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On the Design of Walcott-Zak Sliding Mode Observer

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Abstract—**This paper explores the design of Walcott -Zak sliding mode observer (WZSMO) for a class of system satisfying some positive real conditions. A new sufficient and necessary existence condition only involving the original system parameters is proposed to simplify the design of WZSMO. The full freedom degree of the design of nonlinear injection of WZSMO is studied. The fact that the linear feedback can be designed independently is discovered. An optimizing design procedure of the nonlinear injection of SMO is proposed to sub-minimize the control cost of nonlinear injection. At last, a numerical example is presented to illustrate the proposed design procedure.**

I. INTRODUCTION

Sliding mode observer (SMO) has received large attentions in recent years, since it owns more robust than linear Luenberger observer [7]. By injecting a nonlinear discontinuous term, SMO makes the trajectory of the estimating error remain on a surface in the error space after finite time such that the estimating error system is completely insensitive to the disturbances. Due to this perfect feature, SMO has been utilized not only on state estimation [3] [9] [10] [14], but also on input estimation [2] and fault detection and isolation [4]. There are two kinds of SMO: one is the equivalent control based method and the other is Lyapunov based method. The former was first proposed by Utkin [11] and named Utkin SMO. As stated in [10] and [14], the Utkin SMO owns bounded estimation error for bounded modeling errors. So the recently works mostly focused on the latter, which was first suggested by Walcott and Zak in [12] and named Walcott-Zak SMO (WZSMO) in [3]. However, WZSMO was formulated a constraint Lyapunov equation, which is difficult to solve. Corless and Tu [2] presented a sufficient and necessary condition for the existence of WZSMO that the system should satisfy some positive real conditions. They also developed a canonical form on which a solving method was given. But their method is based on an optimal solution of a non-strict linear matrix inequality which is very difficult to cope with in practice. Tan and Edwards [9] presented another canonical form on which a sufficient condition in terms of linear matrix inequality (LMI) is proposed. But thrice coordinates transformations which are not trivial, should be used to obtain the canonical form. Hence, their method is somewhat complex, although their method is explicit. Moreover, they did not discover the key action of nonlinear injection and not explore the freedom degree of design of nonlinear injection. The relation between linear feedback part and nonlinear injection was also not addressed.

In this paper, a new sufficient and necessary existence condition in terms of LMIs for WZSMO is presented. This condition only involves the original system matrices and endows the design of WZSMO with simpleness. The action and influence of linear feedback part and nonlinear injection of WZSMO is studied deeply. The nonlinear injection design as a key problem of WZSMO is explored with full freedom degrees. This was neglected in previous work [9]. An optimal procedure where the linear feedback part is designed independently and nonlinear injection is designed to sub-minimize the control cost is also concluded. The notations in this paper are standard. Both the Euclidean norm of a vector and the induced spectral norm of a matrix are given by $\|\bullet\|$; $M^{\perp}$ denotes the orthogonal complete matrix of full column rank of matrix M ; The abbreviation of 's.p.d.' stands for symmetric positive definite.

II. EXISTENCE CONDITION OF WZSMO

Consider the following uncertain system

$$x(t) = Ax(t) + Bu(t) + Gd(x,u,t) ,$$

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$$y(t) = Cx(t), \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, $y \in \Re^p$ is the output vector and $d \in \Re^q$ represents the uncertain term including internal/external disturbances and model uncertainties. A, B, C and G are constant matrices of appropriate dimensions with $rank(C) = p$ and $rank(G) = q$. The WZSMO for system (1) has the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_l(y(t) - C\hat{x}(t)) - K_n v(t), \tag{2}$$

where $\hat{x} \in \Re^n$ is the estimator vector, K_l and K_n are linear feedback gain and nonlinear feedback gain, respectively, $v(t)$ is the nonlinear input lying on the uncertain term $d(x,u,t)$. The term $K_n v(t)$ is called nonlinear injection. The gain matrix K_l should satisfy the following equations [13]:

$$P(A - K_l C) + (A - K_l C)^T P = -Q,$$
$$G^T P = FC, \tag{3}$$

where P and Q are s. p. d. matrices, F is a matrix variable and will be used later to determined the nonlinear input $v(t)$.

