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Dynamic Coverage Optimal Control for Interferometric Imaging Spacecraft Formations (Part II): The Nonlinear Case

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***Abstract*— This is the second of a two paper series on the dynamic coverage problem first introduced in [1]. Dynamic coverage optimal control is a new class of optimal control problems motivated by multi-spacecraft interferometric imaging applications. The dynamics is composed of N second order differential equations representing N fully actuated particles. To be minimized is a cost functional that is a weighted sum of the total fuel expenditure, the relative speeds between the particles and the measure of a given set whose size is a function of the particles' trajectories. In this paper, we extend the analysis to formations evolving on non-linear manifolds. We derive the necessary optimality conditions and give an example of a non-rigid two spacecraft formation evolving on a paraboloidal surface.**

I. Introduction

The use of geometric control methods for spacecraft formation flying has received little attention, whereas extensive investigations have been conducted in the field of robotic path planning (see Section (IV) in [2]). This work is an attempt to use geometric optimal control theory for spacecraft formation motion planning for imaging applications.

Let M be a smooth ($\mathcal{C}^\infty$) *Riemannian manifold* with the *Riemannian metric* denoted by $\langle\cdot,\cdot\rangle_p$ for a point $p \in M$. Thus the length of a tangent vector $v \in T_pM$ is denoted by $\|v\|_p = \langle v, v\rangle_p^{1/2}$, where T_pM is the tangent space of M at p. The *Riemannian connection* on M, denoted ∇, is a mapping that assigns to any two smooth vector fields X and Y in M a new vector field, $\nabla_X Y$. For the properties of ∇, we refer the reader to [3] and [4]. We take ∇ to be the Levi-Civita connection and is, hence, assumed to be symmetric throughout the entirety of this paper. The operator ∇_X, which assigns to every vector field Y the vector field $\nabla_X Y$, is called the *covariant derivative of* Y *with respect to* X. We will denote by $[X, Y]$ the *Lie bracket* of the vector fields X and Y and is defined by the identity: $[X, Y]f = X(Yf) - Y(Xf)$. Given vector fields X, Y and Z on M, define the vector field $R(X, Y)Z$ by the identity

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \quad (1.1)$$

R is trilinear in X, Y and Z and is thus a tensor of type $(1, 3)$, which is called the *curvature tensor* of M.

We consider the general class of problems described by a system of N particles satisfying dynamics of the form:

$$\frac{\mathrm{D}\mathbf{c}_i}{\mathrm{d}t}(t) = \mathbf{v}_i(t), \; \frac{\mathrm{D}\mathbf{v}_i}{\mathrm{d}t}(t) = \mathbf{u}_i(t), \quad (1.2)$$

$i = 1, \ldots, N$, where $\mathbf{c}_i : [0, T] \to M$ is a curve on M, $\mathbf{v}_i(t) \in T_{\mathbf{c}_i(t)}M$ and $\mathbf{u}_i(t) \in TT_{\mathbf{c}_i(t)}M$. $\mathrm{D}\mathbf{Y}/\mathrm{d}t$ denotes the covariant time derivative of the vector field $\mathbf{Y}$.

Let $\mathbf{u}_i(t) \in TT_{\mathbf{c}_i(t)}M$ be given by

$$\mathbf{u}_i(t) = \sum_{j=1}^{m} u_i^j(t) Y_j(\mathbf{c}_i(t)), \quad (1.3)$$

where $m \le n$ and Y_j, $j = 1, \ldots, n$, satisfy $\langle Y_j, Y_k\rangle = \delta_{jk}$. In other words, Y_j is an orthonormal set of vector fields on $T_{\mathbf{c}_i(t)}M$. Mathematically, this assumption limits the class of manifolds we consider (to parallelizable manifolds) for the general problem formulation, but is satisfied for the special case where we deal with systems of particles in space. $m = n$ corresponds to the fully actuated system, whereas $m < n$ corresponds to the under-actuated situation. Here we only consider fully actuated systems.

Assumption I.1. Each particle is fully actuated in all n directions. That is to say $m = n$.

II. Imaging and the Coverage Problem

Equations (1.2) represent the spacecraft dynamics, treating each spacecraft as a point particle. Hence, we ignore attitude dynamics and assume all spacecraft are perfectly aligned and are pointing towards the target. Results presented in this paper can be extended to include rigid body dynamics, which is the main reason for using language and tools from geometric control theory. This, however, is the subject of current research. In interferometric imaging, we are interested in the relative position dynamics as projected onto a plane perpendicular to the line of sight. This plane is called the observation plane, denoted by $O \subset \mathbb{R}^2$. Hence, we are interested in the *projected relative curves*:

$$\tilde{\mathbf{c}}_{ij}(t) \;=\; \frac{1}{\lambda}\mathbb{P}_O\left(\mathbf{c}_j(t) - \mathbf{c}_i(t)\right), \quad (2.1)$$

where λ is the optical wavelength and $\tilde{\mathbf{c}}_{ij} : [0, T] \to \tilde{O}$ are curves on $\tilde{O}$, the frequency (or, u-v) plane, and $\mathbb{P}_O$ is the operator that projects relative trajectories in M onto the observation plane O. Hence, O is the plane on which motion is projected and $\tilde{O}$ is the u-v frequency plane. Let $\ll\cdot,\cdot\gg$ denote the inner product on O.

In multi-aperture interferometry, there are two main imaging goals. The first is simply referred to as frequency domain (or u-v plane) coverage. Here, we only state the coverage goal and refer the reader to [5] for a more detailed discussion. We are interested in having the resolution disc

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as defined by the set $\mathcal{D}_R = \left\{(u,v) : \sqrt{u^2+v^2} \leq 1/\theta_r\right\}$ be completely covered by some ball of radius r_p centered at $\tilde{\mathbf{c}}_{ij}(t)$, for some $t \in [0,T]$, i and j, where θ_r is the angular resolution. An image is said to be successfully completed if a maneuver $\mathcal{M}$ satisfies the following condition.

