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Parameter Identification of Affine Time Varying Systems Using Traditional and High Order Sliding Modes

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Abstract— **Time varying parameter identification of affine systems with bounded parameters is addressed via traditional and high order sliding mode parameter observers. Numerical examples illustrate the effectiveness of the proposed parameter estimation based on sliding mode control (SMC) algorithms.**

I. INTRODUCTION

In this work we focus on parameter identification in affine systems with unknown time-varying bounded parameters. Using extended *Kalman Filters* [1] usually yields nonlinear high order equations of the filter, which stability guarantees just locally. In the discrete time case, *Matrix* (non adaptive as well as adaptive) *Forgetting Factor* is introduced in the Least Square Method [2,3] to ensure an acceptable level of nonstationary parametric identification. Use of SMC is beneficial in parameter identification of affine systems [4-6], since it can provide for a global convergence of the estimation error to zero with no immeasurable noises in system dynamics. Sliding mode observers and differentiators [4-10] are also used for state and parameter identification. In particular, state/parameter identification algorithms based on the sliding mode differentiators [5-7] are studied in [8]. Analysis of SMC state observers upon measurement noise is available in [11],[13]. In this paper we propose the parameter identification algorithm based on a direct application of "equivalent control" [4-7] to estimating the right hand side of the differential equations. It is worth noting that estimating the right hand side of a differential equation is equivalent to differentiating the state that allows interpreting the studied parameter estimation algorithms from the standpoint of state differentiation as well. Both traditional [4-7] and second order [9],[10] sliding mode (SOSM) control techniques are used. SOSM does not require a low pass filter to estimate equivalent control, since continuous control is generated directly [9].

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II. THE PROBLEM FORMULATION

Dynamics of an affine system with bounded parameters can be realized as a system of first order differential equations.

$$\begin{cases} \dot{x} = A(t)\varphi(x,t) + \psi(x,t) \\ y = x + \omega(t) \end{cases} \tag{1}$$

with $x \in R^n$, $\varphi(x,t) \in R^n$ and $\psi(x,t) \in R^n$ are a known Lipschitz vector fields, $y \in R^n$ is a measurement vector, $\omega(t) \in R^n$ is an immeasurable bounded noise. The matrix $A(t)$ is given by

$$A(t) = \{a_{ij}(t)\} \forall i = \overline{1,n} \ \forall j = \overline{1,n} \tag{2}$$

with $a_{ij}(t)$ to be time-varying unknown bounded parameters. Also, it is assumed that $\|A(t)\| \le A^+$.

The problem is to design a sliding mode parameter observer for the system (1), (2) that provides at least asymptotic estimation error dynamics, when $\omega(t) \equiv 0$, i.e.

$$\lim_{t\to\infty} \left\| A(t) - \hat{A}(t) \right\| = 0 \tag{3}$$

and zone convergence, when $\|\omega(t)\| \le \varepsilon$, i.e.

$$\lim_{t\to\infty} \left\| A(t) - \hat{A}(t) \right\| \le \delta(\varepsilon), \quad \delta > 0 \tag{4}$$

where $\hat{A}(t)$ estimates $A(t)$.

III. CONSATANT MATRIX A OBSERVATION

In this section we consider a particular case of the plant (1) that obeys the following assumption:

Assumption 1. The following conditions are assumed to be held for the plant (1)

(a) $\omega(t) \equiv 0$

(b) $A = A(t)$ is a constant matrix

(c) $\|\varphi(x,t)\| \le a + b\|x\|$

(d) a matrix $\Phi(t_1,...t_n) \in R^{n\times n}$ given by

$$\Phi(t_1,...,t_n) = [\varphi(x(t_1),t_1), \varphi(x(t_2),t_2),..., \varphi(x(t_n),t_n)] \tag{5}$$

is nonsingular for $t_1 < ... < t_n$, with $t_1 > t_0 > 0$

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A. Constant Matrix A Observation Algorithm Based on Traditional Sliding Mode

The parameter estimation algorithm for the plant given by (1) is formulated in the following *Theorem.*

