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Output Feedback Control for Spacecraft Formation Flying with Coupled Translation and Attitude Dynamics

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Abstract— In this paper, we address an output feedback tracking control problem for the coupled translation and attitude motion of a follower spacecraft relative to a leader spacecraft. It is assumed that *i)* the leader spacecraft is tracking a given desired translation and attitude motion trajectory and *ii)* the translation and angular velocity measurements of the two spacecraft are not available for feedback. First, the mutually coupled translation and attitude motion dynamics of the follower spacecraft relative to a leader spacecraft are described. Next, a suitable high-pass filter is employed to estimate the follower spacecraft relative translation and angular velocities using measurements of its relative translational position and attitude orientation. Using a Lyapunov framework, a nonlinear output feedback control law is designed that ensures the semi-global asymptotic convergence of the follower spacecraft relative translation and attitude position tracking errors, despite the lack of translation and angular velocity measurements of the two spacecraft. Finally, an illustrative numerical simulation is presented to demonstrate the effectiveness of the proposed control design methodology.

I. Introduction

Spacecraft formation flying (SFF) has the potential to enhance space-based imaging/interferometry missions through the use of distributed apertures. Specifically, by combining the imaging apertures placed on several separated spacecraft and by appropriately configuring the formation geometry, the sensing aperture can be enlarged beyond the capability of a single spacecraft. However, effective utilization of the SFF technology necessitates highly maneuverable spacecraft to be precisely controlled in a formation so as to maintain a meaningful separation and orientation. Thus, development of a systematic SFF control design framework incorporating the six degree-of-freedom (DOF) coupled translation and attitude motion dynamics of spacecraft is of paramount importance.

The study of dynamics and control for six-DOF spacecraft has received scant attention in the current literature. Some recent exceptions include [8], [9]. These control methods require the use of translation and angular velocity measurements of spacecraft for feedback. Unfortunately, cost/weight constraints may not permit the use of translation and angular velocity sensors. In prior research, several authors have addressed the problem of output feedback control of spacecraft. Specifically, using the four-parameter quaternion representation of the spacecraft attitude, an adaptive output feedback attitude tracking controller was developed in [3]. Furthermore, in [10], an adaptive output feedback position tracking controller was developed for SFF. However, the output feedback control problem for the six-DOF coupled translation and attitude motion of spacecraft formations remains to be addressed.

In this paper, we address the output feedback tracking control problem for the six-DOF motion of a follower spacecraft relative to a leader spacecraft using the coupled translation and attitude dynamics of a leader-follower spacecraft pair developed in [7]. A high-pass filter is employed to generate a velocity-related signal from the translational position and attitude orientation measurements. A judicious modification of the generally recommended [4] filter is implemented to overcome the complexity arising from the mutual coupling of the follower spacecraft translation and attitude motion dynamics. Using a suitable Lyapunov function, our nonlinear output feedback control law guarantees asymptotic convergence of the translation and attitude position tracking errors, despite the lack of translation and angular velocity feedback.

II. Mathematical Preliminaries

Throughout this paper, several reference frames are employed to characterize the translation and attitude motion dynamics of a spacecraft. Each reference frame used in this paper is assumed to consist of three basis vectors which are right-handed, mutually perpendicular, and of unit length. Let $\mathcal{F}$ denote a reference frame and let $\vec{i}$, $\vec{j}$, and $\vec{k}$ denote the three basis vectors of $\mathcal{F}$. Then $\vec{F} \triangleq \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$ denotes the vectrix of the reference frame $\mathcal{F}$ [5]. A vector $\vec{A}$ can be expressed in the reference frame $\mathcal{F}$ as $\vec{A} \triangleq a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$, where a_1, a_2, and a_3 denote the components of $\vec{A}$ along $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. Frequently, we will assemble these components as $A \triangleq [a_1\ a_2\ a_3]^T$. Using the above vectrix formalism and the usual vector inner product, a vector $\vec{A}$ can be expressed in the reference frame $\mathcal{F}$ as $\vec{A} = A^T \vec{F} = \vec{F}^T A$.

The vectrix $\vec{F}$ has the following two properties: *i)* $\vec{F} \cdot \vec{F}^T = I_3$, where "·" denotes the usual vector

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dot product and I_n denotes an n dimensional identity matrix and *ii)* $\vec{F} \times \vec{F}^T = \begin{bmatrix} \vec{0} & \vec{k} & -\vec{j} \\ -\vec{k} & \vec{0} & \vec{i} \\ \vec{j} & -\vec{i} & \vec{0} \end{bmatrix}$, where "$\times$" denotes the usual vector cross product. Thus, it follows that $A = \vec{F} \cdot \vec{A} = \vec{A} \cdot \vec{F}$. Next, let $\vec{B}$ be a vector, which is expressed in $\mathcal{F}$ by $\vec{B} \triangleq b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$, and let $B \triangleq [b_1\ b_2\ b_3]^T$. Then, it follows that $\vec{A} \cdot \vec{B} = A^T B = B^T A$ and $\vec{A} \times \vec{B} = \vec{F}^T A^{\times} B$, where $A^{\times} \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$.

Throughout this paper, various vectors will be expressed in two or more different reference frames using the rotation matrix concept. For example, consider a vector $\vec{A}$ expressed in the reference frame $\mathcal{F}_{\mathrm{u}}$ has components A_{u}, i.e., $\vec{A} = \vec{F}_{\mathrm{u}}^T A_{\mathrm{u}}$, where $\vec{F}_{\mathrm{u}}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{u}}$. Similarly, consider the vector $\vec{A}$ expressed in the reference frame $\mathcal{F}_{\mathrm{v}}$ has components A_{v}, i.e., $\vec{A} = \vec{F}_{\mathrm{v}}^T A_{\mathrm{v}}$, where $\vec{F}_{\mathrm{v}}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{v}}$. Then, it follows that $A_{\mathrm{v}} = \vec{F}_{\mathrm{v}} \cdot \vec{F}_{\mathrm{u}}^T A_{\mathrm{u}} = C_{\mathrm{v}}^{\mathrm{u}} A_{\mathrm{u}}$, where $C_{\mathrm{v}}^{\mathrm{u}} \triangleq \vec{F}_{\mathrm{v}} \cdot \vec{F}_{\mathrm{u}}^T \in SO(3)$ denotes the rotation matrix that transforms the components of a vector expressed in $\mathcal{F}_{\mathrm{u}}$ (*viz.*, A_{u}) to the components of the same vector expressed in $\mathcal{F}_{\mathrm{v}}$ (*viz.*, A_{v}). Here the notation $SO(3)$ represents the set of all 3×3 rotation matrices.

Let us consider the vector $\vec{A}$ expressed in another reference frame $\mathcal{F}_{\mathrm{w}}$ has components A_{w}, i.e., $\vec{A} = \vec{F}_{\mathrm{w}}^T A_{\mathrm{w}}$, where $\vec{F}_{\mathrm{w}}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{w}}$. Then, as above, $A_{\mathrm{u}} = \vec{F}_{\mathrm{u}} \cdot \vec{F}_{\mathrm{w}}^T A_{\mathrm{w}} = C_{\mathrm{u}}^{\mathrm{w}} A_{\mathrm{w}}$ and $A_{\mathrm{v}} = \vec{F}_{\mathrm{v}} \cdot \vec{F}_{\mathrm{w}}^T A_{\mathrm{w}} = C_{\mathrm{v}}^{\mathrm{w}} A_{\mathrm{w}}$, where $C_{\mathrm{u}}^{\mathrm{w}} \triangleq \vec{F}_{\mathrm{u}} \cdot \vec{F}_{\mathrm{w}}^T \in SO(3)$ and $C_{\mathrm{v}}^{\mathrm{w}} \triangleq \vec{F}_{\mathrm{v}} \cdot \vec{F}_{\mathrm{w}}^T \in SO(3)$. Finally, it follows that $A_{\mathrm{v}} = C_{\mathrm{v}}^{\mathrm{u}} C_{\mathrm{u}}^{\mathrm{w}} A_{\mathrm{w}}$ and $C_{\mathrm{v}}^{\mathrm{w}} = C_{\mathrm{v}}^{\mathrm{u}} C_{\mathrm{u}}^{\mathrm{w}}$.

III. Follower Spacecraft Relative Dynamic Model

In this section, we review the translation and attitude motion dynamics of a *follower* spacecraft relative to a *leader* spacecraft [7]. Each spacecraft is modeled as a rigid body with actuators that provide body-fixed forces and torques about three mutually perpendicular axes that define a body-fixed reference frame (i.e., $\mathcal{F}_{\mathrm{b}\ell}$ and $\mathcal{F}_{\mathrm{bf}}$ located at the mass center of the leader and follower spacecraft, respectively) as shown in Figure 1. We fully account for the mutual coupling between the translation and attitude motion of each spacecraft. For given desired translation and attitude motion trajectories of the follower spacecraft relative to the leader spacecraft, we develop the relative translational position and attitude error dynamics. Finally, we state our control objectives for the translation and attitude motion of the follower spacecraft relative to the leader spacecraft.

A. Follower Spacecraft Relative Translation Motion Dynamics

Let $\mathcal{F}_{\mathrm{i}}$ be an inertial reference frame fixed at the center of the earth and let $\vec{A}$ be an arbitrary vector measured with respect to the origin of $\mathcal{F}_{\mathrm{i}}$. Then, in this paper, $\dot{\vec{A}}(t)$ denotes the time derivative of $\vec{A}(t)$ measured in $\mathcal{F}_{\mathrm{i}}$. Using this notation, the translation motion dynamics of the leader spacecraft is given by [5]

$$\dot{\vec{R}}_{\ell} = \vec{V}_{\ell}, \qquad m_{\ell} \dot{\vec{V}}_{\ell} = \vec{f}_{\mathrm{e}\ell} + \vec{f}_{\mathrm{d}\ell} - \vec{f}_{\ell}, \tag{1}$$

where m_{ℓ} denotes the mass of the leader spacecraft, $\vec{R}_{\ell}(t)$ and $\vec{V}_{\ell}(t)$ denote the position and velocity of its mass center, respectively, $\vec{f}_{\mathrm{e}\ell}(t)$ denotes the inverse-square gravitational force that leads to an elliptical orbit [2], [5], $\vec{f}_{\mathrm{d}\ell}(t)$ denotes the attitude-dependent disturbance force that causes the leader spacecraft trajectory to deviate from an ellipse [5], and $\vec{f}_{\ell}(t)$ denotes the external control force. The inverse-square gravitational force and the attitude-dependent disturbance force are given as $\vec{f}_{\mathrm{e}\ell} = -\frac{\mu m_{\ell}}{||\vec{R}_{\ell}||^3} \vec{R}_{\ell}$ and $\vec{f}_{\mathrm{d}\ell} = -\frac{3\mu}{2||\vec{R}_{\ell}||^4} \left[\left\{ \mathrm{tr}\,(J_{\ell})\, \vec{I} + 2\, \vec{J}_{\ell} \right\} \cdot \vec{Z}_{\ell} - 5 \left(\vec{Z}_{\ell} \cdot \vec{J}_{\ell} \cdot \vec{Z}_{\ell} \right) \vec{Z}_{\ell} \right]$, respectively, where $\mu \triangleq MG$ with M being the mass of the earth and G being the universal gravitational constant, J_{ℓ} is the constant, positive-definite, symmetric inertia matrix of the leader spacecraft expressed in $\mathcal{F}_{\mathrm{b}\ell}$, $\vec{J}_{\ell} \triangleq \vec{F}_{\mathrm{b}\ell}^T J_{\ell} \vec{F}_{\mathrm{b}\ell}$ denotes the central inertia dyadic of the leader spacecraft [5], $\vec{I}$ denotes the dyadic of a 3×3 identity matrix, $\vec{Z}_{\ell} \triangleq \frac{\vec{R}_{\ell}}{||\vec{R}_{\ell}||}$ with $\| \vec{R}_{\ell} \| \triangleq \sqrt{\vec{R}_{\ell} \cdot \vec{R}_{\ell}}$, and $\mathrm{tr}\,(\cdot)$ denotes the trace of a matrix.