Definition II.1. (**Successful Imaging Maneuver**) An imaging maneuver $\mathcal{M}$ is said to be successful if, for each $(u,v) \in \mathcal{D}_R$, there exists a time $t \in [0,T]$ and some $i,j = 1,\ldots,N$ such that $(u,v) \in \bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(t))$, where $\bar{B}_x(\mathbf{y})$ is a *closed* ball in $\mathbb{R}^2$ of radius x centered at $\mathbf{y}$. r_p is proportional to the size of the telescope's airy disc.

The second objective is that all frequencies in $\mathcal{D}_R$ must be sampled while maximizing the signal-to-noise ratio (SNR). SNR can be controlled by controlling the relative speeds between the spacecraft in the formation [5]. As the projected relative speed between a spacecraft pair is reduced, the achievable SNR increases. Intuitively, as a spacecraft moves more slowly, it spends more time in the neighborhood of a relative position state in space. This leads to more photon collection at that state, resulting in a stronger signal.

III. Dynamic Coverage Optimal Control

Based on the above discussion, we wish to minimize three quantities: (1) the fuel expended by each spacecraft in the constellation, (2) the projected relative speeds between the spacecraft of the system and (3) the amount of uncovered points in $\mathcal{D}_R$. The constraints we have are the dynamics (1.2) and boundary conditions on the position and velocity vectors of each spacecraft. Motion constraints (as defined in [6]) are not treated in this paper, though they can be easily incorporated in the analysis. We now state the coverage optimal control problem considered in this paper.

Problem III.1. ***Coverage Optimal Control.*** *Minimize*

$$\mathcal{J}(\mathbf{c}_i,\mathbf{u}_i,t) = \int_0^T \frac{1}{2}\Bigg\{\sum_{j=1}^N \Bigg[\langle \mathbf{u}_j,\mathbf{u}_j\rangle + \tau^2 \sum_{k=1}^N \ll \frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}, \frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t} \gg \Bigg]\Bigg\} + \kappa^2 \ \mathrm{meas}\,(\Psi)\,\mathrm{d}t, \tag{3.1}$$

where Ψ is the mapping that returns the set of uncovered frequency points in $\mathcal{D}_R$ up to time t; $\Psi : (t,\tilde{\mathbf{c}}_{ij}; i,j = 1,\ldots,N) \to \{(u,v) \in \mathcal{D}_R : \forall\sigma \in [0,t]$ and $\forall i,j \in 1,\ldots,N, (u,v) \notin \bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(\sigma))\}$ *and the function* $\mathrm{meas}\,(\Lambda)$ *is a measure function of some set Λ. The constraints are the dynamics (1.2), the boundary conditions*

$$\mathbf{c}_i(0) = \mathbf{c}_i^0,\ \mathbf{c}_i(T) = \mathbf{c}_i^T,\ \mathbf{v}_i(0) = \mathbf{v}_i^0,\ \mathbf{c}_i(T) = \mathbf{v}_i^T, \tag{3.2}$$

$i = 1,\ldots,N$, and the relationship in Equation (2.1).

In Equation (3.1), we have used the simple derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ to differentiate the quantity $\tilde{\mathbf{c}}_{jk}$ since $\tilde{\mathbf{c}}_{jk}$ belongs to $\mathbb{R}^2$. Note that when $\kappa = 0$ the problem reduces to that discussed in [6]. In this case, the terminal boundary conditions alone drive the system. When $\kappa \neq 0$ the system is driven to also minimize the set of uncovered points in $\mathcal{D}_R$. Whenever $\mathrm{meas}\,(\Psi)$ becomes zero, the only drive is to meet the terminal conditions in (3.2). For a discussion on the properties of the measure function $\mathrm{meas}(\cdot)$, see [1]. We assume that

Assumption III.1. The function meas is differentiable with respect to both arguments t and $\tilde{\mathbf{c}}$.

IV. Necessary Conditions for Optimality

To obtain necessary optimality conditions we first append the dynamic constraints in Equations (1.2) to the Lagrangian of the cost functional (3.1) by introducing the Lagrange multipliers λ_1^j and λ_2^j, $j = 1,\ldots,N$ into the cost functional $\mathcal{J}$. Collecting terms with the same indexes, Equation (3.1) becomes:

$$\begin{aligned}\mathcal{J}(\mathbf{c}_i,\mathbf{u}_i) = \int_0^T \sum_{j=1}^N \Bigg[\frac{1}{2}\langle \mathbf{u}_j,\mathbf{u}_j\rangle + \lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t} - \mathbf{v}_j\right) \\ + \lambda_2^j\left(\frac{\mathrm{D}\mathbf{v}_j}{\mathrm{d}t} - \mathbf{u}_j\right) + \frac{\tau^2}{2}\sum_{k=1}^N \ll \frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}, \frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t} \gg \Bigg] \\ + \kappa^2 \ \mathrm{meas}\left[\Psi\left(\tilde{\mathbf{c}}_{jk}(t); j,k = 1,\ldots,N\right)\right]\mathrm{d}t\end{aligned} \tag{4.1}$$

We introduce the one-parameter variations for the curves $\mathbf{c}_i$:

$$\begin{aligned}\mathbf{c}_i(t,0) &= \mathbf{c}_i(t),\\ \frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(t,0) &= \mathbf{W}_i(t)\\ \frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(0,0) &= \frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(T,0) = 0,\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(t,0) &= \frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_i(t) \text{ is continuous on } [0,T]\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(0,0) &= \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(T,0) = 0,\end{aligned} \tag{4.2}$$

