For control design, one requires the computation of the $\mathcal{H}^\infty$ norm, defined by

$$\|T(s)\|_\infty=\sup_{\omega\in\mathbb{R}}\sigma_{max}\{T(j\omega)\}. \tag{4}$$

The standard way to compute the $\mathcal{H}^\infty$ norm is to bring the system (1) in a first-order state-space form (2) and employ a computationally demanding scheme to approximate this norm iteratively, for example using a bisection method [2]. If further optimization of the actuator/sensor locations via a robustness measure is desired, one would then arrive at a numerically intensive scheme with the obvious burden on computational resources. This work is motivated by the results in [5] that provide an analytical calculation of the $\mathcal{H}^\infty$ norm of symmetric systems using a simple explicit formula. Based on this earlier result, we obtain the following bound for the $\mathcal{H}^\infty$ norm of the vector second-order system (1). An analytic solution of the $\mathcal{H}^\infty$ control problem for internally symmetric systems has been presented in [5]. For vector second-order systems the results in [4] present an analytical expression for a bound on the $\mathcal{H}^\infty$ norm of the system and analytical expressions for the static output feedback symmetric controllers that achieve a give bound. The results have obvious computational advantages compared to standard $\mathcal{H}^\infty$ analysis and synthesis methods, especially for large-scale systems.

Theorem 2.1: Consider the vector second-order system (1). The system has an $\mathcal{H}^\infty$ (open loop) norm that satisfies

$$\gamma<\gamma_0=\lambda_{max}\left(B_0^TD^{-1}B_0\right) \tag{5}$$

Proof: The proof of Theorem 2.1 is based on the Bounded Real Lemma (BRL) and Finsler's Lemma (see [4]) and is summarized in [6]. ■

III. The $\mathcal{H}^\infty$ Control Synthesis Problem

We now consider the controlled structural system

$$\begin{aligned} M\ddot{q}+D\dot{q}+Kq &= B_0(u+w) \\ z &= B_0^T\dot{q} \\ y &= B_0^T\dot{q}, \end{aligned} \tag{6}$$

where $y(t)\in\mathbb{R}^m$ is the measured output, $z(t)\in\mathbb{R}^m$ is the performance output vector and $w(t)\in\mathbb{R}^m$ is the disturbance input. The $\mathcal{H}^\infty$ control synthesis problem is to design a symmetric static output feedback gain $G=G^T$ such that the output feedback control

$$u=-Gy, \tag{7}$$

renders the closed-loop system stable with an $\mathcal{H}^\infty$ norm less than a given scalar $\gamma>0$. The resulting closed-loop system (6), (7) is then given by

$$\begin{aligned} M\ddot{q}+\left(D+B_0GB_0^T\right)\dot{q}+Kq &= B_0w \\ z &= B_0^T\dot{q}. \end{aligned} \tag{8}$$

We use the following results that provide an explicit expression of the output feedback gains which guarantee a closed-loop $\mathcal{H}^\infty$ norm less than γ.

Theorem 3.1: Consider the vector second order system (6). For any $\gamma>0$ there exists a symmetric output feedback control law (7) to provide a closed-loop $\mathcal{H}^\infty$ norm less than γ.

(a) If B_0 is square and invertible then G can be selected as

$$G\geq\frac{1}{\gamma}I-B_0^{-1}DB_0^{-1T}. \tag{9}$$

(b) If B_0 is singular then G can be selected as

$$G \geq B_0^{+}\Big[DB_0^{\perp T}\left(B_0^{\perp}DB_0^{\perp T}\right)^{-1}B_0^{\perp}D - D + \frac{1}{\gamma}B_0B_0^T\Big]B_0^{+T} \tag{10}$$

where B_0^{+} denotes the Moore-Penrose inverse [4], and $B_0^{\perp}$ is a matrix such that $B_0^{\perp}B_0 = 0$ and $B_0^{\perp}B_0^{\perp T} > 0$ (i.e. left null space of B_0).