Theorem 1. If Assumption 1 is met for the plant (1) then the constant matrix A can be exactly estimated $\forall t > t_n$ by

$$A = \left[U_{eq}(t_1,...t_n) - \Psi(t_1,...,t_n)\right]\Phi(t_1,...t_n)^{-1} \quad (6)$$

$$\begin{aligned}&\Psi(t_1,...,t_n) = \left[\psi(x(t_1),t_1),...,\psi(x(t_n),t_n)\right] \in R^{n\times n}\\ &U_{eq}(t_1,...t_n) = \left[u_{eq}(t_1),...u_{eq}(t_n)\right] \in R^{n\times n}\end{aligned} \quad (7)$$

with $t_1 > t_r$, $t_r \le \dfrac{\|\sigma(0)\|}{\overline{\rho}}$ and $u_{eq}(t) \in R^n$ standing for equivalent control [4-6] in the observer

$$\begin{cases}\dot{\hat{x}} = u, \ \sigma = y - \hat{x}, \ \hat{x}(0) = 0\\ u = \psi(x,t) + \left(\overline{\rho} + A^+(a + b\|x\|)\right)\dfrac{\sigma}{\|\sigma\|}\end{cases} \quad (8)$$

Proof of Theorem 1 is given in Appendix.

Remark 1. Equivalent control $u_{eq}(t)$ in (6), (7) can be estimated by low pass filtering (LPF) [4-6] of high frequency switching part of control u in (8) and adding it to the known continuous vector field $\psi(x,t)$:

$$\begin{aligned}&\hat{u}_{eq}(t) = \psi(x,t) + LPF(v)\\ &v = \left(\overline{\rho} + A^+(\alpha + \beta\|x\|)\right)\frac{\sigma}{\|\sigma\|}\end{aligned} \quad (9)$$

Equivalent control estimate $\hat{u}_{eq}(t)$ may converge to $u_{eq}(t)$ only asymptotically due to dynamical properties of LPF. Usually, only zone convergence can be achieved, i.e.

$$\left\|\hat{u}_{eq}(t) - u_{eq}(t)\right\| < O(\tau) \quad (10)$$

as time increases. For instance, if LPF is given by a transfer function $1/(1+\tau s)$, then the transient response to a unit step function will reach 98.2% of its steady state value in $T = 4\tau$, which can be treated as a settling time, since by this time the transient response is practically over. So, eq. (10) holds approximately for $t \ge t_n + T$, where $T \ge 4\tau$ with τ standing for a time constant of the LPF [12]. Therefore, using equivalent control estimate (9), the matrix A can be also estimated only approximately, i.e.

$$\hat{A} = \left[\hat{U}_{eq}(t_1,...t_n) - \Psi(t_1,t_2,...,t_n)\right]\Phi(t_1,...t_n)^{-1} \quad (11)$$

where $\hat{U}_{eq}(t_1,...t_n) = \left[\hat{u}_{eq}(t_1),...\hat{u}_{eq}(t_n)\right] \in R^{n\times n}$, $t_1 \ge t_n + T$, and $\left\|\hat{A} - A\right\| < O(\tau)$.

Remark 2. In the observer (8) $\hat{x}(t) = x(t)$ holds in the sliding mode $\sigma = 0$, and $u_{eq}(t)$ estimates $\dot{x}(t)$, i.e. $\dot{x}(t) = u_{eq}(t) \quad \forall t > t_r$. So, the algorithm in (8) and (9) represents a traditional SMC-based differentiator [5,6].

B. Constant Matrix A Observation Algorithm Based on Second Order Sliding Mode

In order to avoid LPF in (9) that yields only approximate estimation of the matrix A in (1), it is beneficial to use second order sliding mode con1trol (SOSM) [9,10], which provides for a finite time convergence to the sliding variable σ and its derivative $\dot{\sigma}$ by means of continuous control in the parameter observation algorithm. The SOSM-based parameter estimation algorithm for the plant given by (1) that obeys Assumption 1 is formulated in the following *Theorem.*

