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Optimized Discrete-Time State Dependent Riccati Equation Regulator

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Abstract – **The state dependent Riccati equation was originally developed for the continuous time systems. In the paper the optimality of a discrete time version of the State Dependent Riccati Equation is considered. The derivation of the optimal control strategy is based on the Hamiltonian optimal solution for the non-linear optimal control problem. The new form of the Discrete State Dependent Riccati Equation with a correction tensor is derived. The prediction of the future trajectory is used in the derivation.**

1. INTRODUCTION

In recent years a number of papers based on the State-Dependent Riccati Equation (SDRE) method emerged. It was reported [1] that the SDRE method has many practical advantages over other non-linear design methods. The method involves an approximation and gives, a suboptimal locally stabilizing solution of the infinite horizon minimization problem of a quadratic (in control) cost-function, subject to non-linear differential constraints [2]. For scalar systems the solution of the SDRE yields an optimal solution [3]. For systems of higher order the optimality of the solution is determined by the state-dependent parameterization of the system matrix [4]. The proper choice of that parameterization may be difficult, if not impossible, since that may require the solution of Hamilton-Jacobi-Bellman equation.

The stability issue for the SDRE method has recently attracted interest. The local stability at the origin of the closed loop system results from the stabilizing properties of the solution of Algebraic Riccati Equation. Unfortunately, so far, one of the most efficient methods of assessing the stability of the SDRE controller is by simulation. Recent work in the field of stability analysis [6,7,8] for the SDRE method gives conditions that are difficult to check, or impose requirements that are difficult to fulfill.

The previous derivation of the discrete form of the SDRE controller with the focus on analysis of the sampling period time is given in [9]. The receding horizon control philosophy, used in connection with the SDRE was previously presented in [10] for a continuous time systems.

In this paper, the discrete version of the SDRE method is analyzed. The attention is focused on the optimality of the solution. The discrete SDRE is not guaranteed to give an optimal solution of the minimization of the performance index. The method of recovering optimality will be described. This may also be seen as an alternative to the numerical optimization for the finite horizon problems. Similar approach for predictive control was described in [11]. This paper will concentrate on the SDRE formulation.

The rest of the paper is organized as follows: Section 2 introduces discrete time version of the Discrete-time SDRE method. Section 3 gives a short description of the predictive DSDRE extension, in section 4 the derivation of the optimal discrete time controller is presented and the Optimized DSDRE controller is introduced. Section 5 contains a simulation example.

2. DISCRETE TIME SDRE METHOD

The SDRE method was originally developed for continuous time systems [2], [3]. The solution is a direct result of adopting the linear continuous time optimal control methods that are based on the algebraic Riccati equation [12]. Thus, the theory that is well established for linear systems may be used in the context of non-linear systems.

The attention is focused here on linear and non-linear discrete time systems. For the linear discrete time systems the control minimizing an infinite horizon quadratic performance index is given by the solution of the Discrete Algebraic Riccati Equation (DARE). In a similar to the original SDRE manner the methodology employing the solution of the DARE may be used for non-linear discrete time systems [5]. The non-linear discrete time system is given by the following difference equation:

$$x_{n+1} = f(x_n) + B(x_n)u_n \tag{1}$$

The model is re-arranged and the state-dependent form of the system is obtained:

$$x_{n+1} = A(x_n)x_n + B(x_n)u_n \tag{2}$$

An assumption on point-wise controllability must be made here, i.e. $\forall_x \left(A(x_n)\,B(x_n)\right)$ is controllable. The cost function to be minimized is given by the following expression:

$$J_n = \frac{1}{2}\sum_{i=n}^{\infty}\left\{x_i^T Q\, x_i + u_i^T R u_i\right\} \tag{3}$$