Analogous to the leader spacecraft, the nonlinear translation motion dynamics of the follower spacecraft is given by [5]

$$\dot{\vec{R}}_{\mathrm{f}} = \vec{V}_{\mathrm{f}}, \qquad m_{\mathrm{f}} \dot{\vec{V}}_{\mathrm{f}} = \vec{f}_{\mathrm{ef}} + \vec{f}_{\mathrm{df}} - \vec{f}_{\mathrm{f}}, \tag{2}$$

where m_{f}, $\vec{R}_{\mathrm{f}}(t)$, $\vec{V}_{\mathrm{f}}(t)$, $\vec{f}_{\mathrm{ef}}(t)$, $\vec{f}_{\mathrm{df}}(t)$, and $\vec{f}_{\mathrm{f}}(t)$ are defined similar to the case of the leader spacecraft.

Next, we develop the translation motion dynamics of the follower spacecraft relative to the leader spacecraft. Before proceeding, for convenience, we introduce the notation

$$\vec{\rho}_{\mathrm{R}} \triangleq \vec{R}_{\mathrm{f}} - \vec{R}_{\ell}, \qquad \vec{\rho}_{\mathrm{V}} \triangleq \vec{V}_{\mathrm{f}} - \vec{V}_{\ell}\,. \tag{3}$$

Let $\vec{\omega}_{\mathrm{bf}}(t)$ denote the angular velocity of $\mathcal{F}_{\mathrm{bf}}$ relative to $\mathcal{F}_{\mathrm{i}}$. In this paper, $\overset{\diamond}{\vec{A}}(t)$ denotes the time derivative of an arbitrary vector $\vec{A}$ measured in $\mathcal{F}_{\mathrm{bf}}$. Using this notation, the time derivative of vector $\vec{A}$ measured in $\mathcal{F}_{\mathrm{i}}$ is given by

$$\dot{\vec{A}} = \overset{\diamond}{\vec{A}} + \vec{\omega}_{\mathrm{bf}} \times \vec{A}\,. \tag{4}$$

Following (4) for vectors $\dot{\vec{R}}_{\mathrm{f}}(t), \dot{\vec{V}}_{\mathrm{f}}(t), \dot{\vec{\rho}}_{\mathrm{R}}(t)$, and $\dot{\vec{\rho}}_{\mathrm{V}}(t)$ yields $\dot{\vec{R}}_{\mathrm{f}} = \overset{\diamond}{\vec{R}}_{\mathrm{f}} + \vec{\omega}_{\mathrm{bf}} \times \vec{R}_{\mathrm{f}}$, $\dot{\vec{V}}_{\mathrm{f}} = \overset{\diamond}{\vec{V}}_{\mathrm{f}} + \vec{\omega}_{\mathrm{bf}} \times \vec{V}_{\mathrm{f}}$, $\dot{\vec{\rho}}_{\mathrm{R}} = \overset{\diamond}{\vec{\rho}}_{\mathrm{R}} + \vec{\omega}_{\mathrm{bf}} \times \vec{\rho}_{\mathrm{R}}$, and $\dot{\vec{\rho}}_{\mathrm{V}} = \overset{\diamond}{\vec{\rho}}_{\mathrm{V}} + \vec{\omega}_{\mathrm{bf}} \times \vec{\rho}_{\mathrm{V}}$, respectively.

In this paper, we assume that the desired translational position of the follower spacecraft relative to the leader spacecraft, denoted by $\vec{\rho}_{\mathrm{R_d}}(t)$, is given. In addition, we assume that the time derivative of

$\vec{\rho}_{\mathrm{R_d}}$ measured in $\mathcal{F}_{\mathrm{i}}$ and denoted by $\vec{\rho}_{\mathrm{V_d}}(t) \triangleq \overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ is given. Note that $\vec{\rho}_{\mathrm{R_d}}(t)$ and its first two time derivatives are assumed to be bounded functions of time. Next, following (4) for vectors $\overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ and $\overset{\bullet}{\vec{\rho}}_{\mathrm{V_d}}$ yields $\overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}} = \overset{\diamond}{\vec{\rho}}_{R_{\mathrm{d}}} + \vec{\omega}_{\mathrm{bf}} \times \vec{\rho}_{\mathrm{R_d}}$ and $\overset{\bullet}{\vec{\rho}}_{\mathrm{V_d}} = \overset{\diamond}{\vec{\rho}}_{V_{\mathrm{d}}} + \vec{\omega}_{\mathrm{bf}} \times \vec{\rho}_{\mathrm{V_d}}$, respectively.

Now we develop the error dynamics of the translation motion of the follower spacecraft relative to the leader spacecraft. We begin by introducing the notation

$$\vec{e}_{\mathrm{R_r}} \triangleq \vec{\rho}_{\mathrm{R_d}} - \vec{\rho}_{\mathrm{R}}, \qquad \vec{e}_{\mathrm{V_r}} \triangleq \vec{\rho}_{\mathrm{V_d}} - \vec{\rho}_{\mathrm{V}}. \tag{5}$$

Computing the time derivative of both sides of (5) measured in $\mathcal{F}_{\mathrm{i}}$ and performing simple manipulations, we get

$$\overset{\bullet}{\vec{e}}_{\mathrm{R_r}} = \overset{\diamond}{\vec{\rho}}_{R_{\mathrm{d}}} - \overset{\diamond}{\vec{\rho}}_{\mathrm{R}} + \vec{\omega}_{\mathrm{bf}} \times \vec{e}_{\mathrm{R_r}}, \quad \overset{\bullet}{\vec{e}}_{\mathrm{V_r}} = \overset{\diamond}{\vec{\rho}}_{V_{\mathrm{d}}} - \overset{\diamond}{\vec{\rho}}_{\mathrm{V}} + \vec{\omega}_{\mathrm{bf}} \times \vec{e}_{\mathrm{V_r}}. \tag{6}$$

It follows from (5) that $\overset{\diamond}{\vec{e}}_{\mathrm{R_r}} = \overset{\diamond}{\vec{\rho}}_{R_{\mathrm{d}}} - \overset{\diamond}{\vec{\rho}}_{\mathrm{R}}$ and $\overset{\bullet}{\vec{e}}_{\mathrm{R_r}} = \overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}} - \overset{\bullet}{\vec{\rho}}_{\mathrm{R}}$. Similarly, it follows from (5) that $\overset{\diamond}{\vec{e}}_{\mathrm{V_r}} = \overset{\diamond}{\vec{\rho}}_{V_{\mathrm{d}}} - \overset{\diamond}{\vec{\rho}}_{\mathrm{V}}$ and $\overset{\bullet}{\vec{e}}_{\mathrm{V_r}} = \overset{\bullet}{\vec{\rho}}_{\mathrm{V_d}} - \overset{\bullet}{\vec{\rho}}_{\mathrm{V}}$. Combining (1)–(3), (5), and (6), we obtain the translation error dynamics of the follower spacecraft relative to the leader spacecraft given by

$$\begin{aligned} \overset{\diamond}{\vec{e}}_{\mathrm{R_r}} &= \vec{e}_{\mathrm{V_r}} - \vec{\omega}_{\mathrm{bf}} \times \vec{e}_{\mathrm{R_r}}, & (7)\\ \overset{\diamond}{\vec{e}}_{\mathrm{V_r}} &= \overset{\bullet}{\vec{\rho}}_{\mathrm{V_d}} - \vec{\omega}_{\mathrm{bf}} \times \vec{e}_{\mathrm{V_r}} + \frac{1}{m_\ell}\left(\vec{f}_{\mathrm{e}\ell} + \vec{f}_{\mathrm{d}\ell}\right) - \frac{1}{m_\ell}\vec{f}_\ell \\ &\quad - \frac{1}{m_{\mathrm{f}}}\left(\vec{f}_{\mathrm{ef}} + \vec{f}_{\mathrm{df}}\right) + \frac{1}{m_{\mathrm{f}}}\vec{f}_{\mathrm{f}}. & (8) \end{aligned}$$