$i = 1,\ldots,N$. Likewise, we may define variations in $\mathbf{v}_i(t)$, $\mathbf{u}_i(t)$ and $\lambda_k^i(t)$, $k = 1,2$, $i = 1,\ldots,N$, by $\mathbf{v}_i(t,\epsilon)$, $\mathbf{u}_i(t,\epsilon)$ and $\lambda_k^i(t,\epsilon)$, $k = 1,2$, $i = 1,\ldots,N$, as follows:

$$\begin{aligned}\mathbf{u}_i(t,\epsilon) &= \sum_{j=1}^m u_i^j(t,\epsilon)Y_j(\mathbf{c}_i(t,\epsilon)) \in TT_{\mathbf{c}_i(t,\epsilon)}M\\ \mathbf{v}_i(t,\epsilon) &= \sum_{j=1}^n v_i^j(t,\epsilon)Y_j(\mathbf{c}_i(t,\epsilon)) \in T_{\mathbf{c}_i(t,\epsilon)}M\\ \lambda_k^i(t,\epsilon) &= \sum_{j=1}^n \lambda_k^{ij}(t,\epsilon)\omega_j(\mathbf{c}_i(t,\epsilon)) \in T^*_{\mathbf{c}_i(t,\epsilon)}M\end{aligned}$$

where ω_j, $j = 1,\ldots,n$, are co-vector fields such that $\omega_l(Y_j) = \delta_{lj}$. Taking variations in $\mathbf{u}_i$ and $\mathbf{v}_i$, we have:

$$\begin{aligned}\frac{\mathrm{D}\mathbf{u}_i}{\partial\epsilon}(t,\epsilon)\bigg|_{\epsilon=0} &= \delta\mathbf{u}_i(t) + \left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t)) \in TTM\\ \frac{\mathrm{D}\mathbf{v}_i}{\partial\epsilon}(t,\epsilon)\bigg|_{\epsilon=0} &= \delta\mathbf{v}_i(t) + \left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t)) \in TTM\end{aligned}$$

where, for instance,

$$\begin{aligned}\delta\mathbf{u}_i(t) &= \sum_{j=1}^m \frac{\partial u_i^j}{\partial\epsilon}(t,0)Y_j(\mathbf{c}_i(t))\\ \left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t)) &= \sum_{j=1}^m u_i^j(t)\left(\nabla_{\mathbf{W}_i}Y_j\right)(\mathbf{c}_i(t)).\end{aligned}$$

Similar expressions can be obtained for $\frac{\mathrm{D}\mathbf{v}_i}{\partial\epsilon}$ and $\frac{\mathrm{D}\lambda_i^j}{\partial\epsilon}$, $j = 1,2$, $i = 1,\ldots,N$. $\mathbf{B}(\cdot,\cdot)$ is a bilinear form that we introduce in order to be able to separate variations in the components of $\mathbf{u}_i$, $\mathbf{v}_i$ and λ_i^j from variations in the basis vector fields. It is important to separate these terms since the variations $\delta\mathbf{u}_i$, $\delta\mathbf{v}_i$ and $\delta\lambda_j^i$, $i=1,\ldots,N$, $j=1,2$, are independent from each other as well as from $\mathbf{W}_i$. For variations in $\tilde{\mathbf{c}}_{ij}(t)$, let

$$\tilde{\mathbf{c}}_{ij}(t,\epsilon) \quad = \quad \sum_{k=1}^{2}\tilde{c}_{ij}^k(t,\epsilon)\mathbf{Z}_k(\tilde{\mathbf{c}}_{ij}(t,\epsilon)) \in T_{\tilde{\mathbf{c}}_{ij}(t,\epsilon)}\tilde{O},$$

where $\mathbf{Z}_k$, $k=1,2$, is an orthonormal set of vector fields on $T_{\tilde{\mathbf{c}}_{ij}(t,\epsilon)}\tilde{O}$. The set $\mathbf{Z}_k$, $k=1,2$, may be taken to be the standard set of vector fields spanning $\mathbb{R}^2$.

In [1], we make the assumption that the manifold $M\equiv O$. This assumption simplified the analysis quite significantly. In this paper, the spacecraft are free to move on some given non-flat surface M. Such surfaces are usually dictated by some further imaging specifications. For example, having the formation evolve on a virtual paraboloid surface provides improved focusing properties [7].

Thus, for $i,j=1,\ldots,N$ we have

$$\begin{aligned}
\tilde{\mathbf{c}}_{ij}(t,0) &= \tilde{\mathbf{c}}_{ij}(t) = \frac{1}{\lambda}\mathbb{P}_O\left(\mathbf{c}_j(t)-\mathbf{c}_i(t)\right),\\
\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(t,0) &= \frac{1}{\lambda}\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\cdot\left(W_j(t)-W_i(t)\right)\\
\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(0,0) &= \frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(T,0)=0, \qquad (4.3)\\
\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(t,0) &= \frac{1}{\lambda}\frac{\mathrm{D}}{\mathrm{d}t}\left[\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\cdot\left(W_j(t)-W_i(t)\right)\right]\\
\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(0,0) &= \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(T,0)=0
\end{aligned}$$

where $\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}: M\to\mathbb{R}^2$ is viewed as a transformation on M. Hence, the right hand sides of the second and fourth equations represent the *projected* variational vector fields.