The difficulty in finding the solution of (3) limited the use of WZSMO. In [13], the solution of (3) is assumed to be found *a prior*. Corless and Tu [2] pointes out that the feasibility of (3) amounts to some positive real condition, which is recalled as follows

Lemma 1[2]: There exists s. p. d. matrices P and Q, matrices K_l and F satisfying (3) if and only if $rank(CG) = rank(G)$ and the triplet $\{A, G, C\}$ is minimum phase.

Remark 1: The equations (3) and Lemma 1 are also thought as the existence condition of WZSMO.

Based on a canonical form, Tan and Edwards [9] gave a LMI approach for WZSMO design, but the canonical form needs thrice transformations, which brings some complexities on design and limits the extensions of their results. In the following, we shall present a novel condition, which is much simpler than before, for the solution of (3).

Theorem 1: The following propositions are equivalent:

P1) There exist s. p. d. matrices P and Q, matrices K_l and F satisfying (3).

P2) $rank(CG) = rank(G)$ and the triplet $\{A, G, C\}$ is minimum phase.

P3) There are symmetric matrices $W_1 \in \Re^{(n-q)\times(n-q)}$ and $W_2 \in \Re^{p\times p}$ and matrix $Y \in \Re^{n\times p}$ such that the following inequalities hold.

$$G^{\perp} W_1 G^{\perp T} + C^T W_2 C > 0, \tag{4}$$

$$\begin{aligned} &G^{\perp} W_1 G^{\perp T} A + C^T W_2 CA - YC \\ &+ (G^{\perp} W_1 G^{\perp T} A + C^T W_2 CA - YC)^T < 0 \end{aligned}. \tag{5}$$

Moreover, one can get the solution of equations (3) are:

$$P = G^{\perp} W_1 G^{\perp T} + C^T W_2 C, \tag{6}$$

$$K_l = (G^{\perp} W_1 G^{\perp T} + C^T W_2 C)^{-1} Y, \tag{7}$$

$$F = G^T C^T W_2, \tag{8}$$

Proof: Lemma 1 shows P1) $\Leftrightarrow$ P2), so we only need to prove P1) $\Leftrightarrow$ P3). Suppose P3) holds. Substituting (6)~(8) into (3), it is straightforwardly seen that the equation (3) holds.

Suppose P1) holds. Define a matrix $\Phi \in \Re^{n\times(n-q+p)}$ as $\Phi = \begin{bmatrix} G^{\perp} & C^T \end{bmatrix}$. Note that

$$\begin{bmatrix} G^{\perp T} \\ G^T \end{bmatrix} \Phi = \begin{bmatrix} I_{n-q} & G^{\perp T} C^T \\ 0 & G^T C^T \end{bmatrix}, \tag{9}$$

$\begin{bmatrix} G^{\perp} & G \end{bmatrix}$ is nonsingular, and $rank(CG) = q$, it follows that $rank(\Phi) = n$. Then it can be easily done to draw out $(n-p)$ columns from $G^{\perp}$ to construct the matrix $H \in \Re^{n\times(n-p)}$ such that the matrix $\begin{bmatrix} H & C^T \end{bmatrix}$ is nonsingular. Hence, one can express any s. p. d. matrix P by

$$P = \begin{bmatrix} H & C^T \end{bmatrix} \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix} \begin{bmatrix} H^T \\ C \end{bmatrix}, \tag{10}$$

where $P_1 \in \Re^{(n-p)\times(n-p)}$, $P_2 \in \Re^{p\times p}$ and $P_3 \in \Re^{(n-p)\times p}$. From the second equation of (3), and noting $G^T H = 0$, we have

$$G^T C^T P_3^T H^T + G^T C^T P_2 C = FC, \tag{11}$$

which is equivalent to

$$\begin{bmatrix} G^T C^T P_3^T & G^T C^T P_2 - F \end{bmatrix} \begin{bmatrix} H^T \\ C \end{bmatrix} = 0. \tag{12}$$