Now using the framework of Section II, various vectors of interest can be expressed in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as follows

$$\Big[R_{\mathrm{f}}\ V_{\mathrm{f}}\ \rho_{\mathrm{R}}\ \dot{\rho}_{\mathrm{R}}\ \rho_{\mathrm{V}}\ \dot{\rho}_{\mathrm{V}}\ \rho_{\mathrm{R_d}}\ \dot{\rho}_{\mathrm{R_d}}\ \rho_{\mathrm{V_d}}\ DV\ e_{\mathrm{R_r}}\ \dot{e}_{\mathrm{R_r}}\ e_{\mathrm{V_r}}\ \dot{e}_{\mathrm{V_r}}\ \omega_{\mathrm{bf}}\ f_{\mathrm{f}}\ f_{\mathrm{ef}}\ f_{\mathrm{df}}\Big] \triangleq \vec{F}_{\mathrm{bf}} \cdot \Big[\vec{R}_{\mathrm{f}}\ \vec{V}_{\mathrm{f}}\ \vec{\rho}_{\mathrm{R}}\ \overset{\diamond}{\vec{\rho}}_{\mathrm{R}}\ \vec{\rho}_{\mathrm{V}}\ \overset{\diamond}{\vec{\rho}}_{\mathrm{V}}\ \vec{\rho}_{\mathrm{R_d}}\ \overset{\diamond}{\vec{\rho}}_{R_{\mathrm{d}}}\ \vec{\rho}_{\mathrm{V_d}}\ \overset{\bullet}{\vec{\rho}}_{\mathrm{V_d}}\ \vec{e}_{\mathrm{R_r}}\ \overset{\diamond}{\vec{e}}_{\mathrm{R_r}}\ \vec{e}_{\mathrm{V_r}}\ \overset{\diamond}{\vec{e}}_{\mathrm{V_r}}\ \vec{\omega}_{\mathrm{bf}}\ \vec{f}_{\mathrm{f}}\ \vec{f}_{\mathrm{ef}}\ \vec{f}_{\mathrm{df}}\Big], \tag{9}$$

where $R_{\mathrm{f}}(t)$, $V_{\mathrm{f}}(t)$, $\rho_{\mathrm{R}}(t)$, $\dot{\rho}_{\mathrm{R}}(t)$, $\rho_{\mathrm{V}}(t)$, $\dot{\rho}_{\mathrm{V}}(t)$, $\rho_{\mathrm{R_d}}(t)$, $\dot{\rho}_{\mathrm{R_d}}(t)$, $\rho_{\mathrm{V_d}}(t)$, $DV(t)$, $e_{\mathrm{R_r}}(t)$, $\dot{e}_{\mathrm{R_r}}(t)$, $e_{\mathrm{V_r}}(t)$, $\dot{e}_{\mathrm{V_r}}(t)$, $\omega_{\mathrm{bf}}(t)$, $f_{\mathrm{f}}(t)$, $f_{\mathrm{ef}}(t)$, $f_{\mathrm{df}}(t) \in \mathbb{R}^3$ and $\vec{F}_{\mathrm{bf}}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{bf}}$. Similarly, various vectors of interest can be expressed in $\mathcal{F}_{\mathrm{b}\ell}$ as follows $[R_\ell\ V_\ell\ f_\ell\ f_{\mathrm{e}\ell}\ f_{\mathrm{d}\ell}] \triangleq \vec{F}_{\mathrm{b}\ell} \cdot \left[\vec{R}_\ell\ \vec{V}_\ell\ \vec{f}_\ell\ \vec{f}_{\mathrm{e}\ell}\ \vec{f}_{\mathrm{d}\ell}\right]$, where $R_\ell(t), V_\ell(t), f_\ell(t), f_{\mathrm{e}\ell}(t), f_{\mathrm{d}\ell}(t) \in \mathbb{R}^3$ and $\vec{F}_{\mathrm{b}\ell}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{b}\ell}$. Next, it follows from Section II that various vectors of interest originally expressed in $\mathcal{F}_{\mathrm{b}\ell}$ can be expressed in $\mathcal{F}_{\mathrm{bf}}$ using the rotation matrix $C_{\mathrm{bf}}^{\mathrm{b}\ell}$, which is given in the sequel. Thus, using (3), (5), and the vectrix formalism of Section II, we obtain

$$R_{\mathrm{f}} = \rho_{\mathrm{R_d}} - e_{\mathrm{R_r}} + C_{\mathrm{bf}}^{\mathrm{b}\ell} R_\ell, \ V_{\mathrm{f}} = \rho_{\mathrm{V_d}} - e_{\mathrm{V_r}} + C_{\mathrm{bf}}^{\mathrm{b}\ell} V_\ell. \tag{10}$$

Finally, an application of the vectrix formalism of Section II on (7) and (8) yields

$$\dot{e}_{\mathrm{R_r}} = e_{\mathrm{V_r}} - \omega_{\mathrm{bf}}^{\times} e_{\mathrm{R_r}}, \tag{11}$$

$$\begin{aligned} \dot{e}_{\mathrm{V_r}} &= DV - \omega_{\mathrm{bf}}^{\times} e_{\mathrm{V_r}} - \frac{1}{m_\ell} C_{\mathrm{bf}}^{\mathrm{b}\ell}\Bigg[\frac{\mu m_\ell}{||\vec{R}_\ell||^3} R_\ell + \Big\{\mathrm{tr}\,(J_\ell)\, I_3 \frac{3\mu}{2||\vec{R}_\ell||^4} \\ &+ 2J_\ell - \frac{5R_\ell^T J_\ell R_\ell}{||\vec{R}_\ell||^2} I_3\Big\}\frac{R_\ell}{||\vec{R}_\ell||} + f_\ell\Bigg] + \frac{1}{m_{\mathrm{f}}}\Bigg[\frac{\mu m_{\mathrm{f}}}{||\vec{R}_{\mathrm{f}}||^3} R_{\mathrm{f}} \\ &+ \frac{3\mu}{2||\vec{R}_{\mathrm{f}}||^4}\Big\{\mathrm{tr}\,(J_{\mathrm{f}})\, I_3 + 2J_{\mathrm{f}} - \frac{5R_{\mathrm{f}}^T J_{\mathrm{f}} R_{\mathrm{f}}}{||\vec{R}_{\mathrm{f}}||^2} I_3\Big\}\frac{R_{\mathrm{f}}}{||\vec{R}_{\mathrm{f}}||}\Bigg] + \frac{1}{m_{\mathrm{f}}} f_{\mathrm{f}}. \end{aligned} \tag{12}$$

Remark 3.1: We assume that the desired translation motion dynamics of the follower spacecraft relative to the leader spacecraft will be typically specified in the earth-fixed inertial reference frame $\mathcal{F}_{\mathrm{i}}$. Let $\vec{F}_{\mathrm{i}} = \left[\vec{i}\ \ \vec{j}\ \ \vec{k}\right]^T$ denote the vectrix of the inertial reference frame $\mathcal{F}_{\mathrm{i}}$. In addition, let $x(t), y(t), z(t) \in \mathbb{R}$ denote the components of $\vec{\rho}_{\mathrm{R_d}}$ along $\vec{i}$, $\vec{j}$, and $\vec{k}$, respectively. Then, $\vec{\rho}_{\mathrm{R_d}} = \vec{F}_{\mathrm{i}}^T D_{\mathrm{r}}$, where $D_{\mathrm{r}}(t) \in \mathbb{R}^3$, $D_{\mathrm{r}} \triangleq [x\ y\ z]^T$. With $\overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ and $\overset{\bullet\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ denoting the first and second derivatives, respectively, of $\vec{\rho}_{\mathrm{R_d}}$ measured in $\mathcal{F}_{\mathrm{i}}$, we can express $\overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ and $\overset{\bullet\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ in $\mathcal{F}_{\mathrm{i}}$ as $\dot{D}_{\mathrm{r}} \triangleq \vec{F}_{\mathrm{i}}^T \cdot \overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}$ and $\ddot{D}_{\mathrm{r}} \triangleq \vec{F}_{\mathrm{i}}^T \cdot \overset{\bullet\bullet}{\vec{\rho}}_{\mathrm{R_d}}$, respectively, where $\dot{D}_{\mathrm{r}}(t), \ddot{D}_{\mathrm{r}}(t) \in \mathbb{R}^3$. Now using the rotation matrix framework of Section II, we transform $D_{\mathrm{r}}, \dot{D}_{\mathrm{r}}$, and $\ddot{D}_{\mathrm{r}}$ to the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as follows $\rho_{\mathrm{R_d}} = C_{\mathrm{bf}}^{\mathrm{i}} D_{\mathrm{r}}$, $\rho_{\mathrm{V_d}} = C_{\mathrm{bf}}^{\mathrm{i}} \dot{D}_{\mathrm{r}}$, and $DV = C_{\mathrm{bf}}^{\mathrm{i}} \ddot{D}_{\mathrm{r}}$, where $C_{\mathrm{bf}}^{\mathrm{i}}$ is given in the sequel.

B. Follower Spacecraft Relative Attitude Dynamics

We begin by characterizing the attitude dynamics of the leader and follower spacecraft. The attitude dynamics of the leader spacecraft is given by

$$\overset{\bullet}{\vec{h}}_\ell = \vec{\tau}_{\mathrm{g}\ell} + \vec{\tau}_\ell, \tag{13}$$

where $\vec{h}_\ell(t)$ given by $\vec{h}_\ell = \vec{J}_\ell \cdot \vec{\omega}_{\mathrm{b}\ell}$ denotes the angular momentum of the leader spacecraft about its mass center, $\vec{\tau}_{\mathrm{g}\ell}(t)$ given by $\vec{\tau}_{\mathrm{g}\ell} = \frac{3\mu}{||\vec{R}_\ell||^3} \vec{Z}_\ell \times \vec{J}_\ell \cdot \vec{Z}_\ell$ denotes the gravity gradient torque [5], and $\vec{\tau}_\ell(t)$ denotes the control torque.

Next, let $\vec{\omega}_{\mathrm{b}\ell}(t)$ denote the angular velocity of $\mathcal{F}_{\mathrm{b}\ell}$ relative to $\mathcal{F}_{\mathrm{i}}$ and let $\overset{\circ}{\vec{A}}(t)$ denote the time derivative of an arbitrary vector $\vec{A}(t)$ measured in $\mathcal{F}_{\mathrm{b}\ell}$. Then, the time derivative of $\vec{h}_\ell$ measured in $\mathcal{F}_{\mathrm{i}}$ is given by $\overset{\bullet}{\vec{h}}_\ell = \overset{\circ}{\vec{h}}_\ell + \vec{\omega}_{\mathrm{b}\ell} \times \vec{h}_\ell$. Once again, using the framework of Section II, various vectors of interest can be expressed in $\mathcal{F}_{\mathrm{b}\ell}$ as $[h_\ell\ \dot{\omega}_{\mathrm{b}\ell}\ \tau_{\mathrm{g}\ell}\ \tau_\ell] \triangleq \vec{F}_{\mathrm{b}\ell} \cdot \left[\vec{h}_\ell\ \overset{\circ}{\vec{\omega}}_{\mathrm{b}\ell}\ \vec{\tau}_{\mathrm{g}\ell}\ \vec{\tau}_\ell\right]$, where $h_\ell(t), \dot{\omega}_{\mathrm{b}\ell}(t), \tau_{\mathrm{g}\ell}(t), \tau_\ell(t) \in \mathbb{R}^3$. Now an application of the vectrix formalism of Section II on (13) yields

$$J_\ell \dot{\omega}_{\mathrm{b}\ell} = -\omega_{\mathrm{b}\ell}^{\times} J_\ell \omega_{\mathrm{b}\ell} + \tau_{\mathrm{g}\ell} + \tau_\ell. \tag{14}$$