Remark IV.1. $\tilde{\mathbf{c}}_{ij}$ belongs to a flat space O. Then

$$\frac{\mathrm{D}}{\partial\epsilon}\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial t}=\frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial\epsilon}+\tilde{R}\left(\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial\epsilon},\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial t}\right)\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial t}=\frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}\tilde{\mathbf{c}}}{\partial\epsilon} \qquad (4.4)$$

since the curvature of O, $\tilde{R}$, is zero everywhere.

Theorem IV.1. *Under Assumptions (I.1) and (III.1), taking first order variations of the expression in Equation (4.1) leads to the following relationship:*

$$\begin{aligned}
&\frac{\partial\mathcal{J}}{\partial\epsilon}(\mathbf{c}_i(t,\epsilon),\mathbf{u}_i(t,\epsilon),t;i=1,\ldots,N)\Big|_{\epsilon=0} =\\
&=\int_0^T\sum_{j=1}^{N}\langle\mathbf{u}_j,\mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\rangle-\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(W_j\right)\\
&-\lambda_1^j\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)-\lambda_2^j\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\right)\\
&+\lambda_2^j\left(R\left(\mathbf{W}_j,\mathbf{v}_j\right)\mathbf{v}_j\right)-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\\
&+\sum_{k=1}^{N}\frac{\tau^2}{\lambda}\ll\frac{\mathrm{d}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2},\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}W_j\gg-\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}W_j\right)\mathrm{d}t\\
&+\int_0^T\sum_{j,k=1}^{N}-\frac{\tau^2}{\lambda}\ll\frac{\mathrm{d}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2},\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}W_k\gg \qquad (4.5)\\
&+\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}W_k\right)\mathrm{d}t\\
&+\int_0^T\sum_{j=1}^{N}-\lambda_2^j(\delta\mathbf{u}_j)+\langle\mathbf{u}_j,\delta\mathbf{u}_j\rangle\,\mathrm{d}t\\
&+\int_0^T\sum_{j=1}^{N}-\lambda_1^j\left(\delta\mathbf{v}_j\right)-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\delta\mathbf{v}_j\right)\mathrm{d}t.
\end{aligned}$$

Proof In Equation (4.1), we replace $\tilde{\mathbf{c}}_{jk}(t)$, $\mathbf{u}_j(t)$ and $\mathbf{v}_j(t)$ with the perturbed variables $\tilde{\mathbf{c}}_{jk}(t,\epsilon)$, $\mathbf{u}_j(t,\epsilon)$ and $\mathbf{v}_j(t,\epsilon)$, respectively. To prove the theorem, we compute $\partial\mathcal{J}/\partial\epsilon$ on a term by term basis as follows. First, we have:

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\int_0^T\frac{1}{2}\langle\mathbf{u}_j(t,\epsilon),\mathbf{u}_j(t,\epsilon)\rangle\,\mathrm{d}t\Big|_{\epsilon=0}\\
&=\int_0^T\langle\mathbf{u}_j,\delta\mathbf{u}_j+\mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\rangle\,\mathrm{d}t, \qquad (4.6)
\end{aligned}$$

where a summation over j is understood. For the fourth term in Equation (4.1), we use the fourth identity in Equations (4.3) and integrate by parts to obtain

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\int_0^T\frac{\tau^2}{2}\sum_{j,k=1}^{N}\ll\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}\gg\mathrm{d}t\Big|_{\epsilon=0}\\
&=\int_0^T\frac{\tau^2}{\lambda}\sum_{j,k=1}^{N}\ll\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\left(W_k-W_j\right)\right]\gg\mathrm{d}t\\
&=-\int_0^T\frac{\tau^2}{\lambda}\sum_{j,k=1}^{N}\ll\frac{\mathrm{d}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2},\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\left(W_k-W_j\right)\gg\mathrm{d}t\\
&+\sum_{j,k=1}^{N}\frac{\tau^2}{\lambda}\ll\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\left(W_k-W_j\right)\gg\Big|_0^T,
\end{aligned}$$

where use has been made of Equation (4.4). The second term vanishes due to the fixed boundary conditions (4.2). Thus, for the fourth term in (4.1) we have

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\int_0^T\frac{\tau^2}{2}\sum_{j,k=1}^{N}\ll\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{d}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}\gg\mathrm{d}t\Big|_{\epsilon=0} \qquad (4.7)\\
&=-\int_0^T\frac{\tau^2}{\lambda}\sum_{j,k=1}^{N}\ll\frac{\mathrm{d}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2},\frac{\partial\mathbb{P}_O}{\partial\mathbf{c}}\left(W_k-W_j\right)\gg\mathrm{d}t.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\int_0^T\lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t}-\mathbf{v}_j\right)\mathrm{d}t\\
&=\int_0^T\lambda_1^j\left(\frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_j-\delta\mathbf{v}_j-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t.
\end{aligned}$$

For the first term in parenthesis, we integrate by parts:

$$\int_0^T\lambda_1^j\left(\frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_j\right)\mathrm{d}t=\lambda_1^j\left(\mathbf{W}_j\right)\Big|_0^T-\int_0^T\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{W}_j\right)\mathrm{d}t.$$

The first term on the right hand side vanishes by virtue of the boundary conditions (4.2) imposed on $\mathbf{W}_j$. We then obtain

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\int_0^T\sum_{j=1}^{N}\lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t}-\mathbf{v}_j\right)\mathrm{d}t \qquad (4.8)\\
&=\int_0^T\sum_{j=1}^{N}-\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{W}_j\right)-\lambda_1^j\left(\delta\mathbf{v}_j+\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t.
\end{aligned}$$