Since the matrix $\begin{bmatrix} H & C^T \end{bmatrix}$ is nonsingular, the following equations can be derived from (12),

$$C^T P_3^T = G^{\perp} \Gamma_1, \tag{13}$$

where $\Gamma_1 \in \Re^{(n-q)\times(n-p)}$. Note that the matrix H can be also expressed by

$$H = G^{\perp} \Gamma_2, \tag{14}$$

where $\Gamma_2 \in \Re^{(n-q)\times(n-p)}$. Now using (13) and (14), one can expand (10) by

$$P = G^{\perp} (\Gamma_2 P_1 \Gamma_2^T + \Gamma_1 \Gamma_2^T + \Gamma_2 \Gamma_1^T) G^{\perp T} + C^T P_2 C. \tag{15}$$

Let $W_1 = \Gamma_2 P_1 \Gamma_2 + \Gamma_1 \Gamma_2^T + \Gamma_2 \Gamma_1^T$ and $W_2 = P_2$, then W_1 and W_2 are obviously symmetric. Set $Y = (G^{\perp} W_1 G^{\perp T} + C^T W_2 C) K_l$, then (4)~(8) can be straightforwardly obtained from (3). This completes the proof.

Remark 2: There are many choices for possible orthogonal complement of the matrix G. It does not matter to choose any one of them as $G^{\perp}$ because W_1 is used as design parameter.

Remark 3: The formula (4) and (5) are standard LMIs, which can be straightforwardly solved by the present well-built LMI-tools [6]. It can be seen that only original system parameters are involved in (4) and (5) without any coordinates transform. This simplifies the synthesis procedure of WZSMO.

Remark 4: Noting P2) $\Leftrightarrow$ P3), Theorem 1 implies a simple and tractable LMI judge for strict positive real transform function. This judge is novel and first explored.

III. NONLINEAR INJECTION DESIGN

After the linear feedback gain K_l was chosen, the error system will be reduced to

$$\dot{e}(t) = A_0 e(t) + Gd(x,u,t) + K_n v(t), \tag{16}$$

where $A_0 = A - K_l C$ and $e(t) = x(t) - \hat{x}(t)$ is the error vector. Since only the output error $e_y = Ce$ is known information, the remained problem to design the nonlinear injection of WZSMO can be thought as sliding mode output feedback control (SMOFC) problem but the input distribution is a designed parameter not fixed one. In [9], the nonlinear gain K_n is assumed to be of fixed form *a prior*. In the following, we will explore the freedom degree of design of K_n from the viewpoint of SMOFC.

According to the intrinsic robust property of sliding mode control theory, one had better set the rank of K_n as high as possible. But due to the limitation of known information, the reachable highest dimension of K_n is p, the dimension of output vector. To make the closed-loop system (16) insensitive to the uncertain term $d(x,u,t)$, it is necessary that the matching condition holds, i.e., there is a matrix $\Gamma_3 \in \Re^{p\times q}$ such that $G = K_n \Gamma_3$. By the theory of SMOFC [1], there exists a sliding mode control to make the closed-loop error system (16) asymptotically stable if and only if there are s. p. d. matrix $P \in \Re^{n\times n}$ and positive scalar σ satisfying

$$A_0^T P + PA_0 - \sigma PK_n K_n^T P < 0,\ K_n^T P = C. \tag{17}$$

From the above statements, the formula for the nonlinear gain matrix K_n is summarized as follows:

$K_n = Q^{-1}C^T$ with Q satisfying

$$QA_0 + A_0^T Q - \sigma C^T C < 0,\ G^T Q = \Gamma_3^T C. \tag{18}$$

It can be seen that (18) is similar to (3) with $F = \Gamma_3^T$. Thus, the technique developed in Theorem 1 can be directly extended to solve (18).

Theorem 2: For WZSMO (2), assume that the uncertain term $d(x,u,t)$ is bounded by

$$\|d(x,u,t)\| \le \alpha(t,u,y), \tag{19}$$

where $\alpha(t,u,y)$ is a known scalar-value function. If the nonlinear injection part is chosen as

$$K_n = Q^{-1}C^T, \tag{20}$$

$$v(t) = -\sigma e_y - \left(\|\Gamma_3\| \alpha(t,u,y) + \eta\right)\frac{e_y}{\|e_y\|}, \tag{21}$$

where η is a positive scalar, Q is the solution of the following LMIs over the variables $W_3 = W_3^T \in \Re^{(n-q)\times(n-q)}$, $W_4 = W_4^T \in \Re^{p\times p}$ and $\sigma \in \Re$:

$$Q = G^{\perp} W_3 G^{\perp T} + C^T W_4 C > 0, \tag{22}$$

$$QA_0 + A_0^T Q - \sigma C^T C < 0, \tag{23}$$

and $\Gamma_3 = G^T C^T W_4$, then the error system (16) can be forced into the sliding surface $e_y = 0$ in finite time and remained on it subsequently, and the closed-loop error system is asymptotically stability.