Next, we characterize the kinematic equation that relates the time derivative of the leader spacecraft angular orientation to its angular velocity as follows [5]

$$\begin{bmatrix}\dot{\varepsilon}_{\mathrm{b}\ell}\\ \dot{\zeta}_{\mathrm{b}\ell}\end{bmatrix}=E(\varepsilon_{\mathrm{b}\ell},\zeta_{\mathrm{b}\ell})\omega_{\mathrm{b}\ell},\ E(\varepsilon_{\mathrm{b}\ell},\zeta_{\mathrm{b}\ell})\triangleq\frac{1}{2}\begin{bmatrix}\varepsilon_{\mathrm{b}\ell}^{\times}+\zeta_{\mathrm{b}\ell}I_3\\ -\varepsilon_{\mathrm{b}\ell}^T\end{bmatrix}, \tag{15}$$

where $(\varepsilon_{\mathrm{b}\ell}(t),\,\zeta_{\mathrm{b}\ell}(t))\in\mathbb{R}^3\times\mathbb{R}$ represents the quaternion, which characterizes the attitude of $\mathcal{F}_{\mathrm{b}\ell}$ with respect to $\mathcal{F}_{\mathrm{i}}$. By construction, $(\varepsilon_{\mathrm{b}\ell},\,\zeta_{\mathrm{b}\ell})$ must satisfy the unit norm constraint $\varepsilon_{\mathrm{b}\ell}^T\varepsilon_{\mathrm{b}\ell}+\zeta_{\mathrm{b}\ell}^2=1$. Following [5], the rotation matrix $C_{\mathrm{b}\ell}^{\mathrm{i}}\in SO(3)$, which brings the inertial frame $\mathcal{F}_{\mathrm{i}}$ onto the spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$, is given as $C_{\mathrm{b}\ell}^{\mathrm{i}}=C(\varepsilon_{\mathrm{b}\ell},\zeta_{\mathrm{b}\ell})\triangleq\left(\zeta_{\mathrm{b}\ell}^2-\varepsilon_{\mathrm{b}\ell}^T\varepsilon_{\mathrm{b}\ell}\right)I_3+2\varepsilon_{\mathrm{b}\ell}\varepsilon_{\mathrm{b}\ell}^T-2\zeta_{\mathrm{b}\ell}\varepsilon_{\mathrm{b}\ell}^{\times}$. The dynamic and kinematic equations of (14) and (15) represent the attitude dynamics of the leader spacecraft.

The attitude dynamics of the follower spacecraft is analogously given by (13) with subscript ℓ replaced by f. In addition, various vectors of interest can be expressed in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as $[h_{\mathrm{f}}\ \dot{\omega}_{\mathrm{bf}}\ \tau_{\mathrm{gf}}\ \tau_{\mathrm{f}}]\triangleq\vec{F}_{\mathrm{bf}}\cdot\left[\vec{h}_{\mathrm{f}}\ \overset{\diamond}{\vec{\omega}}_{\mathrm{bf}}\ \vec{\tau}_{\mathrm{gf}}\ \vec{\tau}_{\mathrm{f}}\right]$. The attitude dynamics of the follower spacecraft is then characterized by the following dynamic and kinematic equations

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{bf}} = -\omega_{\mathrm{bf}}^{\times}J_{\mathrm{f}}\omega_{\mathrm{bf}}+\tau_{\mathrm{gf}}+\tau_{\mathrm{f}}, \tag{16}$$

$$\begin{bmatrix}\dot{\varepsilon}_{\mathrm{bf}}\\ \dot{\zeta}_{\mathrm{bf}}\end{bmatrix} = E(\varepsilon_{\mathrm{bf}},\zeta_{\mathrm{bf}})\omega_{\mathrm{bf}}. \tag{17}$$

Following the definition of $C_{\mathrm{b}\ell}^{\mathrm{i}}$, the rotation matrix $C_{\mathrm{bf}}^{\mathrm{i}}\in SO(3)$, which brings the inertial frame $\mathcal{F}_{\mathrm{i}}$ onto the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$, is given as $C_{\mathrm{bf}}^{\mathrm{i}}\triangleq C(\varepsilon_{\mathrm{bf}},\zeta_{\mathrm{bf}})$. The dynamic and kinematic equations of (16) and (17) represent the attitude dynamics of the follower spacecraft.

Next, we develop the attitude dynamics of the follower spacecraft relative to the leader spacecraft. Let $(\varepsilon_{\mathrm{r}}(t),\,\zeta_{\mathrm{r}}(t))\in\mathbb{R}^3\times\mathbb{R}$ denote the unit quaternion characterizing the mismatch between the orientation of the follower spacecraft $\mathcal{F}_{\mathrm{bf}}$ and the orientation of the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$. In addition, $(\varepsilon_{\mathrm{r}},\,\zeta_{\mathrm{r}})$ can be characterized using $(\varepsilon_{\mathrm{bf}},\,\zeta_{\mathrm{bf}})$ and $(\varepsilon_{\mathrm{b}\ell},\,\zeta_{\mathrm{b}\ell})$ as $\left[\varepsilon_{\mathrm{r}}^T\ \zeta_{\mathrm{r}}\right]^T=F(\varepsilon_{\mathrm{bf}},\zeta_{\mathrm{bf}},\varepsilon_{\mathrm{b}\ell},\zeta_{\mathrm{b}\ell})\triangleq\begin{bmatrix}\zeta_{\mathrm{b}\ell}\varepsilon_{\mathrm{bf}}-\zeta_{\mathrm{bf}}\varepsilon_{\mathrm{b}\ell}+\varepsilon_{\mathrm{bf}}^{\times}\varepsilon_{\mathrm{b}\ell}\\ \zeta_{\mathrm{b}\ell}\zeta_{\mathrm{bf}}+\varepsilon_{\mathrm{b}\ell}^T\varepsilon_{\mathrm{bf}}\end{bmatrix}$. The corresponding rotation matrix $C_{\mathrm{bf}}^{\mathrm{b}\ell}\in SO(3)$, which brings the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ onto the follower spacecraft body-fixed frame $\mathcal{F}_{\mathrm{bf}}$, is given as $C_{\mathrm{bf}}^{\mathrm{b}\ell}=C(\varepsilon_{\mathrm{r}},\zeta_{\mathrm{r}})$ and satisfies $C_{\mathrm{bf}}^{\mathrm{b}\ell}=C_{\mathrm{bf}}^{\mathrm{i}}C_{\mathrm{b}\ell}^{\mathrm{i}\,T}$.

Next, let $\vec{\omega}_{\mathrm{r}}\,(t)$ denote the angular velocity of $\mathcal{F}_{\mathrm{bf}}$ relative to $\mathcal{F}_{\mathrm{b}\ell}$. Then, it follows that $\vec{\omega}_{\mathrm{r}}=\vec{\omega}_{\mathrm{bf}}-\vec{\omega}_{\mathrm{b}\ell}$. Now using the framework of Section II, we express $\vec{\omega}_{\mathrm{r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as

$$\omega_{\mathrm{r}} \triangleq \vec{F}_{\mathrm{bf}}\cdot\vec{\omega}_{\mathrm{r}},\quad \omega_{\mathrm{r}}(t)\in\mathbb{R}^3. \tag{18}$$

In addition, we can obtain

$$\omega_{\mathrm{r}} = \omega_{\mathrm{bf}}-C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}, \tag{19}$$

$$\overset{\diamond}{\vec{\omega}}_{\mathrm{r}} = \overset{\diamond}{\vec{\omega}}_{\mathrm{bf}}-\vec{\omega}_{\mathrm{bf}}\times\vec{\omega}_{\mathrm{r}}-\overset{\circ}{\vec{\omega}}_{\mathrm{b}\ell}\,. \tag{20}$$

Expressing $\overset{\diamond}{\vec{\omega}}_{\mathrm{r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as $\dot{\omega}_{\mathrm{r}}\triangleq\vec{F}_{\mathrm{bf}}\cdot\overset{\diamond}{\vec{\omega}}_{\mathrm{r}},\ \dot{\omega}_{\mathrm{r}}(t)\in\mathbb{R}^3$, we can now express (20) in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$. Finally, multiplying the resultant expression by J_{f} on both sides, we obtain

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{r}} = J_{\mathrm{f}}\dot{\omega}_{\mathrm{bf}}-J_{\mathrm{f}}\omega_{\mathrm{bf}}^{\times}\omega_{\mathrm{r}}-J_{\mathrm{f}}C_{\mathrm{bf}}^{\mathrm{b}\ell}\dot{\omega}_{\mathrm{b}\ell}. \tag{21}$$

We now use (14), (16), (19), and (21) to obtain the following attitude dynamics of the follower spacecraft relative to the leader spacecraft

$$\begin{aligned}J_{\mathrm{f}}\dot{\omega}_{\mathrm{r}}=&-\left(\omega_{\mathrm{r}}+C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times}J_{\mathrm{f}}\left(\omega_{\mathrm{r}}+C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)-J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times}\omega_{\mathrm{r}}\\ &-J_{\mathrm{f}}C_{\mathrm{bf}}^{\mathrm{b}\ell}J_{\ell}^{-1}\left(-\omega_{\mathrm{b}\ell}^{\times}J_{\ell}\omega_{\mathrm{b}\ell}+\tau_{\mathrm{g}\ell}+\tau_{\ell}\right)+\tau_{\mathrm{gf}}+\tau_{\mathrm{f}}.\end{aligned} \tag{22}$$

In addition, the attitude kinematics of the follower spacecraft relative to the leader spacecraft is given by

$$\begin{bmatrix}\dot{\varepsilon}_{\mathrm{r}}\\ \dot{\zeta}_{\mathrm{r}}\end{bmatrix} = E(\varepsilon_{\mathrm{r}},\zeta_{\mathrm{r}})\omega_{\mathrm{r}}. \tag{23}$$

Next, we characterize the desired orientation of the follower spacecraft relative to the leader spacecraft using a desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$. Let $\vec{\omega}_{\mathrm{r_d}}\,(t)$ denote the desired angular velocity of $\mathcal{F}_{\mathrm{r_d}}$ with respect to $\mathcal{F}_{\mathrm{b}\ell}$ and let $\overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}\,(t)$ denote the time derivative of $\vec{\omega}_{\mathrm{r_d}}$ measured in $\mathcal{F}_{\mathrm{r_d}}$. Using the framework of Section II, we express $\vec{\omega}_{\mathrm{r_d}}$ and $\overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}$ in the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ as follows $\left[\omega_{\mathrm{r_d}}\quad\dot{\omega}_{\mathrm{r_d}}\right]\triangleq\vec{F}_{\mathrm{r_d}}\cdot\left[\vec{\omega}_{\mathrm{r_d}}\quad\overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}\right]$, where $\omega_{\mathrm{r_d}}(t),\dot{\omega}_{\mathrm{r_d}}(t)\in\mathbb{R}^3$ and $\vec{F}_{\mathrm{r_d}}$ denotes the vectrix of the reference frame $\mathcal{F}_{\mathrm{r_d}}$. The angular orientation of the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ with respect to the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ is characterized by the desired unit quaternion $(\varepsilon_{\mathrm{r_d}}(t),\,\zeta_{\mathrm{r_d}}(t))\in\mathbb{R}^3\times\mathbb{R}$, whose kinematics is governed by

$$\begin{bmatrix}\dot{\varepsilon}_{\mathrm{r_d}}\\ \dot{\zeta}_{\mathrm{r_d}}\end{bmatrix} = E(\varepsilon_{\mathrm{r_d}},\zeta_{\mathrm{r_d}})\omega_{\mathrm{r_d}}. \tag{24}$$

