Generalized LQR Control and Kalman Filtering with Relations to Computations of Inner-Outer and Spectral Factorizations

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Abstract— **We investigate the generalized LQR control where the dimension of the control input is strictly greater than the dimension of the controlled output, and the weighting matrix on the control signal is singular. The dual problem is the generalized Kalman filtering where the dimension of the input noise process is strictly smaller than the dimension of the output measurement, and the covariance of the observation noise is singular. These two problems are intimately related to inner-outer factorizations for non-square stable transfer matrices with square inners of the smaller size. Such inner-outer factorizations are in turn related to spectral factorizations for power spectral density (PSD) matrices whose normal ranks are not full. We propose iterative algorithms and establish their convergence for inner-outer and spectral factorizations, which in turn solve the generalized LQR control and Kalman filtering.**

1. Introduction

In the standard LQR control, the dimension of the control input is no greater than the dimension of the controlled output, and the weighting matrix on the control signal is nonsingular. For the standard Kalman filtering, the dimension of the input noise process is no smaller than the dimension of the outout measurement, and the covariance of the observation noise is nonsingular. The standard LQR control and Kalman filtering are well studied, and their solutions and properties are well documented [2], [8]. It is interesting to observe that these two optimization problems are related to, and have applications to computations of inner-outer and spectral factorizations [1], [5].

In this paper we study generalized LQR control and Kalman filtering for discrete-time systems in which the aforementioned regular conditions fail. The solutions to these two generalized optimization problems are not difficult to find. Indeed using the same derivations to the regular case, we can obtain similar Riccati equations whose solutions provide the LQR control and Kalman filtering gains. However it is not easy to compute the stabilizing solutions to the algebraic Riccati equations (AREs), associated with the generalized LQR control and Kalman filtering in the stationary case. Such AREs involve pseudo-inverses, and may contain more than one positive semi-definite solutions. In fact for stable systems the two generalized optimization problems can be equivalently converted to inner-outer factorizations for non-square stable transfer matrices whose inners are square and have a smaller size that is in turn related to the spectral factorization for PSD matrices whose normal ranks are not full. Our approach to generalized LQR control and Kalman filtering is through tackling the equivalent inner-outer and spectral factorizations, from which we develop an iterative algorithm for computing the stabilizing solutions to the AREs associated with the two optimization problems. We will prove the convergence of the proposed iterative algorithm.

Spectral factorizations have been widely used in signal processing, control, and communications, due to the need for spectral analysis in signals and for frequency-domain design in systems. There is a large body of literatures devoted to such a topic [1], [4], [7], [11], [14], [15]. The solutions given in [1], [15] are the most general, but both did not consider those PSD matrices whose normal ranks are not full. Such spectral factorizations are less studied, and much harder to compute. Nevertheless its solution helps solve the generalized LQR control, Kalman filtering, and inner-outer factorizations. In addition the blind channel estimation problem emerged in wireless data communications [3], [9], [10] is equivalent to such spectral factorizations. We will follow the state-space approach proposed in [1], and develop convergent iterative algorithms to compute spectral factors for PSD matrices with non-full normal ranks. Due to the space limit, all the proofs are omitted. The complete version of the paper is available upon request.

2. Preliminaries

We will begin with the formulation of the inner-outer and spectral factorizations entailed in this paper. Denote the set of real/complex numbers by $\mathbb{F} = \mathbb{R}/\mathbb{C}$. Let $H(z)$ be a transfer function matrix of size $p \times m$. It is called causal, if its impulse response is causal. Its normal rank is defined as the rank of $H(z)$ for almost all, except a countable set of $z \in \mathbb{C}$. Denote $\bar{a}$ the conjugate of a, and A^* the conjugate and transpose of A. Then the para-hermitian conjugate of $H(z)$ is defined and denoted by $H(z)^\sim = [H(\bar{z}^{-1})]^*$. Assume that the underlying system is finite-dimensional. Then $H(z)$ admits a state-space realization

$$H(z) = D + C(zI - A)^{-1}B =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad (1)$$

by an abuse of notation where $A \in \mathbb{F}^{n\times n}$ and $D \in \mathbb{F}^{p\times m}$. It is clear that B and C have dimensions of $n\times m$ and $p\times n$, respectively. If A is a stability matrix, i.e., all eigenvalues of A are strictly inside the unit circle, then $H(z)$ is stable. If $\forall\ |z| \geq 1$,

$$\text{rank}\left\{\left[\begin{array}{c|c} A - zI & B \\ \hline C & D \end{array}\right]\right\} = n + \min\{p, m\}, \quad (2)$$

then $H(z)$ is strict minimum phase. Notice that the above does not ensure the full rank for D.