For the third term in Equation (4.1), recall the identity [8]:

$$\frac{\mathrm{D}}{\partial\epsilon}\frac{\mathrm{D}}{\partial t}\mathbf{Y}-\frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial\epsilon}\mathbf{Y}=R\left(\frac{\mathrm{D}\mathbf{c}}{\partial\epsilon},\frac{\mathrm{D}\mathbf{c}}{\partial t}\right)\mathbf{Y},$$

where $\mathbf{Y}$ is a vector field along a trajectory $\mathbf{c}(t)$. Then

$$\frac{\partial}{\partial\epsilon}\int_0^T\sum_{j=1}^N\lambda_2^j\left(\frac{\mathrm{D}\mathbf{v}_j}{\mathrm{d}t}-\mathbf{u}_j\right)\mathrm{d}t$$
$$=\int_0^T\sum_{j=1}^N\lambda_2^j\Bigg(R(\mathbf{W}_j,\mathbf{v}_j)\mathbf{v}_j+\frac{\mathrm{D}^2\mathbf{v}_j}{\partial t\partial\epsilon}-\delta\mathbf{u}_j$$
$$-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\Bigg)\mathrm{d}t$$
$$=\int_0^T\sum_{j=1}^N\lambda_2^j\Big(R(\mathbf{W}_j,\mathbf{v}_j)\mathbf{v}_j-\delta\mathbf{u}_j-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\Big)$$
$$-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\delta\mathbf{v}_j+\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t \tag{4.9}$$

where integration by parts has been used to arrive at the last equation. Finally, under Assumption III.1, for the last term we have

$$\frac{\partial}{\partial\epsilon}\int_0^T\kappa^2\mathrm{meas}\left[\Psi\right]\mathrm{d}t=\int_0^T\sum_{j,k=1}^N\kappa^2\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\frac{\partial\tilde{\mathbf{c}}_{jk}}{\partial\epsilon}\mathrm{d}t$$
$$=\int_0^T\sum_{j,k=1}^N\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left[\frac{\mathrm{D}\mathbb{P}_O}{\mathrm{d}\mathbf{c}}\left(W_k-W_j\right)\right]\mathrm{d}t, \tag{4.10}$$

where it is understood that the meas function is applied to the set $\Psi\left(\tilde{\mathbf{c}}_{jk}\right)$ for all $j,k=1,\ldots,N$. Finally, from equations (4.6-4.10), by separating terms involving the coefficients $\mathbf{W}_j$, $\mathbf{W}_k$, $\delta\mathbf{v}_j$ and $\delta\mathbf{u}_j$, we obtain the expression (4.5) and, hance, proving the theorem. ∎

Remark IV.2. Note that $\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}$ constitute the components of the differential form $\mathrm{d}\left(\mathrm{meas}\right)\in T^*O$, the cotangent space on O. Hence, the notation $\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\mathbf{X}\right)$ denotes this form operating on $\mathbf{X}\in TO$.

From Theorem (IV.1) one can extract the necessary optimality conditions as the following theorem states.

Theorem IV.2. *Under Assumptions (I.1) and (III.1), a set of optimal trajectories $\tilde{\mathbf{c}}_i$, $i=1,\ldots,N$, that minimize $\mathcal{J}$ while satisfying the dynamic constraints (1.2) and the boundary conditions (3.2) satisfy the following necessary conditions for an arbitrary vector field $\mathbf{X}$ and for $j=1,\ldots,N$:*

$$\begin{aligned}
\frac{\mathrm{D}\mathbf{c}_i}{\mathrm{d}t} &= \mathbf{v}_i\\
\frac{\mathrm{D}\mathbf{v}_i}{\mathrm{d}t} &= \left(\lambda_2^j\right)^{\#},\\
\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{X}\right) &= \left(R\left(\mathbf{u}_j,\mathbf{v}_j\right)\mathbf{v}_j\right)^{\flat}\left(\mathbf{X}\right)\\
\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\mathbf{X}\right) &= -\lambda_1^j\left(\mathbf{X}\right)\\
\mathbf{u}_j &= \left(\lambda_2^j\right)^{\#}\\
0 &= \sum_{k=1}^N\tau^2\left(\frac{\mathrm{D}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2}\right)^{\flat}\left(\mathbf{X}\right)-\kappa^2\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\mathbf{X}\right)
\end{aligned}$$

where $\mathbf{Y}^{\flat}\left(\mathbf{X}\right)=\langle\mathbf{Y},\mathbf{X}\rangle$ and $\sharp$ is the inverse of $\flat$.

Proof The first equation follows immediately from Equation (1.2). For an optimal solution, the first order necessary condition is that

$$\frac{\partial\mathcal{J}}{\partial\epsilon}\left(\mathbf{c}_i(t,\epsilon),\mathbf{u}_i(t,\epsilon),t;i=1,\ldots,N\right)\Big|_{\epsilon=0}=0. \tag{4.11}$$

The rest of the proof relies on this condition and the fact that $\mathbf{W}_j$, $\mathbf{W}_k$, $\delta\mathbf{u}_j$ and $\delta\mathbf{v}_j$ are independent for all $j,k=1,\ldots,N$. The fourth equation follows immediately from the last integral in Equation (4.5) and the independence of $\delta\mathbf{v}_j$, $j=1,\ldots,N$. The fifth equation follows immediately from condition (4.11), the third integral in Equation (4.5) and the independence of $\delta\mathbf{u}_j$, $j=1,\ldots,N$. The last (algebraic) equation is obtained by studying the second integral in Equation (4.5). Since $\mathbf{W}_k$, $k=1,\ldots,N$, are independent, we then have

$$\sum_{j=1}^N\tau^2\left(\frac{\mathrm{d}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2}\right)^{\flat}\left(\mathbf{X}\right)-\kappa^2\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\mathbf{X}\right)=0,$$