The sub-optimal solution of the minimization problem of (3) is obtained by solving the Discrete-time State Dependent Riccati Equation. If the analytical solution exists the equation is solved once and the non-linear feedback control law may be employed. Otherwise, the equation is solved at each sampling instant numerically. It may also be possible to employ some form of the gain scheduling. The discrete-time state dependent Riccati equation is given by the following equation:

$$\begin{aligned} P(x_n) &= A(x_n)^T\Big[P(x_n) - P(x_n)B(x_n)^T \\ &\times\left(R + B(x_n)^T P(x_n)B(x_n)\right)^{-1} B(x_n)P(x_n)\Big]A(x_n) + Q \end{aligned} \tag{4}$$

The non-linear control action is computed from the following expression:

$$\begin{aligned} u_n &= -K_n x_n \\ K_n &= \left(B(x_n)^T P_n B(x_n) + R\right)^{-1} B(x_n)^T P_n A_n \end{aligned} \tag{5}$$

The solution of DSDRE for the system (1) subject to (3) results in a locally stabilizing control. The optimality of the solution depends on the form of the state-dependent parameterisation (2) and in general the solution is sub-optimal.

3. DISCRETE TIME SDRE WITH PREDICTED TRAJECTORY

The DSDRE method employs the Discrete Algebraic Riccati Equation solution that is based on the matrices of the state-space model (2) frozen at the current state. This implies that the system will remain fixed at the current operating point in the future. It represents a severe approximation, since this is true only for the system in steady state at the origin.

If the system is controllable the state may be driven sufficiently close to the origin in a finite number of steps. It is important to make sure that the DSDRE method is capable of stabilizing the system. The stability issues are analyzed in [6,7,8]. If those methods cannot be applied it is quite common for non-linear systems that the stability is evaluated though simulation.

The non-linear system (1) is time invariant. However, the matrices in the state-dependent linear parameterization (2) are not. The matrices are implicit functions of time through the dependence on state. With the knowledge of the future trajectory the non-linear system may be approximated by a linear time varying system [5]. The future trajectory may be obtained with the state feedback gain obtained in the previous iteration and the system model. The state feedback drives the system (1) to the origin after a finite number of steps. The minimization of the cost function may be split in two parts:

$$\begin{aligned} J_n &= \underbrace{\frac{1}{2}\sum_{i=n}^{n+N-1}\left\{x_i^T Q\, x_i + u_i^T R u_i\right\}}_{J_1} \\ &+ \underbrace{\frac{1}{2}\sum_{i=n+N}^{\infty}\left\{x_i^T Q\, x_i + u_i^T R u_i\right\}}_{J_2} \end{aligned} \tag{6}$$

Assume that within the control horizon N the state of the system is driven to the origin. The solution of the DSDRE at the origin minimizes J_2 part of the cost function (6). The state of the system from the initial to the origin evolves in time therefore the state dependent model matrices also change. The discrete algebraic Riccati equation solution at the origin P_{n+N} is used as a boundary condition for the time-varying optimal control problem for the finite horizon part solution. That is based upon the time-varying approximation of the non-linear system. This requires the following Riccati equation:

$$\begin{aligned} P_i &= A(x_i)^T\Big[P_{i+1} - P_{i+1}B(x_i)^T \\ &\times\left(R + B(x_i)^T P_{i+1}B(x_i)\right)^{-1} B(x_i)P_{i+1}\Big]A(x_i) + Q \end{aligned} \tag{7}$$

The equation is iterated from $i = n + N - 1$ using the solution of the DSDRE $P_{n+N} = P(x_{n+N})$ at the origin (or steady state with $x_{n+N} = 0$) given by equation (4). The iterations of (7) are terminated at $i = n+1$.

The state feedback gains are given by the following expression:

$$\begin{aligned} u_n &= -K_n x_n \\ K_n &= \left(B(x_n)^T P_{n+1} B(x_n) + R\right)^{-1} B(x_n)^T P_{n+1} A_n \end{aligned} \tag{8}$$

The idea behind this control strategy is similar to the dual mode control solution for predictive algorithms [14]. The following algorithm may be employed to refine the DSDRE method and obtain the feedback gain matrix. The receding horizon technique is employed.