A para-hermitian transfer matrix $\Phi(z)$ of size $p \times p$ has the form

$$\Phi(z) = \sum_{k=-\infty}^{\infty} R_k z^{-k}, \quad R_k^* = R_{-k} \in \mathbb{F}^{p\times p}. \quad (3)$$

It follows that $\Phi(z)$ is a hermitian matrix for any z on the unit circle. If in addition $\Phi(z) \geq 0\ \forall\ |z| = 1$, then $\Phi(z)$ qualifies a PSD with $\{R_k\}$ the covariance sequence. Let the normal rank of $\Phi(z)$ be $r < p$. We are interested in spectral factorizations

$$\Phi(z) = W_R(z)^\sim W_R(z) = W_L(z) W_L(z)^\sim \quad (4)$$

where $W_R(z)$ has size $r \times p$, $W_L(z)$ has size $p \times r$, and more importantly both are causal, stable, and strict minimum phase. In other words, all poles and zeros of $W_R(z)$ and $W_L(z)$ are strictly inside the unit circle. In this case $W_R(z)$ and $W_L(z)$ are called the right and left spectral factors of $\Phi(z)$. Extensions can be made for spectral factors to include poles and zeros on the unit circle. But for the sake of simplicity and brevity, we shall not do so in this paper. Instead we assume that $\Phi(z)$ is a bounded hermitian positive matrix with rank r for all z on the unit circle, which excludes the possibilities of poles and zeros on the unit circle for the spectral factors. It is worth to pointing out that most of the existing work on spectral factorizations assume that $r = p$, and there lack effective computational algorithms for spectral factorizations in the case of $0 < r < p$.

In this paper we will also consider more general inner-outer factorizations where $H(z)$ as given in (1) may have zeros strictly outside the unit circle, and its realization is subject to the constraint

$$0 < \text{rank}\{D\} \leq \min\{m, p\}. \quad (5)$$

We investigate inner-outer factorizations for the following two cases:

$$\begin{array}{ll} \text{Case (i) } m > p: & H(z) = H_{\rm i}(z) H_{\rm o}(z) \\ \text{Case (ii) } m < p: & H(z) = H_{\rm o}(z) H_{\rm i}(z) \end{array} \quad (6)$$

where $H_{\rm i}(z)$ is a square inner of the smaller size, and $H_{\rm o}(z)$ is an outer. A square transfer matrix $H_{\rm i}(z)$ is called inner, if it is stable, and $H_{\rm i}(e^{j\omega})$ is a unitary matrix for all $\omega \in \mathbb{R}$. In other words, $H_{\rm i}(z)^\sim H_{\rm i}(z) = I$. A non-square transfer matrix $H_{\rm o}(z)$ is called outer, if it is both stable, and minimum phase. A moment of reflection reveals that all zeros of $H_{\rm i}(z)$ are strictly outside the unit circle, and are thus unstable. On the other hand, zeros of $H_{\rm o}(z)$ are all in the unit disc, including the unit circle. The abovementioned inner-outer factorizations are intimately related to spectral factorizations. In fact, $H_{\rm o}(z)$ is the right spectral factor of $\Phi(z) = H(z)^\sim H(z)$ for Case (i), and is the left spectral factor of $\Phi(z) = H(z)H(z)^\sim$ for Case (ii). The assumption that $D \neq 0$ has no loss of generality, because any causal transfer matrix $H(z)$ can be written as $H(z) = z^{-k}\tilde{H}(z)$ for some $k \geq 0$ and causal transfer matrix $\tilde{H}(z)$ such that $\tilde{D} = \tilde{H}(\infty) \neq 0$. Thus inner-outer factorizations of $\tilde{H}(z)$ can then be studied with z^{-k} subsumed into the inner.

3. Generalized LQR Control and Kalman Filtering

In this section we assume that the regular conditions for LQR control and Kalman filtering fail to hold, and derive their optimal solutions. For finite time horizon, and time-varying state-space systems, the optimal solutions are similar to the standard LQR control and Kalman filtering. But in the stationary case, i.e., the infinite time horizon and time-invariant systems, the optimal solutions require computing the stabilizing solutions to the AREs associated with the generalized LQR control and Kalman filtering. The iterative algorithms proposed in this section are not shown to converge to such stabilizing solutions, which will be proven in the next section.

We will also investigate inner-outer factorizations for non-square transfer matrices. For the interest of this paper, we restrict inners to square, and outers to non-square transfer matrices. It should be mentioned that inner-outer factorizations with square outers are well studied for $\mathcal{H}_\infty$-based robust control, and are associated with standard LQR control, and Kalman filtering. However inner-outer factorizations with square inners are less studied, let alone the singular constraint in (5). In the next two subsections, we generalize the results on optimal control (standard LQR control) and optimal estimation (standard Kalman filtering), and derive an iterative algorithm for computing inner-outer factorizations with square inners.