The corresponding rotation matrix $C_{\mathrm{rd}}^{\mathrm{b}\ell}\in SO(3)$, which brings the leader spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{b}\ell}$ onto the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$, is given by $C_{\mathrm{rd}}^{\mathrm{b}\ell}=C(\varepsilon_{\mathrm{r_d}},\zeta_{\mathrm{r_d}})$. Using (24), it follows that

$$\begin{aligned}\omega_{\mathrm{r_d}} &= 2\left(\zeta_{\mathrm{r_d}}\dot{\varepsilon}_{\mathrm{r_d}}-\dot{\zeta}_{\mathrm{r_d}}\varepsilon_{\mathrm{r_d}}\right)-2\varepsilon_{\mathrm{r_d}}^{\times}\dot{\varepsilon}_{\mathrm{r_d}},\\ \dot{\omega}_{\mathrm{r_d}} &= 2\left(\zeta_{\mathrm{r_d}}\ddot{\varepsilon}_{\mathrm{r_d}}-\ddot{\zeta}_{\mathrm{r_d}}\varepsilon_{\mathrm{r_d}}\right)-2\varepsilon_{\mathrm{r_d}}^{\times}\ddot{\varepsilon}_{\mathrm{r_d}}.\end{aligned} \tag{25}$$

In this paper, we will assume that $\varepsilon_{\mathrm{r_d}}$, $\zeta_{\mathrm{r_d}}$, and their first two time derivatives are all bounded functions of time, which yields boundedness of $\omega_{\mathrm{r_d}}$ and $\dot{\omega}_{\mathrm{r_d}}$ given above.

Now we develop the error dynamics of the attitude motion of the follower spacecraft relative to the leader spacecraft. Let $(e_{\varepsilon_{\mathrm{r}}}(t),\,e_{\zeta_{\mathrm{r}}}(t))\in\mathbb{R}^3\times\mathbb{R}$ denote the unit quaternion characterizing the mismatch between the actual orientation of the follower spacecraft $\mathcal{F}_{\mathrm{bf}}$ relative to the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$ and the desired orientation of the follower spacecraft $\mathcal{F}_{\mathrm{r_d}}$ relative to the leader spacecraft $\mathcal{F}_{\mathrm{b}\ell}$. Next, $(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}})$ can be characterized using $(\varepsilon_{\mathrm{r}}, \zeta_{\mathrm{r}})$ and $(\varepsilon_{\mathrm{r_d}}, \zeta_{\mathrm{r_d}})$ as $\left[e_{\varepsilon_{\mathrm{r}}}^T \; e_{\zeta_{\mathrm{r}}}\right]^T = F(\varepsilon_{\mathrm{r}}, \zeta_{\mathrm{r}}, \varepsilon_{\mathrm{r_d}}, \zeta_{\mathrm{r_d}})$. The corresponding rotation matrix $C_{\mathrm{bf}}^{\mathrm{rd}} \in SO(3)$ that brings the desired, follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{r_d}}$ onto the follower spacecraft body-fixed frame $\mathcal{F}_{\mathrm{bf}}$ is given by [1], [3], [5]

$$C_{\mathrm{bf}}^{\mathrm{rd}} = C(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}}), \tag{26}$$

and satisfies

$$C_{\mathrm{bf}}^{\mathrm{rd}} = C_{\mathrm{bf}}^{\mathrm{b}\ell} C_{\mathrm{rd}}^{\mathrm{b}\ell T}. \tag{27}$$

Next, let $\vec{\omega}_{\mathrm{e_r}}(t)$ denote the angular velocity of $\mathcal{F}_{\mathrm{bf}}$ with respect to $\mathcal{F}_{\mathrm{r_d}}$. Then, it follows that

$$\vec{\omega}_{\mathrm{e_r}} = \vec{\omega}_{\mathrm{r}} - \vec{\omega}_{\mathrm{r_d}}. \tag{28}$$

Now using the framework of Section II, we express $\vec{\omega}_{\mathrm{e_r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as

$$\omega_{\mathrm{e_r}} \triangleq \vec{F}_{\mathrm{bf}} \cdot \vec{\omega}_{\mathrm{e_r}}, \quad \omega_{\mathrm{e_r}}(t) \in \mathbb{R}^3. \tag{29}$$

In addition, we can now obtain

$$\omega_{\mathrm{e_r}} = \omega_{\mathrm{r}} - C_{\mathrm{bf}}^{\mathrm{rd}} \omega_{\mathrm{r_d}}, \tag{30}$$

$$\overset{\diamond}{\vec{\omega}}_{\mathrm{e_r}} = \overset{\diamond}{\vec{\omega}}_{\mathrm{r}} - \vec{\omega}_{\mathrm{bf}} \times \left(\vec{\omega}_{\mathrm{e_r}} - \vec{\omega}_{\mathrm{r}}\right) - \overset{\odot}{\vec{\omega}}_{\mathrm{r_d}}. \tag{31}$$

Expressing $\overset{\diamond}{\vec{\omega}}_{\mathrm{e_r}}$ in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$ as $\dot{\omega}_{\mathrm{e_r}} \triangleq \vec{F}_{\mathrm{bf}} \cdot \overset{\diamond}{\vec{\omega}}_{\mathrm{e_r}}$, $\dot{\omega}_{\mathrm{e_r}}(t) \in \mathbb{R}^3$, noting that $\dot{\omega}_{\mathrm{r}} = \vec{F}_{\mathrm{bf}} \cdot \overset{\diamond}{\vec{\omega}}_{\mathrm{r}}$, and using (9), (18), and (29), we can express (31) in the follower spacecraft body-fixed reference frame $\mathcal{F}_{\mathrm{bf}}$. Finally, multiplying the resultant expression by J_{f} on both sides, we obtain $J_{\mathrm{f}}\dot{\omega}_{\mathrm{e_r}} = J_{\mathrm{f}}\dot{\omega}_{\mathrm{r}} + J_{\mathrm{f}}\omega_{\mathrm{bf}}^{\times}(\omega_{\mathrm{r}} - \omega_{\mathrm{e_r}}) - J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{rd}} \dot{\omega}_{\mathrm{r_d}}$.

We now use (22) and (30) to obtain the following open-loop attitude tracking error dynamics of the follower spacecraft relative to the desired attitude reference frame $\mathcal{F}_{\mathrm{r_d}}$

$$\begin{aligned} J_{\mathrm{f}}\dot{\omega}_{\mathrm{e_r}} = &-\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}} + C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times} J_{\mathrm{f}}\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right. \\ &\left.+ C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right) - J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times}\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right) + J_{\mathrm{f}}\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right. \\ &\left.+ C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times} C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}} - J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{rd}}\dot{\omega}_{\mathrm{r_d}} - J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{b}\ell} J_{\ell}^{-1}\Big(- \omega_{\mathrm{b}\ell}^{\times} J_{\ell}\omega_{\mathrm{b}\ell} \\ &+ \frac{3\mu}{\|R_{\ell}\|^5} R_{\ell}^{\times} J_{\ell} R_{\ell} + \tau_{\ell}\Big) + \frac{3\mu}{\|R_{\mathrm{f}}\|^5} R_{\mathrm{f}}^{\times} J_{\mathrm{f}} R_{\mathrm{f}} + \tau_{\mathrm{f}}. \end{aligned} \tag{32}$$

In addition, using (23), (24), (26), and (30), the open-loop attitude tracking error kinematics is given by

$$\begin{bmatrix} \dot{e}_{\varepsilon_{\mathrm{r}}} \\ \dot{e}_{\zeta_{\mathrm{r}}} \end{bmatrix} = E(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}})\omega_{\mathrm{e_r}}. \tag{33}$$

Next, we rearrange the attitude dynamics of (32) to obtain

$$J_{\mathrm{f}}\dot{\omega}_{\mathrm{e_r}} = -\omega_{\mathrm{e_r}}^{\times} J_{\mathrm{f}}\omega_{\mathrm{e_r}} + \tau_{\mathrm{unr}} + \tau_{\mathrm{knr}} + \tau_{\mathrm{f}}, \tag{34}$$

where $\tau_{\mathrm{unr}}(t), \tau_{\mathrm{knr}}(t) \in \mathbb{R}^3$ are defined as

$$\begin{aligned} \tau_{\mathrm{unr}} \triangleq &-\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times} J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}} + C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right) - \left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right. \\ &\left.+ C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times} J_{\mathrm{f}}\omega_{\mathrm{e_r}} - J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times}\left(\omega_{\mathrm{e_r}} + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right) + J_{\mathrm{f}}\left(\omega_{\mathrm{e_r}}\right. \\ &\left.+ C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times}\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right) - J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{b}\ell} J_{\ell}^{-1}\left(- \omega_{\mathrm{b}\ell}^{\times} J_{\ell}\omega_{\mathrm{b}\ell}\right), \end{aligned} \tag{35}$$

$$\begin{aligned} \tau_{\mathrm{knr}} \triangleq &-\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right)^{\times} J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right) + J_{\mathrm{f}}\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right)^{\times}\left(C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right) \\ &- J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{rd}}\dot{\omega}_{\mathrm{r_d}} - J_{\mathrm{f}} C_{\mathrm{bf}}^{\mathrm{b}\ell} J_{\ell}^{-1}\left(\tfrac{3\mu}{\|R_{\ell}\|^5} R_{\ell}^{\times} J_{\ell} R_{\ell} + \tau_{\ell}\right) + \tfrac{3\mu}{\|R_{\mathrm{f}}\|^5} R_{\mathrm{f}}^{\times} J_{\mathrm{f}} R_{\mathrm{f}}. \end{aligned} \tag{36}$$

Remark 3.2: The definition of τ_{unr} in (35) depends on $\omega_{\mathrm{e_r}}$ and $\omega_{\mathrm{b}\ell}$, which are not measured. Thus, τ_{unr} can not be used in the control design. On the other hand, the definition of τ_{knr} in (36) depends on the desired follower spacecraft attitude trajectory, the attitude of the follower spacecraft relative to the leader spacecraft, the translational position R_{ℓ}, the leader spacecraft control torque, and the translational position R_{f}, signals that are assumed to be known/measured. Thus, τ_{knr} can be used in the control design.