Algorithm 1

1. Use the state feedback gains for the finite horizon *N* computed in previous iteration and simulate the closed loop system with the model (1) starting from the current state x_n. This provides prediction of the state trajectory.
2. The solution of the Riccati equation (4) is calculated at the origin. The state-dependent model of the system is time invariant there.
3. Within the finite horizon the state dependent matrices are calculated along the prediction of the state trajectory. This results in the linear time varying model that is an approximation of the non-linear system.
4. Within the finite horizon *N* the equation (7) is iterated and the $P_{n+N}...P_{n+1}$ are computed. Based on that state feedback gains $K_n...K_{n+N-1}$ are obtained and first gain K_n is used for the control.
5. In the next discrete time the algorithm is repeated and remaining gains $K_{n+1}...K_{n+N-1}$ are used. The last gain K_{n+N} required by the algorithm in the next iteration is obtained DSDRE at the origin (from equations (4), (5) with the state prediction x_{n+N}). This forms the receding horizon strategy.

Note that it is assumed that state is driven to the origin within the horizon *N*. The use of the prediction of future trajectory results in better performance of the controller. This is due to more realistic assumptions about future state.

4. OPTIMISED DISCRETE SDRE METHOD

The discrete SDRE method is not guaranteed to provide an optimal control solution. For some systems the state-dependent parameterization giving an optimal solution may not exist at all. The refinement with the predicted trajectory, given by Algorithm 1, brings the improvement. However, the optimality still depends on the state dependent parameterization.

In this section the optimal control for the system (1), with the infinite horizon cost function, is analyzed. Defining the cost function (6) with J_2 as $J_2 = \frac{1}{2} x_N{}^T P_N\, x_N$ the performance index is given by the following expression:

$$J_n = \frac{1}{2}\sum_{i=n}^{n+N-1}\left\{x_i{}^T Q\, x_i + u_i{}^T R\, u_i\right\} + \frac{1}{2} x_{n+N}{}^T P_{n+N}\, x_{n+N} \tag{9}$$

The matrices P_N, Q and R are assumed symmetric and semi-positive and positive definite respectively. The P_N is a final state penalty matrix for the finite horizon optimization. If the system is driven to the origin (or sufficiently close) within the horizon *N* this cost is zero (or close to zero). The value of the terminal penalty matrix may be obtained from the solution of the Discrete Algebraic Riccati Equation for the system linearized around the origin. If the system is driven to the neighborhood of the origin the fixed gain control must be capable of stabilizing the system in this region. The stability region for the system controlled by the linear state feedback controller is determined using Lyapunov functions theory [13], [14].

The control minimizing the performance index (9) is computed. The Hamiltonian for the cost function (9), subject to equality constraints (1), is given by:

$$H_i = \frac{1}{2}\left(x_i{}^T Q\, x_i + u_i{}^T R\, u_i\right) + \lambda_{i+1}^T\left(f\left(x_i\right) + B\left(x_i\right)u_i\right) \tag{10}$$

The optimality conditions for the minimization problem solution are given as follows [12]:

$$\frac{\partial H_i}{\partial u_i} = R\, u_i + B\left(x_i\right)^T \lambda_{i+1} = 0 \tag{11}$$

$$\frac{\partial H_i}{\partial x_i} = Q\, x_i + \left[\frac{\partial f\left(x_i\right)}{\partial x_i} + \frac{\partial B\left(x_i\right)}{\partial x_i} u_i\right]^T \lambda_{i+1} = \lambda_i \tag{12}$$

$$\frac{\partial H_i}{\partial \lambda_{i+1}} = f\left(x_i\right) + B\left(x_i\right)u_i = x_{i+1} \tag{13}$$

The boundary condition for the co-state in the equation (12) is $\lambda_{n+N} = P_{n+N} x_{n+N}$. The initial condition for the state in the equation (13) (the system state) is x_n. The optimization with the initial value for the state equation and final value for the co-state is known as a two point boundary problem.