A. Generalized LQR Control and the Right Spectral Factor

The generalized LQR control assumes the state-space model, with $x(0) = x_0 \neq 0$,

$$x(t+1) = Ax(t) + Bu(t), \quad z(t) = Cx(t) + Du(t) \quad (7)$$

and searches for the control input $u(t)$ to minimize the quadratic performance index

$$J = \sum_{t=0}^{\infty} \|z(t)\|^2 = \sum_{t=0}^{\infty} z^*(t) z(t). \quad (8)$$

We assume that the control input $u(t)$ has size m, the controlled output $z(t)$ has size p, and $m > p$. Stability of A is not assumed for the generalized LQR control, and $\text{rank}\{D\} \leq p$.

This problem differs from the standard LQR problem in that D is a "fat" matrix by $m > p$, and its rank can be strictly smaller than p. That is, the penalty weighting matrix on the control signal is singular. The conventional approach is to consider optimal control over the finite time horizon, and then take the limit to the infinity time horizon. With careful derivation, we can obtain a similar solution to that for the standard LQR control, which is summarized in the following result.

Theorem 3.1: Let the m-input/p-output system be given as in (7), with $m > p \geq \mathrm{rank}\{D\}$. Suppose that (A, B) is stabilizable, and $X = X^* \geq 0$ satisfies the generalized ARE:

$$X = A^*XA + C^*C - \Gamma(D^*D + B^*XB)^+\Gamma^*, \quad (9)$$

where $\Gamma = A^*XB + C^*D$. Then with $u_{\rm opt}(t) = Fx(t)$, the performance index is $J = J_{\min} = x_0^*Xx_0$ where

$$F = -(D^*D + B^*XB)^+(B^*XA + D^*C). \quad (10)$$

Remark 3.2: We make the following remarks:

(a) Different from the standard LQR theory, we can not conclude stability of $A + BF$ despite the fact that (A, B) is stabilizable. That is, the optimal feedback system

$$x(t+1) = (A+BF)x(t), \quad z(t) = (C+DF)x(t), \quad (11)$$

with F in (10), may not be internally stable, even though the energy of the controlled output

$$J_{\min} = \|z\|_2^2 = \sum_{t=0}^{\infty} \|z(t)\|^2 = x_0^*Xx_0$$

is bounded. A careful reflection concludes that any unstable modes of $(A + BF)$ are unobservable based on the controlled output $z(t) = (C + DF)x(t)$. That is, the unstable modes of $(A+BF)$ are also unobservable modes of $(C + DF, A + BF)$.

(b) The ARE (9) may admit more than one positive semi-definite solutions. Each one can be viewed as an equilibrium to the DRE

$$X_t = A^*X_{t+1}A + C^*C - \Gamma_t(D^*D + B^*X_{t+1}B)^+\Gamma_t^*.$$

However there is a unique maximal solution $X_{\max}$, and a unique minimal solution $X_{\min}$ such that any other positive semi-definite solution X to the ARE (9) satisfies

$$0 \leq X_{\min} \leq X \leq X_{\max}.$$

It can be argued that the limit to the DRE exists for any initial value $X^{(0)} \geq 0$, but the limit is dependent on $X^{(0)}$.

(c) Theorem 3.1 provides an algorithm to compute a positive semi-definite solution $X \geq 0$: For $k = 0, 1, \cdots$, with $X^{(0)} \geq 0$ given, do the following:

$$\begin{array}{rcl} F^{(k)} & = & -(D^*D + B^*X^{(k)}B)^+\Gamma_k^*, \\ X^{(k+1)} & = & A_{F_k}^* X^{(k)} A_{F_k} + Q_k \end{array} \quad (12)$$

with $A_{F_k} = A + BF^{(k)}$ and $Q_k = (C + DF^{(k)})^*(C + DF^{(k)})$. The algorithm can be terminated if $\|X^{(N)} - X^{(N+1)}\|$ is smaller than some pre-specified tolerance bound. It is noted that $X^{(k)} = X_{T-k}$ is the solution to the DRE at time $t = T - k$ with $X_T = X^{(0)} \geq 0$.

(d) For the problem of inner-outer factorization in Case (i) of (6), A is assumed to be a stability matrix. If $X^{(0)} = W$ is chosen as the solution to the Lyapunov equation

$$W = A^*WA + C^*C, \quad (13)$$

then $W \geq 0$. Moreover taking the difference between the above Lyapunov equation and the ARE (9) yields

$$(W - X) = A^*(W - X)A + \Gamma(D^*D + B^*XB)^+\Gamma^*.$$

Stability of A implies that $W \geq X$ for any positive semi-definite solution to the ARE (9). Hence the maximal solution to the ARE (9) is likely to be obtained with the iterative algorithm (12) with the initial value $X^{(0)} = W$.