C. Control Objectives

In this paper, the control objective for the translation motion dynamics of the follower spacecraft relative to the leader spacecraft requires that the mass center of the follower spacecraft relative to the mass center of the leader spacecraft track the desired relative translation motion trajectory, i.e., $\vec{\rho}_{\mathrm{R}}(t) \to \vec{\rho}_{\mathrm{R_d}}(t)$ as $t \to \infty$. In addition, it is required that $\overset{\bullet}{\vec{\rho}}_{\mathrm{R}}(t) \to \overset{\bullet}{\vec{\rho}}_{\mathrm{R_d}}(t)$ as $t \to \infty$. Using (1)–(3), (5), and (9), the follower spacecraft relative translation motion tracking control objective can be stated as follows

$$\lim_{t\to\infty} e_{\mathrm{R_r}}(t),\, e_{\mathrm{V_r}}(t) = 0. \tag{37}$$

The control objective for the attitude dynamics of the follower spacecraft relative to the leader spacecraft requires that the actual attitude of the follower spacecraft track the desired attitude trajectory, i.e., the rotation matrix $C_{\mathrm{bf}}^{\mathrm{b}\ell}$ must coincide with the rotation matrix $C_{\mathrm{rd}}^{\mathrm{b}\ell}$ in steady-state. Using (27), this control objective can be equivalently characterized as $\lim_{t\to\infty} C_{\mathrm{bf}}^{\mathrm{rd}} = I_3$. Furthermore, it is required that $\vec{\omega}_{\mathrm{r}}(t) \to \vec{\omega}_{\mathrm{r_d}}(t)$ as $t \to \infty$. With the aid of the unit norm constraint of $(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}})$ and using (26), (28), and (29), the follower spacecraft attitude tracking control objective can be equivalently stated as follows

$$\lim_{t\to\infty} e_{\varepsilon_{\mathrm{r}}}(t),\, \omega_{\mathrm{e_r}}(t) = 0. \tag{38}$$

The control objectives of (37) and (38) are to be met under the constraint of no direct velocity feedback (i.e., V_{f} and ω_{bf} are not measured).

IV. Follower Spacecraft Output Feedback Control Design

In this section, we develop an output feedback controller based on the system dynamics of (11), (12), (33), and (34) such that the tracking error variables $e_{\mathrm{R_r}}$, $e_{\mathrm{V_r}}$, $e_{\varepsilon_{\mathrm{r}}}$, and ω_{r} exhibit asymptotic stability. Before proceeding with the control design, for notational convenience, we introduce two matrices $T(t) \in \mathbb{R}^{3\times 3}$ and $P(t) \in \mathbb{R}^{3\times 3}$ defined as

$$T = \frac{1}{2}\begin{bmatrix} e_{\zeta_{\mathrm{r}}} & -e_{\varepsilon_3} & e_{\varepsilon_2} \\ e_{\varepsilon_3} & e_{\zeta_{\mathrm{r}}} & -e_{\varepsilon_1} \\ -e_{\varepsilon_2} & e_{\varepsilon_1} & e_{\zeta_{\mathrm{r}}} \end{bmatrix}, \quad P = T^{-1}, \quad (39)$$

where $e_{\varepsilon_1}(t), e_{\varepsilon_2}(t), e_{\varepsilon_3}(t) \in \mathbb{R}$ are the components of $e_{\varepsilon_{\mathrm{r}}}$. Using (39), $\dot{e}_{\varepsilon_{\mathrm{r}}}$ of (33) can be written in a compact form as follows

$$\dot{e}_{\varepsilon_{\mathrm{r}}} = T\omega_{\mathrm{e_r}}, \qquad \Rightarrow \qquad P\dot{e}_{\varepsilon_{\mathrm{r}}} = \omega_{\mathrm{e_r}}. \quad (40)$$

Next, we define the position and velocity tracking error variables $r_0(t), v_0(t) \in \mathbb{R}^6$ as follows

$$r_0 \triangleq \begin{bmatrix} e_{\mathrm{R_r}}^T & e_{\varepsilon_{\mathrm{r}}}^T \end{bmatrix}^T, \qquad v_0 \triangleq \begin{bmatrix} e_{\mathrm{V_r}}^T & \dot{e}_{\varepsilon_{\mathrm{r}}}^T \end{bmatrix}^T. \quad (41)$$

Now differentiating r_0 in (41) with respect to time and using (11) and (41), we obtain

$$\dot{r}_0 = \begin{bmatrix} \dot{e}_{\mathrm{R_r}} \\ \dot{e}_{\varepsilon_{\mathrm{r}}} \end{bmatrix} = v_0 - \Omega, \quad (42)$$

where $\Omega(t) \in \mathbb{R}^6$ is defined as $\Omega \triangleq \left[\left(\omega_{\mathrm{bf}}^{\times} e_{\mathrm{R_r}}\right)^T 0_{1\times 3}\right]^T$.

A. Velocity Filter Design

To account for the lack of follower spacecraft translation and angular velocity measurements *viz.*, V_{f} and ω_{bf}, or equivalently the velocity tracking errors *viz.*, $e_{\mathrm{V_r}}$ and ω_{r}, a filtered velocity error signal $e_f(t) \in \mathbb{R}^6$ is produced using a filter. The filter is constructed as shown below

$$e_f = -kr_0 + p, \quad (43)$$

where $k > 0$ is a positive, constant filter gain, $p(t) \in \mathbb{R}^6$ is a pseudo-velocity tracking error generated using

$$\dot{p} = -(k+1)p + k^2 r_0 + \Gamma r_0 + \Delta, \;\; p(0) = kr_0(0), \quad (44)$$

where $\Gamma(t) \in \mathbb{R}^{6\times 6}$ and $\Delta(t) \in \mathbb{R}^6$ are defined as $\Gamma \triangleq \mathrm{diag}\left\{k_0 I_3, \frac{k_1}{\left(1 - e_{\varepsilon_{\mathrm{r}}}^T e_{\varepsilon_{\mathrm{r}}}\right)^2} I_3\right\}$ and $\Delta \triangleq \left[\left(k\left[P\left(e_{f_2} + e_{\varepsilon_{\mathrm{r}}}\right) \quad - C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right]^{\times} e_{\mathrm{R_r}}\right)^T \quad 0_{1\times 3}\right]^T$, respectively, $k_0 > 0$ is a constant, $k_1 > 1$ is a constant, and $e_{f_2}(t) \in \mathbb{R}^3$ is obtained by decomposing e_f as $e_f = \left[e_{f_1}^T \; e_{f_2}^T\right]^T$ with $e_{f_1}(t), e_{f_2}(t) \in \mathbb{R}^3$.

To assist in the development of the filtered velocity error signal e_f dynamics, we introduce an auxiliary tracking error variable $\eta(t) \in \mathbb{R}^6$ as follows

$$\eta \triangleq v_0 + e_f + r_0. \quad (45)$$

Note that using (45) in (42) produces

$$\dot{r}_0 = \eta - e_f - r_0 - \Omega. \quad (46)$$

Next, we decompose η as $\eta = \left[\eta_1^T \; \eta_2\right]^T$, where $\eta_1(t), \eta_2(t) \in \mathbb{R}^3$. Using this decomposition, (45) yields

$$\eta_1 = e_{\mathrm{V_r}} + e_{f_1} + e_{\mathrm{R_r}}, \qquad \eta_2 = \dot{e}_{\varepsilon_{\mathrm{r}}} + e_{f_2} + e_{\varepsilon_{\mathrm{r}}}. \quad (47)$$

In addition, solving for ω_{bf} in (19), and substituting for ω_{r} from (30) in the resulting equation, we obtain

$$\omega_{\mathrm{bf}} = P(\eta_2 - e_{f_2} - e_{\varepsilon_{\mathrm{r}}}) + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}} + C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}, \quad (48)$$

where (40) and (47) have been used. Next, substitute (48) into the definition of Ω and rearrange terms to decompose Ω as

$$\Omega = \Omega_1 + \Omega_2, \quad \Omega_1(t), \Omega_2(t) \in \mathbb{R}^6, \quad (49)$$

where $\Omega_1 = \left[\left(\left(P\eta_2 + C_{\mathrm{bf}}^{\mathrm{b}\ell}\omega_{\mathrm{b}\ell}\right)^{\times} e_{\mathrm{R_r}}\right)^T \;\; 0_{1\times 3}\right]^T$ and $\Omega_2 = \left[\left(\left(-P\left(e_{f_2} + e_{\varepsilon_{\mathrm{r}}}\right) + C_{\mathrm{bf}}^{\mathrm{rd}}\omega_{\mathrm{r_d}}\right)^{\times} e_{\mathrm{R_r}}\right)^T \;\; 0_{1\times 3}\right]^T$.

To obtain the closed-loop dynamics of e_f, we take the time derivative of (43), which yields

$$\dot{e}_f = -k\eta - e_f + \Gamma r_0 + k\Omega_1, \quad (50)$$

where (43), (44), (46), and $\Delta = -k\Omega_2$ have been used.