Proof: Consider the Lyapunov function $V = e^T Q e$. Its derivative with respect to time along the system (16) is

$$\begin{aligned}\dot{V} &= e^T(QA_0 + A_0^T Q)e + 2e^T QGd(x,u,t) \\ &\quad + 2e^T QK_n v(t) < e^T(QA_0 + A_0^T Q - \sigma C^T C)e < 0\end{aligned}. \tag{24}$$

This shows that the closed-loop system is asymptotically stable. Then consider another Lyapunov function for sliding surface, $\dot{V}_y = e_y^T (CQ^{-1}C^T)^{-1} e_y$. Its derivative along the error system (16) is

$$\begin{aligned}\dot{V}_y &= 2e_y^T (CQ^{-1}C^T)^{-1} CA_0 e \\ &\quad + 2e_y^T (CQ^{-1}C^T)^{-1} CGd(x,u,t) - 2e_y^T v(t)\end{aligned}. \tag{25}$$

Substituting (21) into (25) and noting $CG = CQ^{-1}C^T \Gamma_3$, one can get that

$$\dot{V}_y < 2\|e_y\| \left\{ \|(CQ^{-1}C^T)^{-1} CA_0\| \|e\| - 2\eta \right\}. \tag{26}$$

Let $0 < \tilde{\eta} < 2\eta$. Because of the asymptotically stability of the closed-loop error system, after finite time, the error system will enter into the domain

$$\Omega = \left\{ e : \|(CQ^{-1}C^T)^{-1} CA_0\| \|e\| < 2\eta - \tilde{\eta} \right\} \tag{27}$$

in which $\dot{V}_y < -\tilde{\eta}\|e_y\|$. This implies the siding surface $e_y = 0$ can be reached in finite time and remained subsequently. Proof is complete.

Remark 6: Although the first term of $v(t)$ is a linear feedback, we yet put it into the nonlinear injection because it is one part of the input forcing the system to trend into the sliding surface and its action is to cope with the unmeasured part of error vector. In fact, there are various control strategies for nonlinear injection $v(t)$ on SMOFC design [5] and many of them don't include linear feedback part. For example, if η is chosen large enough, then the initial error vector will be contained in the sliding surface attracting domain Ω and thus the first term of $v(t)$ can be dropped hence.

IV. Relation between K_l and K_n

Theorem 3 shows that the nonlinear injection of WZSMO is formulated as LMIs (22)(23). But the formula (23) involved the linear feedback gain K_l, it is interesting whether or not K_l influences the nonlinear part design. In this section, we consider the problem from the two aspects: one is the feasibility of the LMIs (22)(23); the other is the characteristics of the sliding mode motion.

Theorem 3: The LMIs (22) and (23) is feasible if and only if $rank(CG) = rank(G)$ and the triplet $\{A_0, G, C\}$ is minimum phase.

Proof: Necessary is obvious from Theorem 1. Now we prove the Sufficiency.

By Theorem 1, there exists There are symmetric matrices $W_3 \in \Re^{(n-q)\times(n-q)}$ and $W_4 \in \Re^{p\times p}$, matrix $K \in \Re^{n\times p}$, and small enough positive scalar ε such that the following inequalities hold.

$$Q = G^{\perp} W_3 G^{\perp T} + C^T W_4 C > 0, \tag{28}$$

$$QA_0 + A_0^T Q - QKC - C^T K^T Q + \varepsilon I < 0. \tag{29}$$

For arbitrary positive scalar β, the following inequality holds,

$$QKC + C^T K^T Q < \beta C^T C + \beta^{-1} QKK^T Q. \tag{30}$$

Thus, for large enough β, we have

$$QA_0 + A_0^T Q - \beta C^T C < 0, \tag{31}$$

from which it follows that the pair $(\beta^{-1}W_3, \beta^{-1}W_4)$ is the feasible solution of LMIs (22) and (23). This completes the proof.