(e) A solution $X \geq 0$ to the ARE (9) is said to be a stabilizing solution, if $(A + BF)$ is a stability matrix where F has the expression in (10). It can be argued as in the standard LQR control that the stabilizing solution to the ARE (9) is maximal among all positive semi-definite solutions to (9), and thus is $X_{\max}$, if it exists. The existence of the stabilizing solution $X_{\max}$ is hinged to the condition (which is similar to the standard LQR control):

$$\mathrm{rank}\left\{\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n + p \quad \forall\, \theta \in \mathbb{R}, \quad (14)$$

in addition to the stabilizability of (A, B). It will be shown later that $X = X_{\max}$ is what needed for computing the inner-outer factorization for Case (i) in (6). How to obtain $X = X_{\max}$ will be answered in the next section. ∎

The above remarks indicate that the limiting optimal solution X to the generalized LQR control is dependent on the boundary condition $X^{(0)}$. The resultant control law can not be implemented in practice, unless $(A + BF)$ is a stability matrix, in which case $X = X_{\max}$. For ease of the reference, we denote $F_{\rm m}$ as the optimal feedback gain associated with $X_{\max}$ as follows:

$$F_{\rm m} = -(D^*D + B^*X_{\max}B)^+(B^*X_{\max}A + D^*C). \quad (15)$$

In the rest of the section we present our result on inner-outer factorization for Case (i) in (6).

Theorem 3.3: Suppose that $H(z)$ as in (1) has normal rank $p < m$, satisfies the condition (14), and A is a stability matrix. Let $X_{\max} \geq 0$ be the maximal solution to (12), and $F_{\rm m}$ be as in (15). Then there holds the inner-outer factorization $H(z) = H_{\rm i}(z)H_{\rm o}(z)$ where, with $\Omega_{\rm m}^*\Omega_{\rm m} = \Pi = D^*D + B^*X_{\max}B$, the inner and outer are given respectively by

$$\begin{array}{rcl} H_{\rm i}(z) & = & \left[\begin{array}{c|c} A + BF_{\rm m} & B \\ \hline C + DF_{\rm m} & D \end{array}\right]\Omega_{\rm m}^+ \\ H_{\rm o}(z) & = & \Omega_{\rm m}\left[\begin{array}{c|c} A & B \\ \hline -F_{\rm m} & I \end{array}\right] \end{array} \quad (16)$$

We comment that the outer factor $H_{\rm o}(z)$ has no transmission zeros at $z = \infty$, due to the full rank of $\Omega_{\rm m}$ which has size $p \times m$, and the same rank as the normal rank of $H(z)$. The possible transmission zeros of $H(z)$ at $z = \infty$ are now transmission zeros of the inner factor $H_{\rm i}(z)$, which is evident by its expression in (16). In the case when $G(z)$ is strict minimum phase, i.e.,

$$\text{rank}\left\{\begin{bmatrix} A-zI & B \\ C & D \end{bmatrix}\right\} = n+p \quad \forall\, |z| \geq 1,$$

there is a unique positive semi-definite solution $X \geq 0$. In fact $X = 0$, if rank$\{D\} = p$.

B. Generalized Kalman Filtering and the Left Spectral Factor

The results in this subsection are dual to those in the previous subsection. Thus we will only state the results without proofs and derivations. We will present the result on generalized Kalman filtering, and the inner-outer factorization for Case (ii) in (6), which assumes $p > m$. Since $H(z)H(z)^\sim = H_o(z)H_o(z)^\sim$, we seek a left spectral factor of $H(z)H(z)^\sim$, which is related to generalized Kalman filtering. That is, we are given the random process described by

$$x(t+1) = Ax(t) + Bv(t), \quad y(t) = Cx(t) + Dv(t), \tag{17}$$

where $v(t)$ is a WSS (wide-sense stationary) random process, and satisfies

$$E[v(t)] = 0, \quad E[v(t+k)v^*(t)] = \delta(k)I, \tag{18}$$

with $E[\,\cdot\,]$ the expectation. The dimension of the input noise $\{v(t)\}$ is m, and the dimension of the output measurement $\{y(t)\}$ is p. Since $p > m$, the covariance of the observation noise $Dv(t)$ is singular. The objective is to estimate $x(t+1)$, based on the observation $\{y(k)\}_{k=0}^{t}$. The standard Kalman filtering deals with the case when D is "fat" and has the full row rank. However, we have a "tall" D, which may not have a full column rank: $0 < \text{rank}\{D\} \leq m$.