$\forall k=1,\ldots,N$. Since $\mathbf{u}_j=\left(\lambda_2^j\right)^{\#}$ and by interchanging indices ($j\to k$ and $k\to j$), we obtain the last (algebraic) condition. Hence, the last term under the first integral in Equation (4.5) is zero. the third equation in the theorem is obtained from this, the fourth Equation in the theorem, the first integral in Equation (4.5) and the independence of $\mathbf{W}_j$, $j=1,\ldots,N$. The second equation follows from Equation (1.2) and the fifth condition in the theorem. ∎

V. Example: Dual-Spacecraft Interferometry

In this section we demonstrate the above ideas for a two spacecraft formation. As we did in [1], we first derive a single degree of freedom version of the necessary conditions of Theorem (IV.2), which only apply to fully actuated multi-spacecraft systems. Since in the present example one spacecraft is fixed in space, the symmetries exhibited in Theorem (IV.2) are broken (specifically, the fifth algebraic condition in the theorem). Hence, the result presented in this section is not a simple special case of Theorem (IV.2).

The curve $\mathbf{c}(t)\in M$ corresponds to the trajectory of the collector spacecraft on the manifold and $\mathbf{v}(t)\in TM$ corresponds to the relative velocity vector field between the parent and collector spacecraft. The projected relative position is given by $\tilde{\mathbf{c}}=(1/\lambda)\mathbb{P}_O\left(\mathbf{c}\right)$ while the projected relative velocity is given by $\tilde{\mathbf{v}}=(1/\lambda)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}_O\left(\mathbf{c}\right)=\mathbb{P}_O^v(\mathbf{c},\mathbf{v})$, where $\mathbb{P}_O^v:M\times TM\to\mathbb{E}^2$ is a continuously differentiable function on the tangent bundle to M and where $\mathbb{E}^2$ is the tangent space to $\mathbb{R}^2$. Hence, $\tilde{\mathbf{v}}\in\mathbb{E}^2$.

The cost functional to be minimized is given by:

$$\mathcal{J}\quad=\quad\int_0^T\frac{1}{2}\left\langle\mathbf{u},\mathbf{u}\right\rangle+\frac{\tau^2}{2}\ll\tilde{\mathbf{v}},\tilde{\mathbf{v}}\gg+\kappa^2\mathrm{meas}\left(\Psi(\tilde{\mathbf{c}})\right)\mathrm{d}t.$$

One now follows a procedure similar to that found in Section (IV) by appending the Lagrangian with $\lambda_1\left(\dot{\mathbf{c}}-\mathbf{v}\right)+\lambda_2\left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{d}t}-\mathbf{u}\right)$. In order to compute the necessary conditions we observe that

$$\frac{\partial}{\partial\epsilon}\int_0^T\frac{\tau^2}{2}\ll\tilde{\mathbf{v}},\tilde{\mathbf{v}}\gg\mathrm{d}t$$
$$=\int_0^T\tau^2\ll\tilde{\mathbf{v}},\frac{\partial}{\partial\epsilon}\mathbb{P}_O^v(\mathbf{c},\mathbf{v})\gg\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^T \tau^2 \ll \tilde{\mathbf{v}}, \frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}(\mathbf{W}) + \frac{\partial \mathbb{P}_O^v}{\partial \mathbf{v}}\left(\frac{\partial \mathbf{v}}{\partial \epsilon}\right) \gg \mathrm{d}t \\
&= \int_0^T \tau^2 \Bigg[\left\langle \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}\right)^T \tilde{\mathbf{v}}, \mathbf{W} \right\rangle \\
&+ \left\langle \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{v}}\right)^T \tilde{\mathbf{v}}, \delta\mathbf{v} + \mathbf{B}\left(\mathbf{W}, \mathbf{v}\right) \right\rangle \Bigg] \mathrm{d}t, \qquad (5.1)
\end{aligned}
$$

where $\mathbf{W} \in TM$ is the variation vector field corresponding to the curve $\mathbf{c}$ and $\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}$ and $\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}$ are viewed as the components of the differential form: $\mathrm{d}\mathbb{P}_O^v \in T^*TM$. The T in the superscript corresponds to the adjoint (transpose) operation. In obtaining the above expression we have used integration by parts and the fact that $\mathbf{W}(0) = \mathbf{W}(T) = 0$. The inner product on the last line corresponds to the metric on M. The remainder of the derivation is similar to that obtained in Section (IV). We obtain the following:

$$
\begin{aligned}
\frac{\partial \mathcal{J}}{\partial \epsilon}\bigg|_{\epsilon=0} &= \int_0^T \langle \mathbf{u}, \delta\mathbf{u} \rangle - \lambda_2\left(\delta\mathbf{u}\right) \mathrm{d}t \qquad (5.2) \\
&+ \int_0^T -\lambda_1(\delta\mathbf{v}) - \frac{\mathrm{D}\lambda_2}{\mathrm{d}t}\left(\delta\mathbf{v}\right) + \tau^2 \left\langle \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{v}}\right)^T \tilde{\mathbf{v}}, \delta\mathbf{v} \right\rangle \mathrm{d}t \\
&+ \int_0^T \langle \mathbf{u}, \mathbf{B}\left(\mathbf{W}, \mathbf{u}\right) \rangle + \tau^2 \left\langle \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}\right)^T \tilde{\mathbf{v}}, \mathbf{W} \right\rangle \\
&+ \kappa^2 \frac{\partial \mathrm{meas}}{\partial \mathbf{c}}\left(\mathbf{W}\right) - \frac{\mathrm{D}\lambda_1}{\mathrm{d}t}\left(\mathbf{W}\right) - \lambda_1\left(\mathbf{B}\left(\mathbf{W}, \mathbf{v}\right)\right) \\
&- \lambda_2\left(\mathbf{B}\left(\mathbf{W}, \mathbf{u}\right)\right) - \frac{\mathrm{D}\lambda_2}{\mathrm{d}t}\left(\mathbf{B}\left(\mathbf{W}, \mathbf{v}\right)\right) \\
&+ \lambda_2\left(R(\mathbf{W}, \mathbf{v})\mathbf{v}\right) + \tau^2 \left\langle \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{v}}\right)^T \tilde{\mathbf{v}}, \mathbf{B}\left(\mathbf{W}, \mathbf{v}\right) \right\rangle \mathrm{d}t.
\end{aligned}
$$