Remark 7: Since the property that the triplet $\{A_0, G, C\}$ is minimum phase don't vary for any linear feedback gain K_l, the feasibility of LMIs (22) and (23) is independent of K_l.

Another important aspect of WZSMO is specifying the sliding motion dynamic characteristics. Define a transformation matrix

$$T = \begin{bmatrix} C^{\perp T} Q \\ C \end{bmatrix}, \tag{32}$$

where Q is the solution of LMIs (22) and (23). Applying $\begin{bmatrix} e_1 \\ e_y \end{bmatrix} = Te$, the sliding motion dynamic function restricted on the siding surface $e_y = 0$ is

$$\begin{aligned} \dot{e}_1 &= C^{\perp T} Q A_0 C^{\perp} (C^{\perp T} Q C^{\perp})^{-1} e_1 \\ &= C^{\perp T} Q A C^{\perp} (C^{\perp T} Q C^{\perp})^{-1} e_1 \end{aligned}, \tag{33}$$

which means that the sliding motion dynamics characteristics is also independent of K_l an only relies on K_n.

In view of the above statements, we can conclude that *the design of K_l is not restricted by K_n*. So for K_l, one can achieve the optimal design by the linear quadratic Gaussian (LQG) theory [8] not sub-optimal in [9], where the design of K_l is combined with the sliding motion dynamics design.

The LQG optimal observer design method is to solve the following algebraic Ricatti equation (ARE),

$$AQ_{are} + Q_{are}A^T - Q_{are}C^T V^{-1} C Q_{are} + W = 0. \tag{34}$$

The optimal linear gain is $K_l = Q_{are} C^T V^{-1}$. Here, the s. p. d. matrices $W \in \Re^{n\times n}$ and $V \in \Re^{p\times p}$ are the performance weighting matrix of the observer and the co-variance matrix of system's sensor noise, respectively,

Despite that K_l does not influence the sliding motion characteristic and the existence of K_n, it enlarges the domain of the solution (Q, σ) of the LMIs (23), which determines the nonlinear injection of WZSMO. For example, if we fix the scalar σ by $\sigma = 0$, the LMI (23) is infeasible for unstable system matrix A, but it is feasible for A_0. So when one wants to get the optimal design for K_n, the used procedure should rely on A_0 not A. But this do not influence the design of K_l, which still can be designed independently.

It is a rational requirement that we should minimize the cost of nonlinear injection when the sliding motion dynamics characteristics are ensured. But there are many variables in the nonlinear injection $K_n v(t)$, it is difficult to achieve its minimal cost. If we introduce a constraint condition that $\|G^T C^T W_4\| < \kappa$, where κ is a positive scalar, then we have $\|v(t)\| < \|K_n\| \left(\sigma \|e_y(0)\| + \kappa\alpha(t,u,y) + \eta\right)$. According to the initial output error $e_y(0)$ and the value of the bound of the uncertain term $\alpha(t,u,y)$, we can set σ to a suitable weighting value σ_0. After these setting, minimizing $\|K_n\|$ limits the maximal cost of nonlinear injection and in some sense is a sub-optimal design method to minimize the control cost of nonlinear injection. Noting

$\|K_n\| \le \lambda_{\max}(Q^{-1})\|C\|$, the index $\min(\|K_n\|)$ can be approximately replaced by $\min(\delta)$ with $Q^{-1} < \delta I$. The formal statement of the optimal design procedure of WZSMO is shown as follows:

Optimal design algorithm of WZSMO

Step 1): Solve the ARE (34) to get the optimal linear matrix gain K_l .