B. Open-Loop Auxiliary Tracking Error Dynamics

We begin by differentiating η of (45) with respect to time and substituting the time derivative of (40) and (41) to produce

$$\dot{\eta} = \begin{bmatrix} \dot{e}_{\mathrm{V_r}} \\ \dot{T}\omega_{\mathrm{e_r}} + T\dot{\omega}_{\mathrm{e_r}} \end{bmatrix} + \dot{e}_f + v_0 - \Omega, \quad (51)$$

where (42) has been used. Next, we multiply $M(t) \in \mathbb{R}^{6\times 6}$ defined as $M \triangleq \mathrm{diag}\left\{m_{\mathrm{f}} I_3, P^T J_{\mathrm{f}} P\right\}$ on both sides of (51) to yield

$$M\dot{\eta} = \begin{bmatrix} m_{\mathrm{f}}\dot{e}_{\mathrm{V_r}} \\ P^T J_{\mathrm{f}} P\dot{T}\omega_{\mathrm{e_r}} + P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}} \end{bmatrix} + M\left(\dot{e}_f + v_0 - \Omega\right). \quad (52)$$

Using (34) and (39), $P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}}$ in (52) is expressed as $P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}} = P^T\left(J_{\mathrm{f}}\omega_{\mathrm{e_r}}\right)^{\times}\omega_{\mathrm{e_r}} + P^T(\tau_{\mathrm{unr}} + \tau_{\mathrm{knr}} + \tau_{\mathrm{f}})$. Next, we substitute for $\omega_{\mathrm{e_r}}$ from (40) into $P^T J_{\mathrm{f}} P\dot{T}\omega_{\mathrm{e_r}} + P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}}$ to obtain

$$P^T J_{\mathrm{f}} P\dot{T}\omega_{\mathrm{e_r}} + P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}} = P^T\left(J_{\mathrm{f}} P\dot{T} + \left(J_{\mathrm{f}} P\dot{e}_{\varepsilon_{\mathrm{r}}}\right)^{\times}\right)P\dot{e}_{\varepsilon_{\mathrm{r}}} + P^T\left(\tau_{\mathrm{unr}} + \tau_{\mathrm{knr}} + \tau_{\mathrm{f}}\right). \quad (53)$$

To simplify notation, we define an inertia-like matrix $J^{\star}(t) \in \mathbb{R}^{3\times 3}$ as $J^{\star}\left(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}}\right) \triangleq P^T J_{\mathrm{f}} P$ and a coriolis-like matrix $C^{\star}(t) \in \mathbb{R}^{3\times 3}$ as $C^{\star}\left(e_{\varepsilon_{\mathrm{r}}}, e_{\zeta_{\mathrm{r}}}, \dot{e}_{\varepsilon_{\mathrm{r}}}, \dot{e}_{\zeta_{\mathrm{r}}}\right) \triangleq J^{\star}\dot{T}P + P^T(J_{\mathrm{f}} P\dot{e}_{\varepsilon_{\mathrm{r}}})^{\times}P$. Using these definitions, (53) is given by

$$P^T J_{\mathrm{f}} P\dot{T}\omega_{\mathrm{e_r}} + P^T J_{\mathrm{f}} PT\dot{\omega}_{\mathrm{e_r}} = C^{\star}\eta_2 - C^{\star}\left(e_{f_2} + e_{\varepsilon_{\mathrm{r}}}\right) + P^T\left(\tau_{\mathrm{unr}} + \tau_{\mathrm{knr}} + \tau_{\mathrm{f}}\right), \quad (54)$$

where (47) has been used.

Next, to simplify $\dot{e}_f + v_0 - \Omega$ term in (52), we use (49), (50), and v_0 from (45) to produce

$$\dot{e}_f + v_0 - \Omega = -\bar{k}\eta - 2e_f + \bar{\Gamma}r_0 + \bar{k}\Omega_1 - \Omega_2, \quad (55)$$

where $\bar{k} \triangleq k - 1$, $\bar{\Gamma} \triangleq \Gamma - I_6$.

Finally, using (12), (54), and (55), the open-loop dynamics η of (52) yields $M\dot{\eta} = \Lambda - \bar{k}M\eta + N + M\left(\bar{\Gamma}r_0 - 2e_f - \Omega_2\right) + \chi + \bar{k}M\Omega_1 + u_{\mathrm{f}}$, where $N(t)$, $\Lambda(t)$, $\chi(t)$, $u_{\mathrm{f}}(t) \in \mathbb{R}^6$ are defined as $N \triangleq \Big[\Big\{m_{\mathrm{f}}DV - \frac{m_{\mathrm{f}}}{m_\ell}C_{\mathrm{bf}}^{\mathrm{b}\ell}\Big[\frac{\mu m_\ell}{\|R_\ell\|^3}R_\ell + \left(\frac{3\mu}{2\|R_\ell\|^4}\right)\Big\{\mathrm{tr}\left(J_\ell\right)I_3 + 2J_\ell - \frac{5R_\ell^T J_\ell R_\ell}{\|R_\ell\|^2}I_3\Big\}\frac{R_\ell}{\|R_\ell\|} + f_\ell\Big] + \Big[\frac{\mu m_{\mathrm{f}}}{||R_{\mathrm{f}}||^3}R_{\mathrm{f}} + \left(\frac{3\mu}{2||R_{\mathrm{f}}||^4}\right)\Big\{\mathrm{tr}\left(J_{\mathrm{f}}\right)I_3 + 2J_{\mathrm{f}} - \frac{5R_{\mathrm{f}}^T J_{\mathrm{f}} R_{\mathrm{f}}}{||R_{\mathrm{f}}||^2}I_3\Big\}\frac{R_{\mathrm{f}}}{||R_{\mathrm{f}}||}\Big]\Big\}^T \left(P^T\tau_{\mathrm{knr}}\right)^T\Big]^T$, $\Lambda \triangleq [0_{1\times 3}$

$(C^\star\eta_2)^T\Big]^T$, $\chi \triangleq \Big[\big(-m_f\omega_{bf}^\times e_{V_r}\big)^T \quad \big(-C^\star(e_{f_2}+e_{\varepsilon_r}) + P^T\tau_{unr}\big)^T\Big]^T$, and $u_f \triangleq \Big[f_f^T \quad (P^T\tau_f)^T\Big]^T$, respectively.

Remark 4.1: The inertia- and coriolis-like matrices of $J^\star$ and $C^\star$ satisfy the skew-symmetric property of $z^T\left(\frac{1}{2}\dot{J}^\star + C^\star\right)z = 0$, $\forall z \in \mathbb{R}^6$. See [3] for details.

C. Stability Analysis

To facilitate the following stability analysis, we introduce several variables. We define an auxiliary error variable $y(t) \in \mathbb{R}^6$ as $y \triangleq \left[\eta_1^T \; (P\eta_2)^T\right]^T = \left[y_1^T \; y_2^T\right]^T$, where $y_1(t), y_2(t) \in \mathbb{R}^3$. Next, we define a combined error variable $x(t) \in \mathbb{R}^{18}$ as $x \triangleq \left[\begin{array}{cccc} e_{R_r}^T & e_f^T & \frac{\sqrt{k_1}e_{\varepsilon_r}^T}{\sqrt{1-e_{\varepsilon_r}^T e_{\varepsilon_r}}} & y^T \end{array}\right]^T$ and a constant matrix $\hat{M} \in \mathbb{R}^{6\times 6}$ as $\hat{M} \triangleq \operatorname{diag}\{m_f I_3, J_f\}$. In addition, we define $\chi_1(t) \in \mathbb{R}^6$ and $\chi_2(t) \in \mathbb{R}^3$ as

$$\chi_1 \triangleq \left[\begin{array}{cc} I_3 & 0_{3\times3} \\ 0_{3\times3} & (P^T)^{-1} \end{array}\right]\chi, \tag{56}$$

$$\chi_2 \triangleq \left[\left(ke_f^T + \bar{k}\eta^T M\right)\left[\begin{array}{c} \left(P\eta_2 + C_{bf}^{b\ell}\omega_{b\ell}\right)^\times \\ 0_{3\times3} \end{array}\right]\right]^T. \tag{57}$$

Finally, we define some positive constants, λ_1, λ_2, $k_{e_{R_r}}$, $\hat{k}$, k_y, and $k_{\max}$ as $\lambda_1 \triangleq \frac{1}{2}\min\Big\{k_0, 1, m_f, \lambda_{\min}\{J_f\}\Big\}$, $\lambda_2 \triangleq \frac{1}{2}\max\Big\{k_0, 1, m_f, \lambda_{\max}\{J_f\}\Big\}$, $k_{e_{R_r}} \triangleq k_0 - 1$, $\hat{k} \triangleq \bar{k}\lambda_{\min}\Big\{\hat{M}\Big\}$, $k_y \triangleq \hat{k} - 1$, and $k_{\max} \triangleq \max\Big\{k_{e_{R_r}}, k_y\Big\}$, respectively, where $\lambda_{\min}\Big\{X\Big\}$ and $\lambda_{\max}\Big\{X\Big\}$ represent the minimum and maximum eigenvalue, respectively, of a matrix X.

Remark 4.2: Using (56) and (57), it can be shown that χ_1 and χ_2 satisfy $\|\chi_1\| \le \rho_1(\|x\|)\|x\|$ and $\|\chi_2\| \le \rho_2(\|x\|)\|x\|$, respectively, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are some nondecreasing functions. Note that the definition of χ depends on τ_{unr}, which is dependent on the leader spacecraft angular velocity $\omega_{b\ell}$. If the leader spacecraft control inputs f_ℓ and τ_ℓ are designed using the control design framework of [6], or an output feedback extension of [6] in the spirit of this paper, then the leader spacecraft will asymptotically track the desired translation and attitude motion. In this case, it is reasonable to assume that $\omega_{b\ell}(t) \in \mathcal{L}_\infty$, which can be used to show boundedness of χ_1.

Theorem 4.1: The output feedback control law $u_f(t)$ given by

$$u_f = ke_f - N - M\left(\bar{\Gamma}r_0 - 2e_f - \Omega_2\right) - \left[\begin{array}{c} k_0 e_{R_r} \\ \frac{k_1 e_{\varepsilon_r}}{\left(1-e_{\varepsilon_r}^T e_{\varepsilon_r}\right)^2} \end{array}\right], \tag{58}$$

ensures semi-global asymptotic convergence of the follower spacecraft relative translational position and velocity tracking errors and the follower spacecraft relative attitude position and angular velocity tracking errors as delineated by $\lim_{t\to\infty} e_{R_r}(t), e_{V_r}(t), e_{\varepsilon_r}(t), \omega_r(t) = 0$, if the initial condition of e_{ε_r} is selected such that $\|e_{\varepsilon_r}(0)\| \neq 0$ and k, k_0 are selected such that $k_{\max} > \rho_1^2\left(\sqrt{\frac{\lambda_2}{\lambda_1}}||x(0)||\right) + \rho_2^2\left(\sqrt{\frac{\lambda_2}{\lambda_1}}||x(0)||\right)$.

Proof. The proof follows by showing that the time derivative of the positive definite function $V \triangleq \frac{1}{2}k_0 e_{R_r}{}^T e_{R_r} + \frac{1}{2}e_f{}^T e_f + \frac{1}{2}\frac{k_1 e_{\varepsilon_r}^T e_{\varepsilon_r}}{1-e_{\varepsilon_r}^T e_{\varepsilon_r}} + \frac{1}{2}y^T\hat{M}\,y$ is negative-semidefinite, where k, k_0 are selected such that $k_{\max} > \rho_1^2\left(\sqrt{\frac{\lambda_2}{\lambda_1}}||x(0)||\right) + \rho_2^2\left(\sqrt{\frac{\lambda_2}{\lambda_1}}||x(0)||\right)$. Next standard signal chasing arguments are employed to show that all signals in the closed-loop system remain bounded. Finally, Barbalat's Lemma is used to accomplish the result of Theorem 4.1. See [3], [10] for a similar proof.