By duality, let $Y \geq 0$ be a solution to the ARE

$$\begin{aligned} Y &= AYA^* - L_t(DD^* + CYC^*)L_t + BB^* \quad (19) \\ L &= -(AYC^* + BD^*)(DD^* + CYC^*)^+ \quad (20) \end{aligned}$$

Again there are more than one solutions $Y \geq 0$ in general. A positive semi-definite solution $Y \geq 0$ can be obtained iteratively: For $k = 0, 1, \cdots$, with $Y_0 \geq 0$, do the following:

$$\begin{aligned} L_k &= -(AY_kC^* + BD^*)(DD^* + CY_kC^*)^+ \\ Y_{k+1} &= A_{L_k}Y_kA_{L_k}^* + (B + L_kD)(B + L_kD)^* \end{aligned} \tag{21}$$

where $A_{L_k} = (A + L_kC)$. In practice the algorithm is terminated when $\|Y_N - Y_{N+1}\|$ satisfies some pre-specified tolerance bound.

As in the previous subsection, we point out that $(A+LC)$ may not be stable, even though $Y \geq 0$ is a solution to the ARE (19). However the corresponding state estimation error has bounded variance. That is if $(A + LC)$ is unstable, then $(A + LC, B + LD)$ is an unreachable pair, and all unstable modes of $(A + LC)$ are unreachable modes of $(A + LC, B + LD)$, by noting that the ARE (19) can be written into the form of Lyapunov equation

$$Y = (A+LC)Y(A+LC)^* + (B+LD)(B+LD)^*. \tag{22}$$

Moreover there are more than one positive semi-definite solutions to (21), with only one $Y_{\max}$ and one $Y_{\min}$. Any other $Y \geq 0$ satisfies the inequality $Y_{\max} \geq Y \geq Y_{\min} \geq 0$. If in addition there holds the rank condition

$$\text{rank}\left\{\begin{bmatrix} A-e^{j\theta}I & B \\ C & D \end{bmatrix}\right\} = n+m \quad \forall\, \theta \in \mathbb{R}, \tag{23}$$

then $Y_{\max}$ is stabilizing in the sense that with

$$L_{\mathrm{m}} = -(AY_{\max}C^* + BD^*)(DD^* + CY_{\max}C^*)^+, \tag{24}$$

$(A+L_{\mathrm{m}}C)$ is a stability matrix. In this case the generalized Kalman filter is given by

$$\hat{x}(t+1) = (A + L_{\mathrm{m}}C)\hat{x}(t) - L_{\mathrm{m}}y(t).$$

As in the previous subsection, $Y_{\max}$ and L_{m} are associated with the inner-outer factorization entailed in Case (ii) of (6). The next result presents the solution to the inner-outer factorization in Case (ii) of (6).

Theorem 3.4: Suppose that $H(z)$ as in (1) has normal rank $m < p$, satisfies the condition (23), and A is a stability matrix. Let $Y = Y_{\max} \geq 0$ be the maximal solution to (21), and L_{m} be as in (24). Then there holds the inner-outer factorization $H(z) = H_{\mathrm{o}}(z)H_{\mathrm{i}}(z)$ where, with $\Omega_{\mathrm{m}}\Omega_{\mathrm{m}}^* = \Pi = DD^* + CY_{\max}C^*$, the inner and outer factors of $H(z)$ are given respectively by

$$\begin{aligned} H_{\mathrm{i}}(z) &= \Omega_{\mathrm{m}}^+\left[\begin{array}{c|c} A + L_{\mathrm{m}}C & B + L_{\mathrm{m}}D \\ \hline C & D \end{array}\right], \\ H_{\mathrm{o}}(z) &= \left[\begin{array}{c|c} A & -L_{\mathrm{m}} \\ \hline C & I \end{array}\right]\Omega_{\mathrm{m}}. \end{aligned} \tag{25}$$

Although iterative algorithms are derived for computing solutions to the AREs in (9) and (19), it is unclear how to choose the boundary conditions $X^{(0)} \geq 0$ and $Y_0 \geq 0$ to (12) and (21), respectively, that will ensure their convergence to the required stabilizing solutions. It turns out that such an issue has to be resolved together with that for spectral factorizations.

