Fault Tolerant Control Design Via the Adaptive Two-Stage LQ Reliable Control

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Abstract— **This paper presents a new fault tolerant control based on LQ design method that may stabilize a given system both in the nominal situation, as well as in the situation where some of the actuators have failed. This new control result basically follows the design guideline of the recently developed adaptive two-stage LQ reliable control (ATSLQRC), and considers the faults not confined to a preselect set of actuators. The unified gain margin constraint of the ATSLQRC is presented. It is also shown that the ATSLQRC serves as a guaranteed cost control provided that its design parameters satisfy two new proposed guaranteed cost gain margin constraints. A numerical example is given to illustrate the effectiveness of the proposed results. A potential application of the proposed results to design a passive fault tolerant control is also addressed.**

I. INTRODUCTION

Fault tolerant controller designs guaranteeing stability while permitting control component failures (i.e., actuator failures and/or sensor failures) have received great attentions in the literature [1]-[8]. However, most of aforementioned results are either devoted to an H_∞ framework or developed for continuous-time systems [1], [2], [3], [5], [6], and few results are dedicated to discrete-time LQ designs [4], [7], [8]. In this paper, we shall study the fault tolerant control design for discrete-time systems which are subject to actuator failures in the framework of LQR design method. Specifically, we shall focus on an extension of the reliable LQ regulator of Veillette [3] to discrete-time systems. In an early attempt [11], [12], the author proposed the two-stage LQ reliable control (TSLQRC) to extend Veillette's reliable LQ design to discrete-time systems. In [13], a modified version of the TSLQRC was proposed to improve the gain margin results. Furthermore, a feasible version of the TSLQRC named as the FTSLQRC [14] was also proposed to overcome the inherently infeasible problem existed in the original control structure of the TSLQRC. Recently [15], to further improve the performance and stability gain margins of the FTSLQRC, an adaptive version of the TSLQRC, i.e., the ATSLQRC, which mainly employing a novel actuator-failure-estimation technique to overcome the aforementioned restricted gain margin results has been proposed. Moreover, the problem that the ATSLQRC can tolerate system uncertainties in the form of norm-bounded parameter uncertainties was also considered in [15].

Although the gain margin results of the ATSLQRC are appealing, they can only tolerate actuator outage within a preselected actuator set. Unfortunately, in practical fault tolerant control applications, actuator failures may exist in any actuator set. Hence, the ATSLQRC may not be considered in systems for which any actuator might fail. Thus, the main aim of this paper is to explore and modify the gain margin properties of the ATSLQRC when further considering the possible actuator outage in the whole actuator set. In other words, this paper presents a new fault tolerant control, based on the previously proposed ATSLQRC, that may stabilize a given system both in the nominal situation, as well as in the situation where some (not confined to a preselect set of actuators) of the actuators have failed. To facilitate the development, the unified gain margin constraint of the ATSLQRC is presented. It is shown that the ATSLQRC serves as a guaranteed cost control provided that its design parameters satisfy two new proposed guaranteed cost gain margin constraints. A numerical example is also given to illustrate the effectiveness of the proposed results. Recently, a corresponding result dedicated to continuous-time systems was given in [9].

As addressed in [8] that the approaches to fault tolerant control can be divided into two main classes: Active fault tolerant control and passive fault tolerant control. Although an active fault tolerant control system might accommodate actuator failures more efficiently, however the involved failure estimating scheme may add the complexity of the overall system. On the other hand, in the passive fault tolerant control approach, a fixed controller is often enough to achieve the system's stability requirement. Although the proposed ATSLQRC is a certain kind of active fault tolerant control scheme, however this paper shows a possibility that the ATSLQRC may be useful to design a fixed controller to solve a passive fault tolerant control design problem in which the system may suffer from the problem that any one of the actuators might fail. Thus, a potential application of the ATSLQRC to design a passive fault tolerant control is also addressed. This result also suggests a possible approach to solve the fault tolerant control problem in which several actuators can fail simultaneously, which appears to be a subject of future research raised by [8].

This paper is organized as follows. Section II introduces the problem and the recently developed ATSLQRC. It is followed by the derivation of the gain margin properties of the ATSLQRC. Section IV presents a guaranteed cost control design via the ATSLQRC. An illustrative example is given in Section V to demonstrate the proposed method. Section VI gives a potential application to a passive fault tolerant control design. Finally, Section VII gives the conclusions.