Theorem 2. If Assumption 1 is met for the plant (1) then the constant matrix A can be exactly estimated $\forall t > t_n$ by

$$A = \left[U(t_1,...t_n) - \Psi(t_1,...,t_n)\right]\Phi(t_1,...t_n)^{-1} \quad (12)$$

$$\begin{aligned}&\Psi(t_1,...,t_n) = \left[\psi(x(t_1),t_1),...,\psi(x(t_n),t_n)\right] \in R^{n\times n}\\ &U(t_1,...t_n) = \left[u(t_1),...u(t_n)\right] \in R^{n\times n}\end{aligned} \quad (13)$$

with $u(t) = \left[u_1(t),...,u_n(t)\right]^T \in R^n$ and

$$\begin{cases}u_i = \alpha_i|\sigma_i|^{1/2}sign(\sigma_i) + \beta_i v_i + \psi_i(x,t)\\ \dot{v}_i = sign(\sigma_i) \quad \forall i = \overline{1,n}\end{cases} \quad (14)$$

$$\begin{aligned}&\alpha_i \ge 0.5\sqrt{A^+\left(a + b\|x\|\right)}\\ &\beta_i \ge 4A^+\left(a + b\|x\|\right) \quad \forall i = \overline{1,n}\end{aligned} \quad (15)$$

standing for control in the observer

$$\dot{\hat{x}} = u, \ \sigma = y - \hat{x}, \ \hat{x}(0) = 0 \quad (16)$$

in the second order sliding mode $\sigma = \dot{\sigma} = 0$.

Proof of Theorem 2 is given in Appendix.

Remark 3. There is no need to filter $u(t)$ while estimating the matrix A in (12), since $u(t)$ in (14) is already continues due to integration of a high frequency switching part of control.

Remark 4. The control function (14), which is called super-twisting control [9,10], exactly estimates $\dot{x}(t)$, i.e. $\dot{x}(t) = u(t)$, in the second order sliding mode $\sigma = \dot{\sigma} = 0$.

So, the algorithm in (14)-(16) is an exact second SOSM-based differentiator [9,10].

IV. Time-Varying Matrix $A(t)$ Observation

The constant matrix estimation algorithms given by Theorems 1 and 2 are applicable for the approximate identification of the slowly varying matrix $A(t)$. In this case the parameter estimation algorithm, let's say for certainty

(12)-(16), can be applied for the matrix $A(t)$ estimation on the consecutive intervals $\Delta t_i = t_{ni} - t_{1i} \quad \forall i = 1,2,...$ assuming Assumption 1 holds approximately at all these time-intervals. It yields a following discrete sequence of the matrix $A(t)$ estimate: $A(t_{n1}), A(t_{n1} + \Delta t_2),..., A(t_{n1} + \Delta t_2 + ... + t_k),...$.

Next, in this section we consider a particular case of the plant (1) with a time-varying matrix $A(t)$, which is not necessarily slow-varying. The plant dynamics under consideration is given by the linear scalar differential equation with unknown bounded time-varying coefficients

$$\begin{aligned} &q^{(n)} + a_{n-1}(t)q^{(n-1)} + ... + a_0(t)q = g(t) \\ &\bar{y} = q + \bar{\varpi}(t) \end{aligned} \tag{17}$$

where $g(t) \in R^1$ is a known Lipschitz function, $\bar{y} \in R^1$ is the output measurement, and $\bar{\varpi} \in R^1$ is a measurement noise. Eq. (17) can be realized in a state variable format (1) [12] with

$$A(t) = \begin{bmatrix} -a_{n-1}(t) & 1 & 0 & ... & 0 \\ -a_{n-2}(t) & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ -a_1(t) & 0 & 0 & ... & 1 \\ -a_0(t) & 0 & 0 & ... & 0 \end{bmatrix} \tag{18}$$

$\varphi(x,t) = x \in R^n$, $\psi(x,t) = [0, 0,...g(t)]^T$

The output $\bar{y} \in R^1$ in (17) is identified

$$\bar{y} = x_1 + \bar{\varpi}(t) \tag{19}$$

In this section we consider a particular case of the plant (1), (18), (19) that obeys the following assumption:

Assumption 2. The following conditions are assumed to be held for the plant (1), (18), (19)

(a) the states $x_2,...,x_n$ of the plant (1), (18) are also measuring available, i.e.