4. Spectral Factorizations

In this section we investigate the spectral factorization problem for the $q \times q$ para-hermitian transfer matrix $\Phi(z)$ which is positive semi-definite on the unit circle. It has the form

$$\Phi(z) = R_0 + C_\Phi(zI - A)^{-1}B_\Phi + B_\Phi^*(z^{-1}I - A^*)^{-1}C_\Phi^* \tag{26}$$

where A is a stability matrix, and the normal rank of $\Phi(z)$ is $\rho < q$. This problem is much harder than the case of full normal rank. Since $\Phi(z)$ is positive semi-definite on the unit circle. there exist minimal degree factorizations [1]

$$\Phi(z) = W_G(z)^\sim W_G(z) = W_K(z)W_K(z)^\sim, \tag{27}$$

where $W_G(z)$ of size $\rho \times q$ and $W_K(z)$ of size $q \times \rho$ are both stable, given by

$$W_G(z) = \left[\begin{array}{c|c} A & B_\Phi \\ \hline G & D_G \end{array}\right], \quad W_K(z) = \left[\begin{array}{c|c} A & K \\ \hline C_\Phi & D_K \end{array}\right], \tag{28}$$

for some (G, D_G) and (K, D_K). The following result is modified from [1].

Lemma 4.1: Consider the positive para-hermitian transfer matrix $\Phi(z)$ in (26) where A is a stability matrix. There exist minimal degree factorizations as in (27) for some $W_G(z)$ and $W_K(z)$ in the form of (28), if and only if

$$P = A^*PA + G^*G, \qquad C_\Phi = D_G^*G + B_\Phi^*PA \quad (29)$$
$$Q = AQA^* + KK^*, \qquad B_\Phi^* = D_K K^* + C_\Phi Q A^* \quad (30)$$
$$R_0 = D_G^*D_G + B_\Phi^*PB_\Phi, \qquad R_0 = D_K D_K^* + C_\Phi Q C_\Phi^*$$

admit solutions (P, G, D_G), and (Q, K, D_K) respectively.

Lemma 4.1 shows that in order to obtain the minimal degree factors $W_G(z)$ and $W_K(z)$ in (28), i.e., (G, D_G) and (K, D_K), we need first solve for P and Q in (29) and (30), respectively, which are two Lyapunov equations. Since A is stable, $P \geq 0$, and $Q \geq 0$, if they exist. In fact (29), and (30) have solutions (P, G, D_G), and (Q, K, D_K), respectively, if and only if $\Phi(z) \geq 0$ for all $|z| = 1$. However more than one set of such solutions (P, G, D_G), or (Q, K, D_K) exist, implying that more than one pair of minimal degree factors exist. However there are unique sets of solutions (P, G, D_G), and (Q, K, D_K) such that both $W_G(z)$ and $W_K(z)$ as in (28) are outer functions, i.e.,

$$\text{rank } \{ W_G(z) \} = \text{rank} \{ W_K(z) \} = \rho \ \ \forall \, |z| \geq 1. \quad (31)$$

The spectral factorization problem in this section is also referred to as minimal degree spectral factorizations, and the spectral factors are unique up to a factor of unitary matrices. Because not every set of solutions to (29) or to (30) yields spectral factors of $\Phi(z)$, our goal is to obtain the right sets of solutions such that the resultant $W_G(z)$, and $W_K(z)$ are spectral factors of $\Phi(z)$, and satisfy (27). For this purpose the results from the previous section play the pivotal role.

Through some derivations we can show that

$$G^*G \;=\; \tilde{G}^*(R_0 - B_\Phi^*PB_\Phi)^+\tilde{G} \quad (32)$$
$$KK^* \;=\; \tilde{K}(R_0 - C_\Phi Q C_\Phi^*)^+\tilde{K}^* \quad (33)$$

where $\tilde{G} = (C_\Phi - B_\Phi^*PA)$ and $\tilde{K} = (B_\Phi - AQC_\Phi^*)$. Hence the two Lyapunov equations in (29) and (30) have the form of AREs:

$$P \;=\; A^*PA + \tilde{G}^*(R_0 - B_\Phi^*PB_\Phi)^+\tilde{G}, \quad (34)$$
$$Q \;=\; AQA^* + \tilde{K}(R_0 - C_\Phi Q C_\Phi^*)^+\tilde{K}^*, \quad (35)$$

respectively. The following result is again modified from [1].

Lemma 4.2: Suppose that $\Phi(z) \geq 0$ for all $|z| = 1$. Then all solutions P and Q to (34), and (35) respectively are non-negative definite. There exist maximal solutions $P_{\max}$, $Q_{\max}$, and minimal solutions $P_{\min}$, $Q_{\min}$ to (34), and (35), respectively. All other solutions P, and Q to (34), and (35), respectively satisfy $P_{\min} \leq P \leq P_{\max}$ and $Q_{\min} \leq Q \leq Q_{\max}$.