Proof: The proof of Theorem 2 follows from the BRL and the Generalized Finsler's Theorem and is presented in [6]. ■

Remark 3.1: The upper value for the closed-loop bound γ in Theorem 3.1 cannot exceed the one for the open loop case, given by γ_0 in (5), as it would produce a closed loop system with identical $\mathcal{H}^\infty$ norm bound as that of its open loop counterpart, i.e. the choice of γ in (9) or (10) must ensure that

$$\left\|B_0^T s\Big(Ms^2+(D+B_0GB_0)s+K\Big)^{-1}B_0\right\|_\infty < \left\|B_0^T s\Big(Ms^2+Ds+K\Big)^{-1}B_0\right\|_\infty.$$

Remark 3.2: Since the optimal gain G from either (9) or (10), depends on the $\mathcal{H}^\infty$ bound γ , one needs to find an acceptable bound for γ. To do so, we consider the uncontrolled system

$$\begin{aligned} M\ddot{q}+D\dot{q}+Kq &= B_0 w \\ z &= B_0^T\dot{q} \end{aligned}$$

and bound its $\mathcal{H}^\infty$ norm using (5) from Theorem 2.1, to arrive at

$$\gamma < \gamma_0 = \lambda_{max}\Big(B_0^T D^{-1} B_0\Big).$$

This then can be used in (9) or (10) as an initial upper bound for γ to calculate the feedback gain G. Once the feedback gain G is found for that initial γ, then the next iterate of γ can be found from

$$-\Big(D+B_0GB_0^T\Big)+\frac{1}{\gamma}B_0B_0^T \leq 0,$$

and continue with the new γ till it satisfies an a priori given stopping criterion.

The algorithm for this iteration is summarized below. For ease of exposition, we consider the case of a square and invertible B_0 and make repeated use of (9) in Theorem 3.1.

<u>*Algorithm 1:*</u>

Step 1. initialize $\gamma_0 = \lambda_{max}(B_0^T D^{-1} B_0)$, (open loop bound)

Step 2. for $k = 0, 1, 2, \ldots$

(i) select G_k as

$$G_k \geq \frac{1}{\gamma_k} I - B_0^{-1} D B_0^{-1T}$$

(ii) set new γ by

$$\gamma_{k+1} = \lambda_{max}\Big(B_0^T\left(D+B_0G_kB_0^T\right)^{-1}B_0\Big)$$

(iii) if $\gamma_k - \gamma_{k+1} > \varepsilon$, where ε is the a priori chosen threshold, then continue with $k \leftarrow k+1$ in step 2, else terminate iteration and exit with the current values γ_{k+1}, G_{k+1}.

IV. Results

We consider a cantilever beam, having a pair of collocated piezoceramic patches forming the actuator/sensor pair. The equation that describe the transverse displacement for this Euler-Bernoulli beam, as taken from Banks *et. al.* [1], is given by

$$\rho(\xi)\frac{\partial^2 x(t,\xi)}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}\left(EI(\xi)\frac{\partial^2 x(t,\xi)}{\partial \xi^2} + c_D I(\xi)\frac{\partial^3 x(t,\xi)}{\partial \xi^2 \partial t}\right) + d_{air}\frac{\partial x(t,\xi)}{\partial t} = \frac{\partial^2}{\partial \xi^2}\left(k_p \chi(\xi_i) u(t)\right),$$

where $u(t)$ denotes the patch voltage, k_p the piezoceramic constant and

$$\chi(\xi_i) = \begin{cases} 1 & \text{if } \xi_i - \varepsilon \leq \xi \leq \xi_i + \varepsilon \\ 0 & \text{otherwise} \end{cases},$$

Using a Galerkin approximation scheme [3] that preserves exponential detectability and stabilizability [1], the above system can be written in a vector second order form

$$\begin{aligned} &M\ddot{q}(t)+D\dot{q}(t)+Kq(t) = B_0 u(t) \\ &y(t) = B_0^T \dot{q}(t) \end{aligned}$$