To find a solution introduce the matrix coefficient $P_i = P\left(x_i, x_{i+1}, ..., x_{n+N}\right)$. Without loss of generality it may be assumed that the following expression for the co-state λ_i holds:

$$\lambda_i = P_i x_i \tag{14}$$

From the system equation (13), stationary condition (11) and the assumption (14) the following may be computed:

$$x_{i+1} = \left(I - B\left(x_i\right) R^{-1} B\left(x_i\right)^T P_{i+1}\right) f\left(x_i\right) \tag{15}$$

From the co-state equation (12), the assumption (14) and the equation (15) the expression is obtained:

$$P_i x_i = \left(\frac{\partial f(x_i)}{\partial x_i} + \frac{\partial B(x_i)}{\partial x_i} u_i \right)^T P_{i+1} \times \left(I - B(x_i) R^{-1} B(x_i)^T P_{i+1} \right) f(x_i) + Q x_i \qquad (16)$$

The equation should hold for all x_i in the state-space. The equation (16) is re-arranged using the matrix inversion Lemma [12]. The state-dependent parameterization of the system (1) given by (2) is employed next. The following equation is obtained:

$$P_i = \left(\frac{\partial f(x_i)}{\partial x_i} + \frac{\partial B(x_i)}{\partial x_i} u_i \right)^T \Big[P_{i+1} - P_{i+1} B(x_i) \times \left(B(x_i)^T P_{i+1} B(x_i) + R \right)^{-1} B(x_i)^T P_{i+1} \Big] A(x_i) + Q \qquad (17)$$

The equation (17) has a similar structure to the difference Riccati equation (7). Only the following term which can be re-written using the state-dependent parameterization is different:

$$\frac{\partial f(x_i)}{\partial x_i} + \frac{\partial B(x_i)}{\partial x_i} u_i = \frac{\partial A(x_i)}{\partial x_i} x_i + A(x_i) + \frac{\partial B(x_i)}{\partial x_i} u_i \qquad (18)$$

The derivatives $\frac{\partial A(x_i)}{\partial x_i}$ and $\frac{\partial B(x_i)}{\partial x_i}$ are tensors (third dimension has to be introduced to accommodate derivatives of each element of A matrix). Note that for the linear system where *A* and *B* are constant or time varying but state-independent equation (16) becomes the ordinary Difference Riccati Equation (7). The same result is obtained if matrices *A* and *B* are frozen.

The optimal control minimizing the cost function (9) may be computed from equations (11), (13), (14), which results in the equation (8).

The value of P_{n+1} is obtained by iterating the equation (17) from $i = n + N - 1$ back in time to $i = n + 1$.

The equations (17), (18) must be used with the predicted future trajectory. For this trajectory tensors $\frac{\partial A(x_i)}{\partial x_i}$ and $\frac{\partial B(x_i)}{\partial x_i}$ as well as $A(x_i), B(x_i)$ are computed.

Algorithm 2

1. Use the state feedback gains for the finite horizon *N* computed in previous iteration and simulate the closed loop system with the model (1) starting from the current state x_n. This provides prediction of the state *and control* trajectory.
2. The solution of the Riccati equation (4) is calculated at the origin. The state-dependent model of the system is time invariant there.
3. Within the finite horizon the state dependent matrices *and tensors* are calculated along the prediction of the state trajectory.
4. Within the finite horizon *N* the equation (17) is iterated and the $P_{n+N} \ldots P_{n+1}$ are computed. Based on that state feedback gains $K_n \ldots K_{n+N-1}$ are obtained and first gain K_n is used for the control.
5. In the next discrete time the algorithm is repeated and remaining gains $K_{n+1} \ldots K_{n+N-1}$ with K_{n+N} are used.