The solution sets corresponding to $P_{\min}$, and $Q_{\min}$ are associated with right, and left spectral factors of $\Phi(z)$, respectively, while $P_{\max}$, and $Q_{\max}$ are associated with those factors $W_G(z)$, and $W_K(z)$, whose transmission zeros are all outside unit circle, respectively. Any other solutions P, and Q being neither minimal, nor maximal correspond to those factors $W_G(z)$, and $W_K(z)$ which contain some non-minimum phase zeros. The computation of $P_{\min}$, and $Q_{\min}$ is the main focus of this section, which yield the minimal degree spectral factors of $\Phi(z)$ in (26). We propose the following iterative algorithm.

- Set initial values $P_0 = 0$, and $Q_0 = 0$.
- For $k = 0, 1, \cdots$, compute

$$P_{k+1} \;=\; A^*P_kA + \tilde{G}^*(R_0 - B_\Phi^*P_kB_\Phi)^+\tilde{G} \quad (36)$$
$$Q_{k+1} \;=\; AQ_kA^* + \tilde{K}(R_0 - C_\Phi Q_k C_\Phi^*)^+\tilde{K}^* \quad (37)$$

- If $\|P_N - P_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{P_k\}$; If $\|Q_N - Q_{N-1}\|$ is smaller than the pre-specified tolerance bound, terminate computation of $\{Q_k\}$.

In the rest of the section we will show that the above algorithm is convergent with limit $P_{\min}$, and $Q_{\min}$. For this purpose define D_{G_m} and D_{K_m} as the minimum Cholesky factors via

$$\begin{aligned} D_{G_\mathrm{m}}^*D_{G_\mathrm{m}} &= R_0 - B_\Phi^*P_{\min}B_\Phi, \\ D_{K_\mathrm{m}}D_{K_\mathrm{m}}^* &= R_0 - C_\Phi Q_{\min} C_\Phi. \end{aligned} \quad (38)$$

Similarly define G_m and K_m as

$$\begin{aligned} G_\mathrm{m} &= (D_{G_\mathrm{m}}^+)^*(C_\Phi - B_\Phi^*P_{\min}A), \\ K_\mathrm{m} &= (B_\Phi - AQ_{\min}C_\Phi^*)(D_{K_\mathrm{m}}^+)^*. \end{aligned} \quad (39)$$

Then $(A, K_\mathrm{m}, C_\Phi, D_{K_\mathrm{m}})$, and $(A, B_\Phi, G_\mathrm{m}, D_{G_\mathrm{m}})$ are realizations associated with left, and right spectral factors of $\Phi(z)$, respectively. That is,

$$\begin{aligned} W_{K_\mathrm{m}}(z) &= \left[\begin{array}{c|c} A & K_\mathrm{m} \\ \hline C_\Phi & D_{K_\mathrm{m}} \end{array}\right], \\ W_{G_\mathrm{m}}(z) &= \left[\begin{array}{c|c} A & B_\Phi \\ \hline G_\mathrm{m} & D_{G_\mathrm{m}} \end{array}\right] \end{aligned} \quad (40)$$

are the left, and right spectral factors of $\Phi(z)$, respectively, and are thus outers. In light of Theorem 3.3, D_{G_m} has rank ρ, and in light of Theorem 3.4, D_{K_m} also has rank ρ. As a result, D_{G_m} and D_{K_m} have dimensions $\rho \times q$ and $q \times \rho$, respectively, and thus have the full rank. Recall that ρ is the normal rank of $\Phi(z)$. However for any other minimal degree factors $W_K(z)$ and $W_G(z)$ as in (28) which are not spectral factors of $\Phi(z)$, the associated D_G and D_K may have ranks strictly smaller than ρ. It is crucial to observe that the right spectral factor of $\Phi(z)$ can be obtained from the inner-outer factorization of $H(z) = W_G(z)$ as in Case (i) of (6), and the left spectral factor of $\Phi(z)$ can be obtained from the inner-outer factorization of $H(z) = W_K(z)$ as in Case (ii) of (6). Hence the following result is true.

Theorem 4.3: Consider $W_G(z)$ of size $\rho \times q$, and $W_K(z)$ of size $q \times \rho$ as in (28), which are not spectral factors of $\Phi(z)$, but satisfy (27) with $\rho < q$, where $\Phi(z) \geq 0$ for all $|z| = 1$. Then for any $X^{(0)} \geq 0$, and $Y_0 \geq 0$ with $T > 0$, the algorithms (12) and (21) have non-negative definite solutions $\{X^{(k)}\}_{k=1}^T$, and $\{Y_k\}_{k=1}^T$, respectively.