Using a value of $\gamma = 1.5$, we computed a dynamic $\mathcal{H}^\infty$ controller and compare the result to a static $\mathcal{H}^\infty$ controller using (10) in Theorem 3.1. The results for three different values of the discretization index $n = 20, 60$ and 120, which result in a $40, 120$ and 240 dimensional system in first order form (2), are presented in Table I. One may easily observe the computational benefits of the proposed static output feedback controller over the case of a full-order dynamic $\mathcal{H}^\infty$ controller; for example, when $n = 120$, it takes 82.6 CPU seconds to compute the dynamic controller versus the 3.875 CPU seconds required to implement the static controller using (10). The resulting closed loop $\mathcal{H}^\infty$ norm given by the dynamic controller is 1.22 and the one obtained by the static controller is 1.4785. This difference in performance is a small price to pay for the huge computational savings which are in the range of $90-95\%$ CPU reduction.

Comparing also the computational time to simply compute the true $\mathcal{H}^\infty$ norm versus the approximate upper bound using (5), also produces surprising results as shown in Table II. The computed upper bound of the $\mathcal{H}^\infty$ norm of the closed loop system is close to the true norm whereas the computational time is about two orders of magnitude less (450 times less).

To further appreciate the benefits of the proposed control design scheme, we plot in Figures 1-3 the magnitude curves

TABLE I

COMPUTATION TIME FOR DYNAMIC $\mathcal{H}^\infty$ AND STATIC $\mathcal{H}^\infty$ CONTROLLERS (CPU SECONDS) FOR $\gamma = 1.5$.

case	dynamic		static	
# elements	time	true $\mathcal{H}^\infty$ norm	time	true $\mathcal{H}^\infty$ norm
20	0.533	1.1696	0.047	1.4767
60	11.956	1.1693	0.531	1.4765
120	82.652	1.2236	3.875	1.4785

TABLE II

COMPUTATION TIME FOR $M\ddot{q} + (D + B_0 G B_0^T)\dot{q} + Kq = B_0 w, z = B_0^T \dot{q}$ USING EXACT $\mathcal{H}^\infty$ NORM AND APPROXIMATE $\mathcal{H}^\infty$ BOUND FOR $n = 120$.

true		approximate using (5)	
CPU time	true $\mathcal{H}^\infty$ norm	CPU time	$\mathcal{H}^\infty$ bound
7.188	0.099105	0.016	0.1

of the open and closed loop systems using $\gamma = 10, 1$ and $\gamma = 0.1$.

Finally, we compare the magnitude plots of the closed loop system using the full-order dynamic $\mathcal{H}^\infty$ controller to the closed loop system using the static output feedback (10) for a design value of $\gamma = 1.5$. Figure 4 depicts the closed loop system with the two different controllers.

V. CONCLUSIONS

An analytical bound approach for $\mathcal{H}^\infty$ control of collocated structural systems has been validated in this paper using the model of a cantilevered beam with a collocated piezoelectric sensor/actuator pair. We have seen that the proposed $\mathcal{H}^\infty$ norm bounds and the corresponding static output feedback control gains provide effective $\mathcal{H}^\infty$ control of this

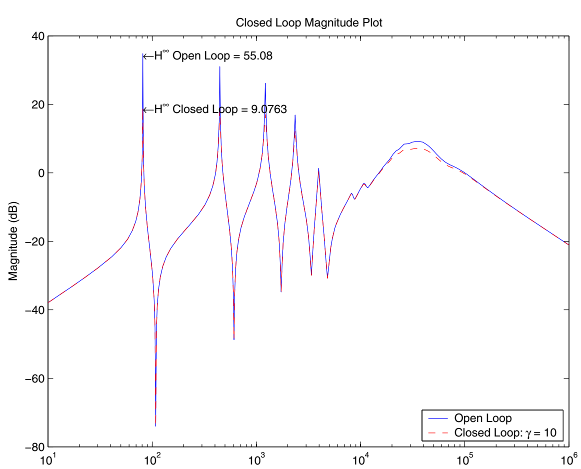

Fig. 1. Magnitude plot for open (solid) and closed (dotted) loop system with $\gamma = 10$.