$$y = x + \omega(t) \tag{20}$$

(b) $|a_i(t)| < \bar{\lambda}_i \quad \forall i = \overline{0, n-1}$

(c) $\omega(t) \equiv 0$

A. Time-varying Matrix $A(t)$ Observation Algorithm Based on Traditional Sliding Mode

The parameter estimation algorithm for the plant given by (1), (18), (20) that obeys Assumption 2 is formulated in the following *Theorem.*

Theorem 3. If Assumption 2 is met for in the plant (1), (18), (20) and $x_1(t) \neq 0 \quad \forall t > t_r$, $t_r \leq \dfrac{\|\sigma(0)\|}{\bar{\rho}}$ then the elements of the time-varying matrix $A(t)$ can be exactly estimated by

$$a(t) = -\frac{u_{eq}(t) - r(t)}{x_1(t)} \tag{21}$$

where

$a(t) = [a_{n-1}(t), a_{n-2}(t),...,a_0(t)]^T$, $r(t) = [\, x_2, x_3, ..., x_n, 0]^T$ with $u_{eq}(t) \in R^n$ and t_r standing for equivalent control and reaching time correspondingly [4-6] in the observer

$$\begin{cases} \dot{\hat{x}} = u + \psi(t), \ \sigma = y - \hat{x}, \ \hat{x}(0) = 0 \\ u(t) = \left(\rho + A^+\|x\|\right)\dfrac{\sigma}{\|\sigma\|} \end{cases} \tag{22}$$

Proof of Theorem 3 is omitted for brevity.

Remark 5. Equivalent control $u_{eq}(t)$ in (21) can be estimated by low pass filtering (LPF) [4-6] of high frequency switching control u in (22)

$$\hat{u}_{eq}(t) = LPF(u(t)) \tag{23}$$

Equivalent control estimate $\hat{u}_{eq}(t)$ may converge to $u_{eq}(t)$ only asymptotically due to dynamical properties of LPF. Usually only zone convergence is achievable, and eq. (10) holds for LPF in (23) as time increases. Practically eq. (10) holds after the transient response in the LPF dies out, and the elements of time-varying matrix $A(t)$ in (1) and (18) are to be estimated for $t > t_r + T$, $T = 4\tau$ via the sliding mode parameter observer (21)-(23) as

$$\hat{a}(t) = -\frac{\hat{u}_{eq}(t) - r(t)}{x_1(t)} \tag{24}$$

and $\|\hat{a}(t) - a(t)\| < O(\tau)$ as time increases.

B. Time-varying Matrix A Observation Algorithm Based on Second Order Sliding Mode

Use of high order sliding mode technique, in particular super-twisting algorithm [9], allows avoiding filtering of equivalent control in (22), (23) and achieving exact time-varying matrix estimation in a finite time. The SOSM-based parameter estimation algorithm for the plant given by (1), (18), (20) that obeys Assumption 2 is formulated in the following *Theorem.*

Theorem 4. If Assumption 2 is met for the plant (1), (18), (20) and $x_1(t) \neq 0 \quad \forall t > t_r$, then the elements of the time-varying matrix $A(t)$ can be exactly estimated by

$$a(t) = -\frac{u(t) - r(t)}{x_1(t)} \tag{25}$$

where

$a(t) = [a_{n-1}(t), a_{n-2}(t),...,a_0(t)]^T$, $r(t) = [\, x_2, x_3, ..., x_n, 0]^T$

with $u(t) = [u_1(t),...,u_n(t)]^T \in R^n$ standing for control [9]

$$\begin{cases} u_i = \alpha_i |\sigma_i|^{1/2} sign(\sigma_i) + \beta_i v_i \\ \dot{v}_i = sign(\sigma_i) \quad \forall i = \overline{1, n} \end{cases} \tag{26}$$

$$\begin{aligned} &\alpha_i \geq 0.5\sqrt{\bar{\lambda}_{n-i}|x_1| + |x_{i+1}|} \\ &\beta_i \geq 4\left(\bar{\lambda}_{n-i}|x_1| + |x_{i+1}|\right) \qquad \forall i = \overline{1, n-1} \end{aligned} \tag{27}$$

$$\begin{array}{l}\alpha_n \ge 0.5\sqrt{\bar{\lambda}_0 |x_1|}, \ \beta_n \ge 4\bar{\lambda}_0 |x_1| \\ \bar{\lambda}_i > |a_i(t)| \quad \forall i = \overline{0, n-1}\end{array} \tag{28}$$

and t_r standing for the reaching time in the observer

$$\dot{\hat{x}} = u + \psi(t), \ \sigma = y - \hat{x}, \ \hat{x}(0) = 0 \tag{29}$$

Proof of Theorem 4 is omitted for brevity.