5. EXAMPLE

As an example a discrete-time model of the driven inverted pendulum is employed. The pendulum is shown in Figure 1.

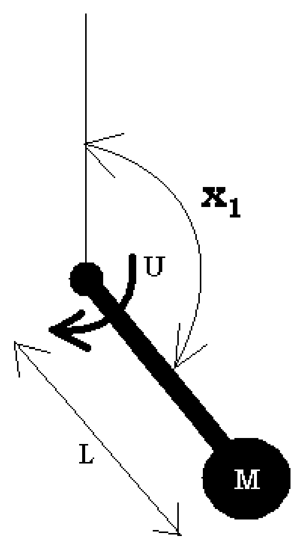

Figure 1: Inverted Pendulum

The control task is to find the optimal control sequence for the pendulum from the certain initial level to the unstable equilibrium point. Assuming that the origin corresponds to the unstable equilibrium the model is given as follows:

$$x_{1,n+1} = x_{1,n} + T_s x_{2,n}$$
$$x_{2,n+1} = \left(1 - \frac{T_s \gamma}{ML}\right) x_{2,n} + \frac{T_s g}{L} \sin(x_{1,n}) + u_n \qquad (19)$$

Where

$T_s = 0.05, M = 0.1,\ L = 0.1,\ g = 10,\ \gamma = 0.05$

The state-dependent parameterization of the system (19) is given as follows:

$$\underline{x_{n+1}} = \begin{bmatrix} 1 & T_s \\ \dfrac{T_s g \sin\left(x_{1,n}\right)}{L x_{1,n}} & \left(1 - \dfrac{T_s \gamma}{ML}\right) \end{bmatrix} \underline{x_n} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_n \quad (20)$$

To avoid the division by zero the $\frac{T_s g \sin\left(x_{1,n}\right)}{L x_{1,n}}$ term in (20) is substituted for $x_{1,n} = 0$ by the limit $\lim_{x_{1,n} \to 0} \frac{T_s g \sin\left(x_{1,n}\right)}{L x_{1,n}} = \frac{T_s g}{L}$. The cost function employed in the example is given by the equation (9). The following weights and control horizon were chosen:

$Q = I,\ R = 1,\ N = 40$

The boundary condition P_N is obtained from the solution of the discrete state dependent Riccati equation at the origin.
The length of the control horizon is chosen such that the state is driven to zero within that time frame.
The following results are obtained. The state trajectories for the DSDRE, Predictive DSDRE (Algorithm 1) and Predictive Optimized DSDRE (Algorithm 2) are shown in Figure 2.

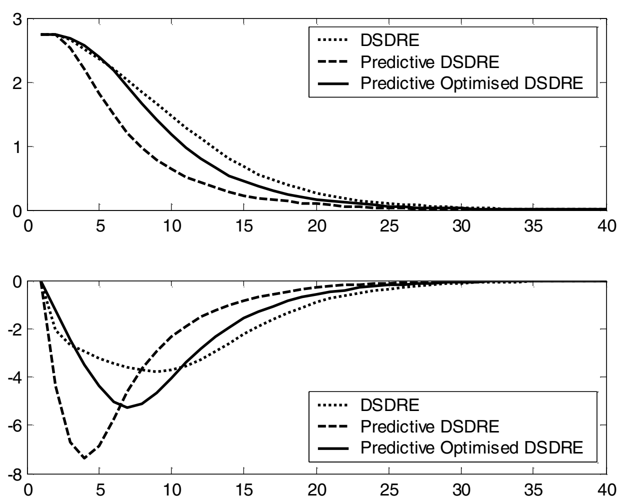

Figure 2: State Trajectory for the DSDRE, P-DSDRE, PO-DSDRE

It may be noticed that speed of response is fastest for the predictive DSDRE algorithm. The optimized predictive DSDRE provides slower response and the DSDRE is the slowest.