Therefore, the necessary conditions are:

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} \\
\frac{\mathrm{D}\mathbf{v}}{\mathrm{d}t} &= \lambda_2^\sharp \\
\left(\frac{\mathrm{D}\lambda_1}{\mathrm{d}t}\right)^\sharp &= R(\mathbf{u}, \mathbf{v})\mathbf{v} - \tau^2 \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{c}}\right)^T \tilde{\mathbf{v}} + \kappa^2 \frac{\partial \mathrm{meas}}{\partial \mathbf{c}} \\
\frac{\mathrm{D}\lambda_2}{\mathrm{d}t} &= -\lambda_1 + \tau^2 \left(\frac{\partial \mathbb{P}_O^v}{\partial \mathbf{v}}\right)^T \tilde{\mathbf{v}}^\flat \\
\mathbf{u} &= \lambda_2^\sharp.
\end{aligned}
$$

These necessary conditions for the problem when $\kappa = 0$ are obtained in local coordinates (the arc length q_1) in [9].

We now look at a class of two-spacecraft formations, where one spacecraft, the "parent", is fixed at the origin and the second, the "collector", is restricted to move along a spiral embedded in a two-dimensional paraboloidal surface. Hence, this system is a one degree of freedom system.

We model each spacecraft as a point mass, each with unit mass. The choice of a paraboloid surface is made because of its improved focusing properties. This type of formation belongs to a class of formations known by Space Technology 3 (ST-3) as one of NASA's Origin's missions. For more on this class of formations, we refer the reader to [7]. Moreover, the collector spacecraft follows a spiral trajectory along the paraboloid. The spiral is designed to ensure that the resulting maneuver is successful in the sense of Definition (II.1). Hence, the spiral embedded on the paraboloid surface will be considered as our one-dimensional manifold M with $n = 1$.

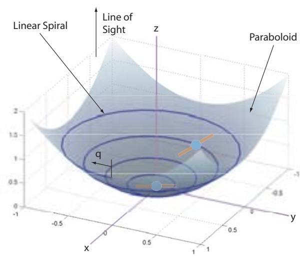

Fig. 1. The basic interferometer.

Refer to figure (1). Let $\mathbf{q} = q$ be the single coordinate we choose to work. We choose q to be the arc length traversed along the spiral. Therefore, $\partial_q = \frac{\partial}{\partial q}$ is the basis vector for T_qM. The velocity vector field is then given by $\mathbf{v} = v_1\partial_q$. The control vector $\mathbf{u} = u_1\partial_q$ is restricted to the tangent space to TM. That is, $\mathbf{u} \in TTM$. In rectangular coordinates, the paraboloid is given by

$$
z = (1/2)\left((\rho^2/\beta^2) - \beta^2\right), \qquad (5.3)
$$

where $\rho = \sqrt{x^2 + y^2}$ and β is a parameter that controls the depth of the paraboloid. Note that vertex of the paraboloid is located at the point $(0, 0, -\beta^2/2)$ in $\mathbb{R}^3$.

In this section we give a brief account of the spiral maneuver. We refer the reader to [10] for further details and background information. In the x-y plane, the projected position may be given in terms of polar coordinates (ρ, θ). One way to ensure full coverage of the resolution disc $\mathcal{D}_R$ is to initialize the second spacecraft such that at $t = 0$ we have $(\rho = \frac{\lambda(m+1)}{2\theta_p}, \theta = 0)$, make it follow a linear spiral as a function of θ, and to impose the terminal condition that at $t = T$ we have $\rho = 0, \theta = \frac{(m+1)\pi}{2})$, where T is the terminal maneuver time. m is an integer that is equal to the number of pixels in the reconstructed image and θ_p is a parameter such that $\theta_p = m\theta_r$. This motion implies that the two coverage balls $\bar{B}_{r_p}(\tilde{\mathbf{p}}_{12})$ and $\bar{B}_{r_p}(\tilde{\mathbf{p}}_{21})$ are initialized such that they lie outside the resolution disc $\mathcal{D}_R$ and move spirally inwards till they overlap the central (fixed) ball $\bar{B}_{r_p}\left(\tilde{\mathbf{p}}_{00}\right) = \bar{B}_{r_p}\left(0, 0\right)$. The spiral path ensures visiting all relative position states that correspond to full coverage of the resolution disc. Thus ρ and θ are constrained to satisfy

$$
\rho(\theta) = k_1\left(k_2\pi - \theta\right), \ \theta \in \left[0, (m+1)\pi/2\right], \qquad (5.4)
$$

where $k_1 = \frac{\lambda}{\pi\theta_p}$ and $k_2 = \frac{(m+1)}{2}$.