Step 2): Let $A_0 = A - K_l C$ and chose suitable scalars $\kappa > 0$ and $\sigma_0 > 0$, then solve the following optimal problem,

$$\min(\delta) \text{, subject to:}$$

$$Q = G^{\perp} W_3 G^{\perp T} + C^T W_4 C ,$$

$$\begin{bmatrix} -\kappa I & G^T C^T W_4 \\ W_4 C G & -\kappa I \end{bmatrix} < 0 , \tag{35}$$

$$\begin{bmatrix} -\delta I & I \\ I & -Q \end{bmatrix} < 0 , \tag{36}$$

$$QA_0 + A_0^T Q - \sigma_0 C^T C < 0 \tag{37}$$

Step 3): Chose a suitable η and then by the formulas (20) and (21), the nonlinear part of WZSMO is obtained.

Remark 8: In [9], the design of linear feedback gain is combined with the design of nonlinear part, so their results are limited. But in this paper it is completely independent design procedure. Thus the extant linear observer theory can be utilized directly to design K_l without any constraints. The proposed procedure means that K_l makes the obtained WZSMO be the LQG optimal observer for nominal system and K_n makes it be sub-optimal observer for the cost of nonlinear injection.

V. NUMERICAL EXAMPLE

The system under consideration has the following data:

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -2 & 1 & -1 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The uncertain term is $d(x,u,t) = 0.1x_3(t) + 0.2u(t)$ and its bound can be chosen as $\alpha(t,u,y) = 0.1\|y(t)\| + 0.2|u(t)|$. The weighting matrices of LQG observer is set as $W = 0.05 I_3$ and $V = 0.1 I_2$.

By the step 1), we can calculate the linear gain matrix

$$K_l = \begin{bmatrix} 0.5410 & -0.3807 \\ 2.8157 & 0.0887 \\ 0.9023 & -0.3640 \\ 0.0887 & 0.4526 \end{bmatrix}, \tag{39}$$

and $A_0 = A - K_l C$. Assume that the initial state vector is $x(0) = \begin{bmatrix} 1 & -2 & 1 & -2 \end{bmatrix}^T$ and input vector is the sum of a sinusoidal signal and a state feedback $u(t) = \sin t + K\hat{x}$, where $K = [2.3974\ 12.0659\ 10.0659\ 2.3974]$ make the original closed-loop system stable and $\hat{x}$ is the state vector of the WZSMO system. Set $\eta = 1$ and $\kappa = 1$. Then a suitable weighting σ_0 can be chosen as $\sigma_0 = 0.1$. A permissible orthogonal complement matrix of G is

$$G^{\perp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

After this, we can get the sub-optimal nonlinear injection part by using step 2).

$$\Gamma_3 = G^T C^T W_4 = \begin{bmatrix} 0.0857 & -0.9963 \end{bmatrix}, \tag{41}$$

$$K_n = Q^{-1} C^T = \begin{bmatrix} 0.0970 & 0.0083 \\ 0.1416 & 0.0122 \\ 0.0892 & -0.9961 \\ 0.0122 & 1.0048 \end{bmatrix}, \tag{42}$$

$$v(t) = -0.1 e_y - \left(0.1\|y(t)\| + 0.2|u(t)| + 1\right)\frac{e_y}{\|e_y\|}, \tag{43}$$

The poles of the sliding motion are $\lambda_s = -1.3580 \pm 0.4323i$, which calculated by $\lambda\left(C^{\perp T} Q A C^{\perp} (C^{\perp T} Q C^{\perp})^{-1}\right)$ of dynamic function (33). By (40)~(43), the WZSMO system (2) can be constructed.

The simulation results are shown in Figure 1 consisting of six pictures. Picture (a) shows that the sliding surface $\boldsymbol{e}_y = 0$ will be remained after 1.7 second. The effectiveness of the WZSMO can be seen in Pictures (c)~(f). x_2 and x_4 , shown in (d) and (f) respectively, are in fact the two entities of the output vector. The estimating system states, $\hat{x}_2$ and $\hat{x}_4$, successfully track the original systems states x_2 and x_4 at 1.7 second, which is consistence with the time when the siding motion begins in Picture (a). Picture (c) and (e) show that the unmeasured states are estimated with no bias in about 4.5 seconds, i.e., after 2.8 second the sliding motion of the error system is close to zero. The spent time approximately coincides with the poles of the sliding motion, $4/|\mathrm{Re}(\lambda_s)|$. Picture (b) shows the control costs of the nonlinear injection $K_n v(t)$ of WZSMO. It can be seen from Picture (b) that by the propose sub-optimal design the maximum value of $\|K_n v(t)\|$ is less than 4 and the whole spent energy of the nonlinear injection $\int_0^7 \|K_n v(t)\|^2 dt \approx 11$.