Suppose that $X^{(0)} \geq 0$ and $Y_0 \geq 0$ are chosen such that $X^{(T)}$ converges to $X_{\max} \geq 0$, and Y_T converges $Y_{\max} \geq 0$, respectively, as $T \to \infty$, satisfying the AREs (9) and (19), respectively. In this case, realizations of the left and right spectral factors in (40) are uniquely specified (up to a factor of unitary matrices) respectively by

$$\begin{aligned} D_{G_{\rm m}}^* D_{G_{\rm m}} &= D_G^* D_G + B_\Phi^* X_{\max} B_\Phi, && (41)\\ G_{\rm m} &= (D_{G_{\rm m}}^+)^* (B_\Phi^* X_{\max} A + D_G^* G), \\ D_{K_{\rm m}} D_{K_{\rm m}}^* &= D_K D_K^* + C_\Phi Y_{\max} C_\Phi^*, && (42)\\ K_{\rm m} &= (A Y_{\max} C_\Phi^* + K D_K^*)(D_{K_{\rm m}}^+)^*, \end{aligned}$$

where $D_{G_{\rm m}}$ and $D_{K_{\rm m}}$ are the minimum rank Cholesky factors.

Theorem 4.3 shows that the minimal solutions $P_{\min} \geq 0$, and $Q_{\min} \geq 0$ to the AREs (34) and (35) can be computed from

$$\begin{aligned} P_{\min} &= A^* P_{\min} A + G_{\rm m}^* G_{\rm m} \\ Q_{\min} &= A Q_{\min} A^* + K_{\rm m} K_{\rm m}^* \end{aligned} \qquad (43)$$

respectively, which are basically the special cases of (29), and (30). It can be shown that

$$\begin{aligned} (X_{\max} + P_{\min}) &= A^*(X_{\max} + P_{\min})A + G^* G && (44)\\ (Y_{\max} + Q_{\min}) &= A(Y_{\max} + Q_{\min})A^* + K K^* && (45) \end{aligned}$$

Comparing the above two Lyapunov equations with those in (29) and (30), respectively concludes that $P = X_{\max} + P_{\min}$, and $Q = Y_{\max} + Q_{\min}$. Note that $X_{\max}$ is dependent on G, while $Y_{\max}$ is dependent on K, but $P_{\min}$ and $Q_{\min}$ are not. Hence we may switch to the notations

$$\begin{aligned} X_{\max} &= X_{\max}(G), \qquad P = P(G), \\ Y_{\max} &= Y_{\max}(K), \qquad Q = Q(K), \end{aligned}$$

respectively. The aforementioned analysis leads to the relation

$$P(G) = X_{\max}(G) + P_{\min}, \quad Q(K) = Y_{\max}(K) + Q_{\min}, \qquad (46)$$

which are associated with $W_G(z)$ and $W_K(z)$ in (28), respectively. The above relation is crucial to prove the main result of this section.

Theorem 4.4: Let $\Phi(z)$ of size $q \times q$ as in (26) have normal rank $\rho < q$. Suppose that A is a stability matrix, and $\Phi(z) \geq 0$ and $\text{rank}\{\Phi(z)\} = \rho$ for all $|z| = 1$. Then the iterative formulas (36) and (37) in the proposed algorithm are convergent with limits $P_{\min}$, and $Q_{\min}$, which are the minimum solutions to the AREs (34) and (35), respectively.

It is noted that the convergence of the proposed spectral factorization algorithm embodied in (36) and (37) is established under the zero initial condition $P_0 = Q_0 = 0$. If $P_0 \geq 0$ and $Q_0 \geq 0$ are arbitrary, then the convergence of the DREs in (36) and (37) remains unknown, that is very different from the inner-outer factorization algorithms in the previous section.

Remark 4.5: In light of (46) and the proof of Theorem 4.4, we also obtain the right initial values $X^{(0)}$ and Y_0 for the iterative algorithms in (12), and (21), respectively, in order to ensure the limits $X_{\max}$ and $Y_{\max}$, respectively. That is, the initial condition $X^{(0)} = W$ with W the solution to the Lyapunov equation (13) can ensure that the iterative algorithm in (12) admits limit $X_{\max}$, as required for the inner-outer factorization in Case (i) of (6); Similarly if Y_0 satisfying $Y_0 = A Y_0 A^* + B B^*$ is chosen, then the iterative algorithm (21) admits the limit $Y_{\max}$, as required for the inner-outer factorization in Case (ii) of (6). ∎

5. Conclusion

This paper considers generalized LQR control and Kalman filtering. The main contributions are the relations between these two optimization problems and computations of inner-outer factorizations (Section 3), and spectral factorizations (Section 4). It is these relations that help develop iterative algorithms, convergent to the stabilizing solutions of the AREs, associated with generalized LQR control and Kalman filtering, which in turn solves the problem of inner-outer factorizations and spectral factorizations. The results are applicable to control, signal processing, and communications. Due to the space limit, examples are skipped in this conference version.

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