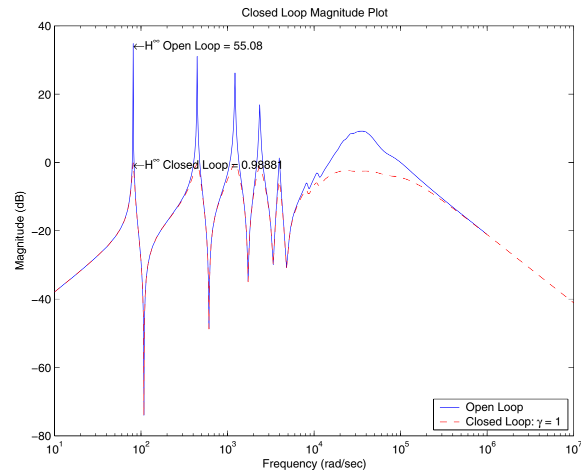

Fig. 2. Magnitude plot for open (solid) and closed (dotted) loop system with $\gamma = 1$.

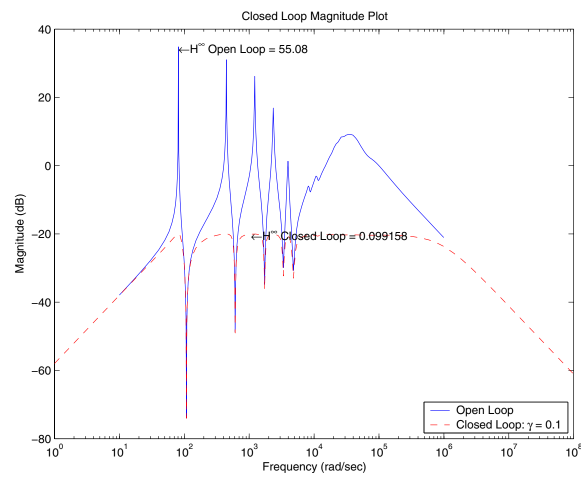

Fig. 3. Magnitude plot for open (solid) and closed (dotted) loop system with $\gamma = 0.1$.

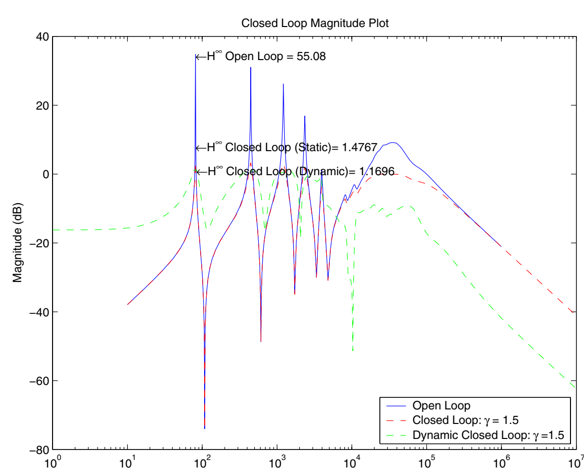

Fig. 4. Magnitude plots using dynamic and static $\mathcal{H}^\infty$ controllers with $\gamma = 1.5$.

model with computational savings up to 95% compared to the standard $\mathcal{H}^\infty$ control approach.

REFERENCES

[1] H. T. BANKS, R. C. SMITH, AND Y. WANG, *Smart Material Structures: Modeling, Estimation and Control*, Wiley-Masson, New York, 1996.

[2] S. BOYD, V. BALAKRISHNAN, AND P. KABAMBA, *A bisection method for computing the H_∞ norm of a transfer function and related problems*, Mathematics of Control, Signals, and Systems, 2 (1989), pp. 207–219.

[3] L. MEIROVITCH, *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[4] R. E. SKELTON, T. IWASAKI, AND K. GRIGORIADIS, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London, UK, 1998.

[5] K. TAN AND K. M. GRIGORIADIS, *Stabilization and H^∞ control of symmetric systems: An explicit solution*, Systems & Control Letters, 44 (2001), pp. 57–72.

[6] ———, *Collocated actuator placement in structural systems using an analytical bound approach*, in Proceedings of the 2004 American Control Conference, Boston, MA, June 30- July 2 2004, pp. 1604–1609.