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II. PROBLEM STATEMENT AND THE ATSLQRC

Consider the following discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where $x_k \in R^n$ is the system state, $u_k \in R^m$ is the control input whose components may fail during system operation, and matrices A and B are known constant matrices. The quadratic performance index associated with the system is given by

$$J = \sum_{k=0}^{\infty}(x_k' Q x_k + u_k' R u_k), \quad (2)$$

where $Q \geq 0$ and $R > 0$ are given weighting matrices, and $'$ denotes transpose.

In this paper, we shall determine whether it is possible to design a LQ regulator that is guaranteed to achieve an upper performance bound for system (1), i.e., $J \leq \bar{J} < \infty$, in case some actuators may fail. Specifically, we consider the actuator failure model as

$$u_k^F = diag\{N_{\bar{\Omega}}, N_{\Omega}\}u_k, \quad (3)$$

where $N_{\bar{\Omega}}$ and N_{Ω} are assumed as follows:

$$N_{\bar{\Omega}} = diag\{n_{\bar{\Omega}}^1, n_{\bar{\Omega}}^2, \cdots, n_{\bar{\Omega}}^p\}, \ \underline{n}_{\bar{\Omega}}^i \leq n_{\bar{\Omega}}^i \leq \bar{n}_{\bar{\Omega}}^i, \quad (4)$$

$$N_{\Omega} = diag\{n_{\Omega}^1, n_{\Omega}^2, \cdots, n_{\Omega}^{m-p}\}, \ \underline{n}_{\Omega}^j \leq n_{\Omega}^j \leq \bar{n}_{\Omega}^j, \quad (5)$$

in which $0 \leq \underline{n}_{\bar{\Omega}}^i \leq 1$, $\bar{n}_{\bar{\Omega}}^i \geq 1$, $0 \leq \underline{n}_{\Omega}^j \leq 1$, and $\bar{n}_{\Omega}^j \geq 1$. Accordingly, the input matrix B and the weighting matrix R are partitioned as $B = \begin{bmatrix} B_{\bar{\Omega}} & B_{\Omega} \end{bmatrix}$ and $R = diag\{R_{\bar{\Omega}}, R_{\Omega}\}$, respectively. Note that in the above formulations, Ω denotes a preselected subset of unreliable actuators within which outages must be tolerated while $\bar{\Omega}$ denotes the complementary subset of actuators. Without loss of generality, in this paper we focus on the following two special actuator failure models: 1) $N_{\bar{\Omega}} \geq 0$ & $N_{\Omega} = 0$ and 2) $N_{\bar{\Omega}} = 0$ & $N_{\Omega} \geq 0$, where $N_S = 0$ denotes the complete outage of the actuators in the set S while $N_S \geq 0$ denotes that there exists at least one actuator in the set S such that outage will not occur.

For easy reference, the recently developed ATSLQRC [15] is summarized as follows:

$$u_k = -\begin{bmatrix} \hat{N}_{\bar{\Omega},k}^{+}\bar{K}_{\bar{\Omega}}\hat{U}_k \\ \hat{N}_{\Omega,k}^{+}\bar{K}_{\Omega} \end{bmatrix} Ax_k, \quad (6)$$

where $\hat{N}_{\bar{\Omega},k}$ and $\hat{N}_{\Omega,k}$ are the estimates of the perturbation matrices $N_{\bar{\Omega}}$ and N_{Ω}, respectively, M^+ is an arbitrary generalized inverse of M satisfying $MM^+M = M$, and

$$\begin{aligned}
\bar{K}_{\bar{\Omega}} &= S_{\bar{\Omega}}^{-1}B_{\bar{\Omega}}'P, \quad S_{\bar{\Omega}} = B_{\bar{\Omega}}'PB_{\bar{\Omega}} + R_{\bar{\Omega}}, & (7)\\
\bar{K}_{\Omega} &= S_{\Omega}^{-1}B_{\Omega}'\bar{P}, \quad S_{\Omega} = B_{\Omega}'\bar{P}B_{\Omega} + R_{\Omega}, & (8)\\
\bar{P} &= P(I - B_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}) = P - PB_{\bar{\Omega}}S_{\bar{\Omega}}^{-1}B_{\bar{\Omega}}'P, & (9)\\
P &= A'P(I - B_{\bar{\Omega}}\Gamma_{\bar{\Omega}}^{\Phi}\bar{K}_{\bar{\Omega}})A + Q - A'E_0A, & (10)\\
E_0 &= P(I - B_{\bar{\Omega}}\Gamma_{\bar{\Omega}}^{\Phi}\bar{K}_{\bar{\Omega}})B_{\Omega}\Gamma_{\Omega}^{\Phi}\bar{K}_{\Omega}, & (11)\\
\hat{U}_k &= I - B_{\Omega}\hat{N}_{\Omega,k}^{\Phi}\bar{K}_{\Omega}, \quad \hat{N}_{\Omega,k}^{\Phi} = \hat{N}_{\Omega,k}\hat{N}_{\Omega,k}^{+}, & (12)
\end{aligned}$$