V. Illustrative Simulation

In this section, we illustrate the output feedback controller of Section IV such that a follower spacecraft with mutually coupled translation and attitude motion dynamics tracks a desired trajectory relative to a leader spacecraft, where the leader spacecraft with mutually coupled translation and attitude motion dynamics also follows a given desired trajectory. In this paper, the desired translation motion trajectory for the leader spacecraft is a natural elliptical orbit around the earth [6]. The following problem data is used in our simulation: $M = 5.974\times10^{24}$ kg, $G = 6.673\times10^{-11}$ m^2/kg $\cdot$ s^2, $m_\ell = 410$ kg, $J_\ell = \operatorname{diag}(17, 20, 18)$ kg $\cdot$ m^2, $a = 4.2223\times10^7$ m, and $e = 0.01$, where a and e denote the semi-major axis and the eccentricity, respectively, of the desired elliptical orbit of the leader spacecraft. Typically, for useful operation, a body-fixed spacecraft axis must point towards a specified direction. Thus, the desired attitude trajectory of the leader spacecraft is generated such that the leader spacecraft body-fixed axis (*viz.*, the $-x$ axis of the desired, leader spacecraft body-fixed reference frame $\mathcal{F}_d$) points towards the earth center [6].

Next, we consider a leader-follower SFF configuration with the following parameters for the follower spacecraft: $m_f = 410$ kg and $J_f = \operatorname{diag}(7, 20, 18)$ kg $\cdot$ m^2. The desired translation motion trajectory of the follower spacecraft relative to the leader spacecraft is selected as $\vec{\rho}_{R_d} = 25\sin(\omega_{tr}t)\left[1-e^{-0.2t^3}\right]\vec{i} + 25\cos(\omega_{tr}t)\left[1-e^{-0.2t^3}\right]\vec{j} - 25\sin(\omega_{tr}t)\left[1-e^{-0.2t^3}\right]\vec{k}$, where $\omega_{tr} = 5\times10^{-5}$ rad/s. We use (9) and the rotation matrix C_{bf}^{i} to obtain the desired follower spacecraft relative translation motion trajectory components, i.e., position ρ_{R_d}, velocity ρ_{V_d}, and acceleration DV, which are expressed in the follower spacecraft body-fixed reference frame $\mathcal{F}_{bf}$. The initial value of C_{bf}^{i} is obtained as outlined below. Finally, noting that $\dot{R}_\ell = V_\ell - \omega_{b\ell}^\times R_\ell$ and using (10), the actual follower spacecraft position and velocity are initialized to $R_f(0) = \rho_{R_d}(0) - [25 \;\; -15 \;\; 15]^T + C_{bf}^{b\ell}(0)R_\ell(0)$ and $V_f(0) = \rho_{V_d}(0) + C_{bf}^{b\ell}(0)V_\ell(0)$, respectively, where $C_{bf}^{b\ell}(0)$ is obtained from $C_{bf}^{i}(0) = C_{bf}^{b\ell}(0)C_{b\ell}^{i}(0)$. Although this desired relative translation motion trajectory may not correspond to a practical scenario, we contend that it demonstrates the efficacy of the proposed controller to track aggressive trajectories, which may arise during the formation reconfiguration process. Alternatively, one can produce a follower spacecraft desired translation motion trajectory based on natural orbital motion.

Next, the desired attitude trajectory of the follower spacecraft relative to the leader spacecraft is specified by the unit quaternion as follows $\varepsilon_{\mathrm{r_d}} = \begin{bmatrix} 0.9165\cos(\omega_{\mathrm{ar}}t) & 0.4472\sin(\omega_{\mathrm{ar}}t) & 0.8\sin(\omega_{\mathrm{ar}}t) \end{bmatrix}^T$, $\zeta_{\mathrm{r_d}} = 0.4$, where $\omega_{\mathrm{ar}} = 1 \times 10^{-4}$ rad/s. For this desired attitude trajectory, $\omega_{\mathrm{r_d}}$ and $\dot{\omega}_{\mathrm{r_d}}$ can be computed using (25). The actual follower spacecraft attitude and angular velocity are initialized to $\varepsilon_{\mathrm{bf}}(0) = \begin{bmatrix} 0.6 & -0.3 & 0.4 \end{bmatrix}$, $\zeta_{\mathrm{bf}}(0) = 0.6245$, and $\omega_{\mathrm{bf}}(0) = \begin{bmatrix} 0.5 & -0.3 & 0.2 \end{bmatrix}^T$ rad/s. These values of $\varepsilon_{\mathrm{bf}}(0)$, $\zeta_{\mathrm{bf}}(0)$ satisfy the unit norm constraint and are used to obtain $C^{\mathrm{i}}_{\mathrm{bf}}(0) = C(\varepsilon_{\mathrm{bf}}(0), \zeta_{\mathrm{bf}}(0))$.

The control gains in (58) are tuned by trial and error to achieve a good tracking response and are given as $k = 60$, $k_0 = 1$, and $k_1 = 1$. For the above problem and design data, the results of numerical simulations are provided in Figure 2. In particular, Figures 2(a) and (b) depict the translational position and velocity tracking errors. Furthermore, Figures 2(c) and (d) depict the relative angular orientation tracking errors (in terms of the error quaternion) and the angular velocity tracking errors. Finally, Figures 2(e) and (f) show the control forces and torques used by the follower spacecraft.

VI. Conclusion

In this paper, we addressed an output feedback tracking control problem for a follower spacecraft with coupled translation and attitude motion when only translational position and attitude orientation measurements are available. A Lyapunov based tracking controller was designed with guaranteed semi-global, asymptotic stability for the position and velocity tracking errors. This control design methodology required only position measurements while estimating velocity error through a high pass filtering scheme. A numerical simulation was presented to illustrate the efficacy of this control design.

References

[1] J. Ahmed, V. T. Coppola, and D. S. Bernstein, "Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification," *JGCD*, 21, 684–91, 1998.

[2] V. A. Chobotov, *Orbital Mechanics*. Washington, DC: AIAA, 1996.

[3] B. T. Costic, D. M. Dawson, M. S. de Queiroz, and V. Kapila, "A quaternion-based adaptive attitude tracking controller without velocity measurements," *JGCD*, 24, 1214–22, 2001.

[4] D. M. Dawson, J. Hu, and T. C. Burg, *Nonlinear Control of Electric Machinery*. New York, NY: Marcel Dekker, 1998.

[5] P. C. Hughes, *Spacecraft Attitude Control*. New York: Wiley, 1986.

[6] H. Pan and V. Kapila, "Adaptive nonlinear control for spacecraft with coupled translational and attitude dynamics," *ASME IMECE*, DSC–24580, 2001.

[7] H. Pan and V. Kapila, "Adaptive nonlinear control for spacecraft formation flying with coupled translational and attitude dynamics," *CDC*, 2057–62, 2001.

[8] D. T. Stansbery and J. R. Cloutier, "Position and attitude control of a spacecraft using the state-dependent Riccati equation technique," *ACC*, 1867–71, 2000.

[9] F. Terui, "Position and attitude control of a satellite by sliding mode control," *ACC*, 217–21, 1998.

[10] H. Wong, V. Kapila, and A. G. Sparks, "Adaptive output feedback tracking control of multiple spacecraft," *IJRNC*, 12, 117-39, 2002.

[11] Q. Yan, G. Yang, V. Kapila, and M. S. de Queiroz, "Nonlinear dynamics, trajectory generation, and adaptive control of multiple spacecraft in periodic relative orbits," *AAS Guid. and Contr. Conf.*, 104, 159–74, 2000.

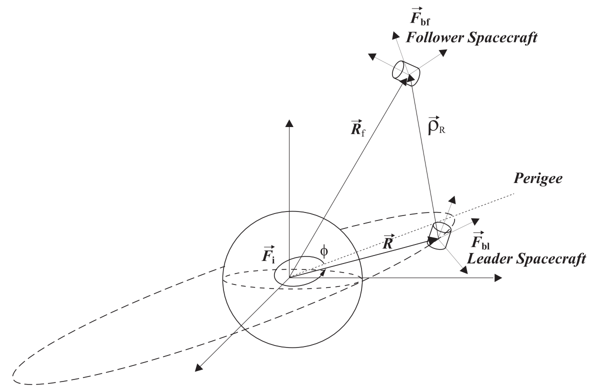
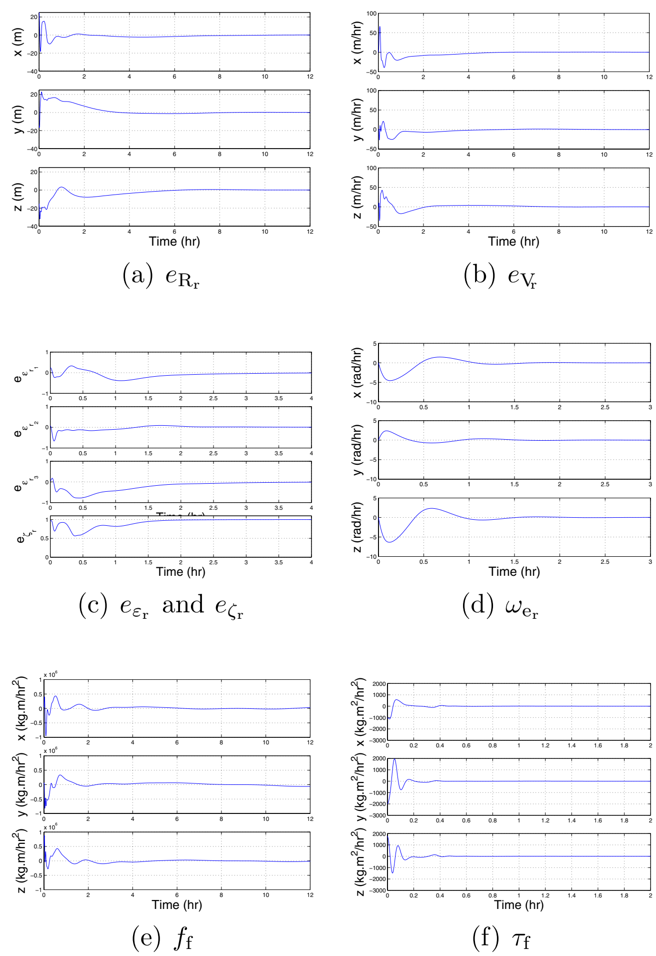


Fig. 1. Schematic representation of the leader-follower spacecraft system



(a) $e_{\mathrm{R_r}}$ (b) $e_{\mathrm{V_r}}$

(c) $e_{\varepsilon_{\mathrm{r}}}$ and $e_{\zeta_{\mathrm{r}}}$ (d) $\omega_{\mathrm{e_r}}$

(e) f_{f} (f) τ_{f}

Fig. 2. Follower spacecraft tracking errors and control inputs