This would suggest that the P-SDRE provides the best performance. However, one may measure the performance of the control system by the equation (9). This performance index is used for the derivation of control algorithm. Thus, it is a good indicator of the controller performance.

The slower response of the optimized predictive DSDRE algorithm may be explained by the lower control effort. The control trajectories are shown in Figure 3.

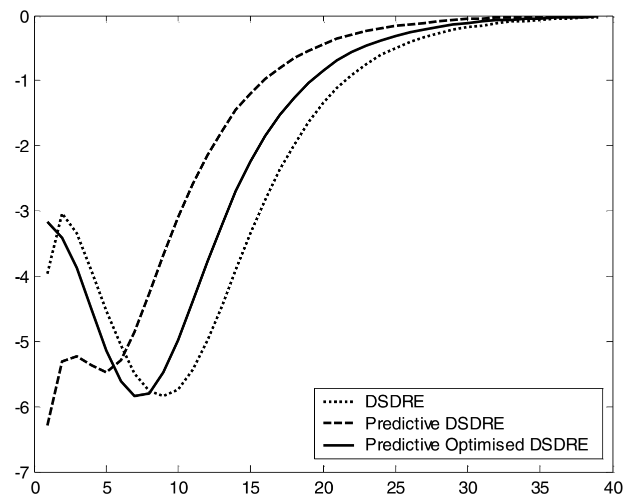

Figure 3: Control Trajectory for the DSDRE, P-DSDRE, PO-DSDRE

The value of the performance index (9) calculated along state and control trajectories for three algorithms is given below.
$J_{\text{DSDRE}} = 557.72$: DSDRE controller,
$J_{\text{P-DSDRE}} = 554.78$: Predictive DSDRE (Algorithm 1),
$J_{\text{PO-DSDRE}} = 541.53$: Predictive Optimized DSDRE (Algorithm 2).

The optimized P-DSDRE algorithm provides the best performance. The P-DSDRE and DSDRE algorithms result in higher costs.
The system trajectories for three controllers plotted in state-space are shown in Figure 4.

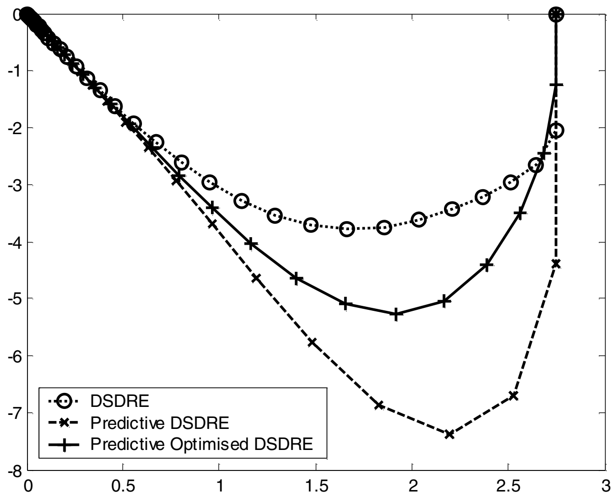

Figure 4: State Trajectory for the DSDRE, P-DSDRE, PO-SDRE method in State Space

6. CONCLUSIONS

The method presented in the paper provides a control law that gives lower values of the cost function. The method is based on the State Dependent Riccati Equation for the discrete-time non-linear systems. Originally the SDRE was developed for the continuous time systems. For these systems, optimality was achievable theoretically, e.g. for scalar case, or for higher order systems - providing the state dependent representation was selected properly. In discrete-time case the DSDRE is not guaranteed to provide optimality even in the simplest first-order case. The method presented in this paper introduces additional term in the Riccati equation that helps to improve the optimality of the method.

It was noticed that if the prediction of the future trajectory was refined iteratively at a given time instant, the method not only decreased the value. Additionally, the control trajectory converged to the optimal sequence minimizing the given cost function. This however was done only for a limited number of examples based on simulation and is subject of ongoing research.

7. ACKNOWLEDGEMENTS

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