where $\Gamma_{\bar{\Omega}}^{\Phi}$ and Γ_{Ω}^{Φ} are two design parameters. Note that E_0 in (11) is a modified version of the original term in [15], which may yield an asymmetric DARE, i.e., the obtained stabilizing solution P by (10) is not symmetric. In this paper we also assume that the estimates $\hat{N}_{\bar{\Omega},k}$ and $\hat{N}_{\Omega,k}$ can be accurately obtained, i.e., $\hat{N}_{\bar{\Omega},k} \approx N_{\bar{\Omega}}$ and $\hat{N}_{\Omega,k} \approx N_{\Omega}$ for time $k \geq T$.

From [15], one knows that the stability and performance gain margins of the above ATSLQRC may be given as follows: $0 < n_{\bar{\Omega}}^i < \infty$ and $0 \leq n_{\Omega}^j < \infty$. However, in practical control applications, the actuators in the set $\bar{\Omega}$ may also encounter actuator outage, i.e., $n_{\bar{\Omega}}^i = 0$ for some i. Thus, the main aim of this paper is to explore the gain margin properties of the ATSLQRC when further considering the possible actuator outage in the set $\bar{\Omega}$. Furthermore, a potential application of the proposed results to solve a passive fault tolerant control design problem is also presented.

III. GAIN MARGIN PROPERTIES OF THE ATSLQRC

To facilitate latter development, in this section we will present the key result which states the unified gain margin constraint of the ATSLQRC in the following Theorem.

Theorem 1. The state-feedback system obtained by applying the ATSLQRC (6)-(12) to system (1), which guarantees the following performance bound

$$J \leq J_0 + x_T'(P + a \cdot \tilde{P}_T)x_T, \quad (13)$$

where $a \in R$ and

$$\begin{aligned}
J_0 &= \sum_{k=0}^{T-1}\left\{x_k'(Q + A'\bar{Q}_kA)x_k\right\}, & (14)\\
\tilde{P}_k &= F_k'\tilde{P}_{k+1}F_k + A'\bar{Q}_kA, & (15)\\
\bar{Q}_k &= \hat{U}_k'\bar{K}_{\bar{\Omega}}'(N_{\bar{\Omega},k}^{\Phi})'R_{\bar{\Omega}}N_{\bar{\Omega},k}^{\Phi}\bar{K}_{\bar{\Omega}}\hat{U}_k \\
&\quad +\bar{K}_{\Omega}'(N_{\Omega,k}^{\Phi})'R_{\Omega}N_{\Omega,k}^{\Phi}\bar{K}_{\Omega}, & (16)\\
F_k &= (I - B_{\bar{\Omega}}N_{\bar{\Omega},k}^{\Phi}\bar{K}_{\bar{\Omega}})\hat{U}_kA \\
&\quad +B_{\Omega}(\hat{N}_{\Omega,k}^{\Phi} - N_{\Omega,k}^{\Phi})\bar{K}_{\Omega}A, & (17)
\end{aligned}$$

in which $N_{\Omega,k}^{\Phi} = N_{\Omega}\hat{N}_{\Omega,k}^{+}$ and $N_{\bar{\Omega},k}^{\Phi} = N_{\bar{\Omega}}\hat{N}_{\bar{\Omega},k}^{+}$, if for time $k \geq T$ the matrices $\Gamma_{\bar{\Omega}}^{\Phi}$, Γ_{Ω}^{Φ}, $N_{\bar{\Omega}}^{\Phi}$, and N_{Ω}^{Φ} satisfy the following unified gain margin constraint:

$$\begin{aligned}
&\bar{K}_{\Omega}'D_{\Omega}\bar{K}_{\Omega} + PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}} + a \cdot \bar{Q}_k \\
\geq\ &\hat{U}_k'\bar{K}_{\bar{\Omega}}'D_{\bar{\Omega}}\bar{K}_{\bar{\Omega}}\hat{U}_k + \Upsilon_k + E_0, & (18)
\end{aligned}$$