Remark 6. Control $u(t)$ is continuous in SOSM that occurs in the observer (29) in a finite time t_r. So, there is no need in using LPF, and the matrix $A(t)$ estimate is exact in (25).

Remark 7. It is worth noting that $\hat{u}_{eq} + \psi(t)$ and $u + \psi(t)$ evaluate derivatives of the vector state of the plant (1), (18) in the traditional and SOSM estimation algorithms, and represent sliding mode differentiators [5],[6],[9],[10].

V. MEASUREMENT NOISE EFFECTS IN PARAMETER OBSERVATION ALGORITMS

In this section we study effects of nonzero bounded measurement noise on matrix $A(t)$ estimation in the plant given by (1), (18) and (20) using traditional and second order sliding mode estimation algorithms discussed in Sections III and IV. This study was done upon the following assumption:

Assumption 3. The following conditions are assumed to be held for the plant (1), (18), (20)

(a) $\omega(t) \neq 0$, $\|\omega\| \le \varepsilon$ in (20)

(b) $|a_i(t)| < \bar{\lambda}_i \quad \forall i = \overline{0, n-1}$

The effect of the bounded measurement noise on SMC-based parameter estimation algorithm is formulated in the following *Theorem.*

Theorem 5. If Assumption 3 is met for the plant (1), (18), (20), $y_1(t) \neq 0 \ \ \forall t > t_r + T$, $t_r \le \frac{\|\sigma(0)\|}{\bar{\rho}}$ and provided

$$\begin{cases}\hat{a}(t) = -\dfrac{\hat{u}_{eq}(t) - \bar{r}(t)}{y_1(t)} \\ \bar{r}(t) = [\, y_2, y_3, ..., y_n, 0]\end{cases} \tag{30}$$

be the estimate of the elements of the time-varying matrix $A(t)$, with $\hat{u}_{eq}$ defined by (22) and (23), then the following inequality holds

$$\|\hat{a}(t) - a(t)\| = O(\tau) + O(\varepsilon) \quad \forall t > t_r + T \tag{31}$$

as time increases.

Proof of Theorem 5 in given in Appendix.

The effect of the bounded measurement noise on SOSM-based parameter estimation algorithm is formulated in the following *Theorem.*

Theorem 6. If Assumption 3 is met for in the plant (1), (18), (20), $y_1(t) \neq 0 \ \ \forall t > t_r$ and provided

$$\hat{a}(t) = -\frac{u(t) - \bar{r}(t)}{y_1(t)}, \ \bar{r} = [y_2, y_3, ... y_n, 0] \tag{32}$$

be the estimate of the elements of the time-varying matrix $A(t)$ with $u(t)$ defined by (26)-(29), then the following inequality holds

$$\|\hat{a}(t) - a(t)\| = O(\varepsilon^{1/2}) \quad \forall t > t_r \tag{33}$$

Proof of Theorem 6 is given in Appendix.

VI. EXAMPLES

A. Time-Varying Matrix $A(t)$ observation

The LTV system in (1), (18) and (20) is considered for $n = 2$, $x_1(0) = 3.0, x_2(0) = 5.6$, $\psi(t) \equiv 0$, and simulated with $\omega(t)$ as a vector-Gaussian noise with zero mean and 0.02 standard deviation. The unknown time-varying parameters $a_0(t)$ and $a_1(t)$ are defined in the simulations

$$\begin{array}{l} a_0(t) = 0.1 - 0.2 \cdot 1(t-2) + 0.2 \cdot 1(t-4) + \\ \qquad [1(t-2.5) - 1(t-3.5)] 0.1 \sin 10t \\ a_1(t) = 0.2 + 0.2 \cdot 1(t-1.5) - 0.3 \cdot 1(t-3) + \\ \qquad [1(t-1.8) - 1(t-2.7)] 0.2 \sin 20t \end{array} \tag{34}$$