If we let $\mathbf{r}$ be the position vector of the collector spacecraft, then the constraints (5.3) and (5.4) imply that

$$
\begin{aligned}
\mathbf{r} = (x, y, z) \quad &= \quad \big((k_1(k_2\pi - \theta)\cos\theta, k_1(k_2\pi - \theta)\sin\theta, \\
&\quad (1/2)\left((k_1^2/\beta^2)(k_2\pi - \theta)^2 - \beta^2\right)\big). \qquad (5.5)
\end{aligned}
$$

The arc length q is obtained as a function of θ using the definition of the arc length of curve in space:

$$
q(\theta) = h(\theta) = \int_0^\theta \|\partial\mathbf{r}(\theta')/\partial\theta'\| \, \mathrm{d}\theta'. \qquad (5.6)
$$

The functional form of h can be obtained explicitly, which we omit for the sake of brevity. By the geometry of the problem described in previous paragraphs, it is easy to see that the function h is both one-to-one and onto. Hence, given a value for θ, one can uniquely solve for q using

$$\theta = h^{-1}(q). \tag{5.7}$$

The metric on the tangent space is determined by computing the line element $\mathrm{d}s^2$ in terms of the coordinate q. Since q is the distance traveled on M, $\mathrm{d}s^2 = \mathrm{d}q^2$ and the single element of the metric g is simply given by

$$g(q) = 1. \tag{5.8}$$

With this, one is now in a position to compute the connection coefficients. The single connection coefficient (or the Christoffel symbol) is given by

$$\Gamma = (1/2g)(\partial g/\partial q) = 0. \tag{5.9}$$

Since M is a one-dimensional manifold, the curvature tensor R is identically zero everywhere on M.

To compute $\tilde{\mathbf{v}}$ in terms of q and $v = \dot{q}_1$, we first need to obtain an expression for $\dot{\theta}$ in terms of q and $\dot{q}_1$. Differentiating Equation (5.6), we obtain

$$v = \dot{q}_1 = (\partial h/\partial \theta)\dot{\theta} = r(q)\dot{\theta}, \tag{5.10}$$

where $r(q) = \frac{\partial h}{\partial \theta}\big|_{\theta = h^{-1}(q)}$ using the relation (5.7). Using this and Equations (5.5) and (5.7), we have:

$$\tilde{\mathbf{v}} = \mathbb{P}_O^v = \dot{x}\partial_x + \dot{y}\partial_y = P_x(q,v)\partial_x + P_y(q,v)\partial_y, \tag{5.11}$$

where

$$\begin{aligned} P_x(q,v) &= (v/r(q))\big[-k_1 \cos\left(h^{-1}(q)\right) \\ &\quad -k_1\left(k_2\pi - h^{-1}(q)\right)\sin\left(h^{-1}(q)\right)\big] \\ P_y(q,v) &= (v/r(q))\big[-k_1 \sin\left(h^{-1}(q)\right) \\ &\quad +k_1\left(k_2\pi - h^{-1}(q)\right)\cos\left(h^{-1}(q)\right)\big]. \end{aligned}$$

Finally, after quite an involved computation, one can show that for such "scripted" (that is, pre-determined) maneuvers, the meas function that ensues from the spiral motion is given in terms of $\theta = h^{-1}(q)$ by:

$$\begin{aligned} \mathrm{meas}(\theta) &= \left(\pi/\left(4\theta_p^2\right)\right) + \frac{1}{2}\left[\frac{m\theta^2}{2\pi\theta_p^2} - \frac{\theta^3}{3\pi^2\theta_p^2}\right], \\ &\quad \theta \in [0,\pi] \\ \mathrm{meas}(\theta) &= \left(\pi(3m+1)/\left(12\theta_p^2\right)\right) \\ &\quad + \frac{1}{\theta_p^2}\left[\frac{(m+1)(\theta-\pi)}{2} - \frac{\theta^2-\pi^2}{2\pi}\right], \\ &\quad \theta \in [\pi, (\pi(m-1)/2)] \\ \mathrm{meas}(\theta) &= \frac{\pi\left(3m^2-7\right)}{24\theta_p^2} \\ &\quad + \frac{1}{2\theta_p^2}\left[\frac{(m+3)(m+1)}{4}\left(\theta - \frac{(m-1)\pi}{2}\right)\right. \\ &\quad - \frac{(m+2)}{2\pi}\left(\theta^2 - \frac{(m-1)^2\pi^2}{4}\right) \\ &\quad \left. + \frac{1}{3\pi^2}\left(\theta^3 - \frac{(m-1)^3\pi^3}{8}\right)\right], \\ &\quad \theta \in [(\pi(m-1)/2), (\pi(m+1)/2)]. \end{aligned}$$

The function meas may now be obtained in terms of q by the relation $q = h(\theta)$.

A final remark is in order. The basic assumption in this work is that interferometry is performed continuously in time as opposed to a "stop and stare" strategy. In a stop and stare strategy, we identify discrete points on the spiral and have the moving spacecraft visit these points, stop and then stare to make a measurement. In the present work, the scheme we envision is one where the way points are assumed to be sufficiently close to each other and lie on the spiral. We also assume that the optical system generates interference patterns on the fly while the formation is dynamically evolving in space.

VI. Conclusion

In this paper we used a differential geometric approach to study extensions to the dynamic coverage optimal control problem (see [1]), where now we study non-coplaner formations. The optimal control problem is defined to achieve maneuvers optimal in both imaging and fuel senses. Optimality conditions were derived and a two spacecraft example was given to illustrate our results for the one degree of freedom case. The geometric approach is pursued since the notation is compact, the result is global and no need for specifying any one set of local coordinates is required. The result of Theorem (IV.2) may also be easily extended to a rigid body treatment (as opposed to the point particle model) of each spacecraft to include attitude control. Future work will aim at the investigation of computer based simulations in an attempt to obtain solutions to the N spacecraft problem and to study the behavior of an optimal solution that satisfies the necessary conditions given in this paper.

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