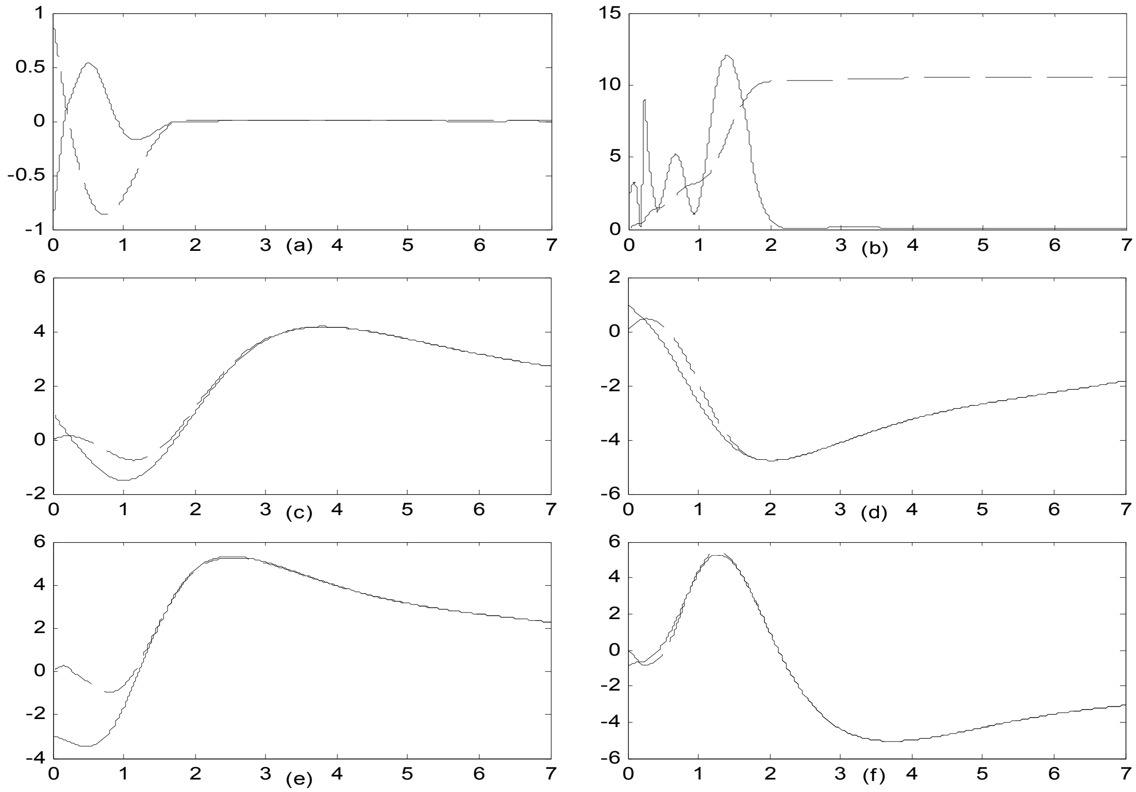

Figure 1. (a) the sliding surface vector e_y ; (b) the nonlinear injection: $\|K_n v(t)\|^2$ (solid line -), $\int_0^t \|K_n v(t)\|^2 dt$ (dashed line --);

(c)~(f) the trajectories of state vector $x_1 \sim x_4$ (solid line -) and estimator vector $\hat{x}_1 \sim \hat{x}_4$ (dashed line --).

VI. CONCLUSION

A simple solution method for WZSMO design is presented in terms of LMIs where only original system parameters are involved. The fact that the linear feedback part of WZSMO can be designed independently is explored. In reported results, its design is limited by the nonlinear injection and sliding motion characteristics. In fact, the WZSMO design is equivalent to the design problem of SMOFC with the input distribution matrix being a selectable parameter. An optimal design procedure followed by a numerical example is presented such that the linear feedback part is designed by the LQG optimal theory and the nonlinear injection is sub-optimized to limit and in some sense minimize the control cost.

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