The simulations were performed using Euler method with a step size 10^{-5}. Traditional SMC-based parameter observer is designed in (21) and (23) with control

$$u_i = 10 sign(\sigma_i) \ \forall i = \overline{1,2} \tag{35}$$

The LPF dynamics is taken in a transfer function format $1/(0.004s + 1)$.

The results of the simulations that demonstrate $a_0(t)$ and $a_1(t)$ estimations via the parameter observer (21)-(23) and (35) are shown in Figures 1 and 2. High accuracy robust to noise estimation with zone convergence is confirmed. High frequency control switching is significantly attenuated by the LPF. SOSM-based parameter observer is designed in (25)-(29) with control

$$\begin{cases} u_i = 40|\sigma_i|^{1/2} sign(\sigma_i) + 10 v_i \\ \dot{v}_i = sign(\sigma_i) \quad \forall i = \overline{1,2} \end{cases} \tag{36}$$

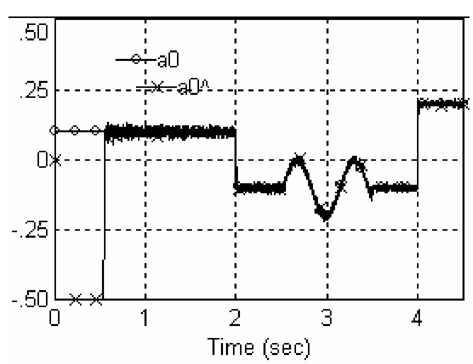

Fig. 1. Estimation of a_0 via the SMC parameter observer

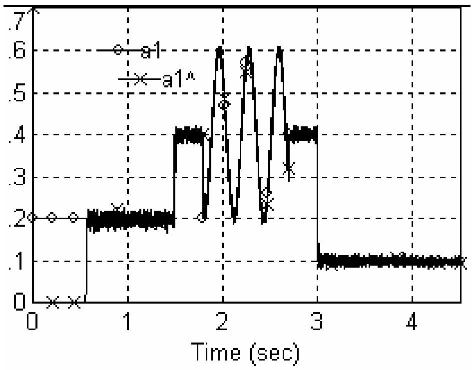

Fig. 2. Estimation of a_1 via the SMC parameter observer

The results of the simulations that demonstrate a_0 and a_1 estimations via the parameter observer (25)-(29) and (36) are similar to the ones obtained by the SMC observer.

B. Constant Matrix A Observation Algorithm

The DC-motor dynamics is given by

$$\frac{d}{dt}\begin{bmatrix}\xi \\ i_a\end{bmatrix} = \begin{bmatrix}-b/J & k_m/J \\ -k_b/L_a & R_a/L_a\end{bmatrix}\begin{bmatrix}\xi \\ i_a\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}V_a$$

$$y_1 = \xi + \omega_1,\ y_2 = i_a + \omega_2$$

where ξ is an angular velocity (rad/s), i_a is an armature current (A); V_a is an armature voltage (V); $J = 2\,kgm^2$ is a moment of inertia; $L_a = 1H$ is an armature inductance; b, k_m, k_b, R_a are viscous friction, motor torque and back electromotive force coefficients and an armature resistance respectively, which are to be estimated; $t_1 = 0.5\,s$, $t_2 = 4.0\,s$. The measurement noises are Gaussian with zero means and 0.02 standard deviations. The unknown parameters are defined for the simulations

$$[b, k_m, k_b, R_a] = [0.5,\ 10,\ 0.1,\ 1] \tag{37}$$

The simulations were performed using Euler method with a step size 10^{-5} via SMC-based (6)-(8) and SOSM-based (12)-(14) parameter estimations algorithms. The LPF dynamics is taken in a transfer function format $1/(0.004s+1)$, which is used in both observers. The results of parameter estimations are presented in Table 1.

Table 1

	$\hat{b}$	$\hat{k}_m$	$\hat{k}_b$	$\hat{R}_a$
SMC parameter observer	0.4804	9.7202	0.0972	1.0228
SOSM parameter observer	0.4968	9.9128	0.1021	0.9683

Based on Figures 1-4 and Table 1 one can conclude that the both algorithms provide robust to noise parameter estimation. The accuracy of the parameter estimation achieved by SOSM-based estimator is higher then the one achieved via traditional SMC-based observer.

VII. Conclusions

Parameter identification of affine time–varying systems is addressed using sliding mode control. A toolbox that consist of traditional SMC-based and SOSM-based parameter observers is developed for the plants with constant unknown parameter matrix and the plants given by n^{th} order LTV differential equation with unknown time-varying coefficients. Effect of measurement noise to the parameter estimation algorithms is revealed as a small-size zone convergence, which is proportional to the upper bound of the measurement noise and to the time constant of LPF for the traditional SMC-based parameter estimation algorithms, and to the square of the upper bound of the measurement noise for the SOSM-based algorithm.

Appendix

Proof of Theorem 1. The existence of the sliding mode in the observer (8) is to be proved. For the sliding variable dynamics obtained from (1) and (8)

$$\dot{\sigma} = A\varphi(x,t) + \psi(x,t) - u \tag{38}$$

the control function (8) makes the inequality

$$\dot{V} \le -\bar{\rho}\sqrt{2V} \tag{39}$$

valid, where

$$V = \frac{1}{2}\sigma^T\sigma \tag{40}$$

is a Lyapunov function for (38). It means that the origin of (38) is globally asymptotically stable with a reaching time

$$t_r \le \frac{\sqrt{2V(0)}}{\bar{\rho}} = \frac{\|\sigma(0)\|}{\bar{\rho}} \tag{41}$$

So, the sliding mode exists in the observer (8) $\forall t > t_r$ and equivalent control $u_{eq}(t)$ that satisfies $\dot{\sigma} = 0$ [4-6] obeys the following equation:

$$u_{eq} = A\varphi(x,t) + \psi(x,t) \tag{42}$$

The validity of (6) is demonstrated. Indeed, after computing $u_{eq}(t)$, $\varphi(x,t)$ and $\psi(x,t)$ in the time instances $t_1 < ... < t_n$ with $t_1 > t_0 = t_r$, eq. (42) is rewritten as

$$U_{eq}(t_1,...,t_n) = A\Phi(t_1,...,t_n) + \Psi(t_1,...,t_n) \tag{43}$$

and yields (6). □

*Proof of Theorem 2. T*he existence of the second order sliding mode in the observer (14)-(16) is to be proved. Indeed, it is well known that a solution $z(t) \in R^1$ and its derivative $\dot{z}(t)$ of the differential equation

$$\dot{z} + \alpha|z|^{1/2}sign(z) + \beta\int sign(z)d\tau = \xi(t) \tag{44}$$

converges to zero in a finite time if $\alpha > 0.5\sqrt{C}$ and $\beta \ge 4C$ with $|\xi(t)| \le C$ [9,10]. The σ_i dynamics are obtained

$$\dot{\sigma}_i = A_i\varphi(x,t) + \psi_i(x,t) - u_i \tag{45}$$

where A_i stands for the i^{th} row of the matrix A and $\psi_i(x,t)$ is the i^{th} coordinate of the vector $\psi(x,t)$. The control function (14) being substituted into (45) yields

$$\dot{\sigma}_i + \alpha_i |\sigma_i|^{1/2} sign(\sigma_i) + \beta_i \int sign(\sigma_i) dt = A_i \varphi(x,t) \qquad (46)$$

By comparing (44) and (46) one can conclude that eq. (46) converges to second order sliding mode $\sigma_i = \dot{\sigma}_i = 0$ in a finite reaching time t_{ri} for α_i and β_i selected from (15). In the second order sliding mode eq. (45) becomes

$$A\varphi(x,t) + \psi(x,t) - u = 0 \qquad (47)$$

The validity of (12) is to be demonstrated. Indeed, after computing $u(t)$, $\varphi(x,t)$ and $\psi(x,t)$ in the time instances $t_1 < ... < t_n$ with $t_1 > t_0 = t_r = \max_{i=1,n} t_{ri}$, eq. (47) is rewritten as

$$A\Phi(t_1,...,t_n) + \Psi(t_1,...,t_n) - U(t_1,...,t_n) = 0 \qquad (48)$$

and yields (18). □

Proof of Theorem 5. Control in (22) is taken in a format

$$u = M(y) SIGN(\sigma) \qquad (49)$$

where $M(y)$ is a scalar positive function

$$M(y) \geq \max_{i=1,n-1} \{ (\rho_i + \bar{\lambda}_{n-i}|y_1| + |y_{i+1}|), (\rho_n + \bar{\lambda}_0 |y_1|) \} = A^+ (\|y\| + \varepsilon + \bar{\rho}) \qquad (50)$$

with $SIGN(\sigma) = [sign(\sigma_1), sign(\sigma_2), ..., sign(\sigma_n)]^T$, $\bar{\rho} > 0$.

Introducing $V(\sigma) = \|x - \hat{x}\|^2$ and taking into account (49) and (50) one can obtain

$$\dot{V} = 2(x-\hat{x})^T (A(t)x - u) \leq 2\|x-\hat{x}\| A^+ (\|y\| + \varepsilon) - 2M(y)(x-\hat{x})^T SIGN(x - \hat{x} + \omega) \qquad (51)$$

Applying the inequality that is derived in [13]

$$z^T SIGN(z+\omega) \geq \sum_{i=1}^{n} |z_i| - 2\sqrt{n}\|\omega\| \qquad (52)$$

to (51) we obtain

$$\dot{V} \leq 2\|x-\hat{x}\| A^+ (\|y\| + \varepsilon) - 2M(y)(\|x-\hat{x}\| - 2\sqrt{n}\varepsilon) = 2\|x-\hat{x}\| (A^+(\|y\| + \varepsilon) - M(y)) + 4M(y)\sqrt{n}\varepsilon \qquad (53)$$

Substituting (50) into (53) implies

$$\dot{V} \leq -2\bar{\rho}\|x-\hat{x}\| + 4\sqrt{n}\varepsilon \left(A^+ \sup_y \|y\| + \varepsilon + \bar{\rho} \right) = -2\bar{\rho}\sqrt{V} + \theta$$
$$\theta := 4\sqrt{n}\varepsilon \left(A \sup_y \|y\| + \varepsilon + \bar{\rho} \right) \qquad (54)$$

Using Lemma 2 from the Appendix in [13], one can conclude that V converges to the domain

$$\Omega : x = \{V \leq \mu\}, \ \mu := \left(\frac{\theta}{2\bar{\rho}} \right)^2 \qquad (55)$$

in a finite time t_f, that is $\left[1 - \frac{\mu}{V} \right]_+ \to 0$ as time increases.

So, the following equation holds within the domain (55)

$$\dot{\sigma} = Ax - u_{eq} = Ay - u_{eq} - A\omega \to \|Ay - u_{eq}\| < O(\varepsilon) \qquad (56)$$

Due to estimation of $\hat{u}_{eq}$ by means of LPF in (23) $\|\hat{u}_{eq}(t) - u_{eq}(t)\| < O(\tau)$, and (28) becomes

$$\|Ay - \hat{u}_{eq}\| < O(\varepsilon) + O(\tau) \qquad (57)$$

Eq. (57) yields (31). □

Proof of Theorem 6. Theorem 4 conditions implies the fact that super-twisting control in (26)-(29) gives an exact estimate of $\dot{x}_i$ in the plant (1), (18) and (20) with zero measurement noise. It is proved in [9] that if x_i is measured with noise ω_i, then the following inequality holds $\forall i = \overline{1,n}$

$$|u_i(t) - \dot{x}_i(t)| = O(\varepsilon^{1/2}) \qquad (58)$$

Equality (58) can be rewritten as follows:

$$\|u(t) - \dot{x}(t)\| = O(\varepsilon^{1/2}) \qquad (59)$$

which, taking into account (32), yields (33). □

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