where

$$\begin{aligned}
D_{\Omega} &= S_{\Omega} - (I - N_{\Omega,k}^{\Phi})'S_{\Omega}(I - N_{\Omega,k}^{\Phi}), & (19)\\
D_{\bar{\Omega}} &= (I - N_{\bar{\Omega},k}^{\Phi})'S_{\bar{\Omega}}(I - N_{\bar{\Omega},k}^{\Phi}), & (20)\\
\Upsilon_k &= \hat{U}_k'(PB_{\bar{\Omega}} - \bar{K}_{\bar{\Omega}}'S_{\bar{\Omega}})(I - N_{\bar{\Omega},k}^{\Phi})\bar{K}_{\bar{\Omega}}\hat{U}_k \\
&\quad +(\bar{K}_{\Omega}'S_{\Omega} - \bar{P}B_{\Omega})N_{\Omega,k}^{\Phi}\bar{K}_{\Omega}, & (21)\\
E_0 &= P(I - B_{\bar{\Omega}}\Gamma_{\bar{\Omega}}^{\Phi}\bar{K}_{\bar{\Omega}})B_{\Omega}\Gamma_{\Omega}^{\Phi}\bar{K}_{\Omega}. & (22)
\end{aligned}$$

Proof: First, we note that the performance index (2), in the presence of actuator failures (3), can be represented as follows:

$$J = J_0 + \sum_{k=T}^{\infty} \left\{ x_k'(Q + A'\bar{Q}_k A)x_k \right\}. \quad (23)$$

Next, using the following substitution: $Q \to Q - A'E_0A$ and applying the same procedures as given in deriving (34) of [14], one obtains

$$\begin{aligned} &F_k'(P + a\tilde{P}_{k+1})F_k - (P + a\tilde{P}_k) + Q + A'\bar{Q}_k A \\ &= -A' \left\{ PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}} + \{\bullet\} + a\bar{Q}_k - E_0 \right\} A, \quad (24) \end{aligned}$$

where

$$\{\bullet\} = \bar{K}_\Omega' D_\Omega \bar{K}_\Omega - \hat{U}_k' \bar{K}_{\bar{\Omega}}' D_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \hat{U}_k - \Upsilon_k. \quad (25)$$

Finally, using (24) one can easily show that the performance index (23) can be represented alternatively as follows:

$$J = J_0 + x_T'(P + a\tilde{P}_T)x_T - \sum_{k=T}^{\infty} (Ax_k)'\{\bullet\}_1 Ax_k, \quad (26)$$

where

$$\{\bullet\}_1 = PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}} + \{\bullet\} - E_0 + a\bar{Q}_k. \quad (27)$$

From (26), the performance bound (13) holds if matrix $\{\bullet\}_1$ is positive semidefinite, which establishes the constraint (18). This completes the proof. ∎

Note that the above unified gain margin constraint (18) is a modified version of the original term in [15] to compensate for the resulted asymmetric DARE (10).

Now, we are in the position to simplify the unified gain margin constraint (18) according to the considered two actuator failure models as below.

Case 1: $N_{\bar{\Omega}} \geq 0$ and $N_\Omega = 0$. In this case, one has $N_{\Omega,k}^{\Phi} = 0$, $\hat{N}_{\Omega,k}^{\Phi} = 0$, and $N_{\bar{\Omega},k}^{\Phi}$ a diagonal matrix in which all principal values are either zero or one and at least one value is one. Constraint (18) then becomes

$$\begin{aligned} &PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}} + a \cdot \bar{Q}_k \\ \geq\ &(PB_{\bar{\Omega}} - \bar{K}_{\bar{\Omega}}' N_{\bar{\Omega},k}^{\Phi} S_{\bar{\Omega}})(I - N_{\bar{\Omega},k}^{\Phi})\bar{K}_{\bar{\Omega}} \\ &+\bar{P}B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega + PB_{\bar{\Omega}}\left(I - \Gamma_{\bar{\Omega}}^{\Phi}\right)\bar{K}_{\bar{\Omega}} B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega, \end{aligned}$$

which can be reformulated as

$$\begin{aligned} &a \cdot \bar{Q}_k - \bar{P}B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega \\ &+PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}}\left(I - B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega\right) \\ \geq\ &(PB_{\bar{\Omega}} - \bar{K}_{\bar{\Omega}}' N_{\bar{\Omega},k}^{\Phi} S_{\bar{\Omega}})(I - N_{\bar{\Omega},k}^{\Phi})\bar{K}_{\bar{\Omega}}. \quad (28) \end{aligned}$$

Note that for the special case $N_{\bar{\Omega},k}^{\Phi} = I$, which denotes that within the set $\bar{\Omega}$ actuator outage is not taken into account by the design, (28) is always satisfied if one chooses $\Gamma_\Omega^\Phi = 0$ and $\Gamma_{\bar{\Omega}}^{\Phi} \leq I$. These are the chosen conditions of the previously proposed modified TSLQRC [13].

Case 2: $N_{\bar{\Omega}} = 0$ and $N_\Omega \geq 0$. In this case, one has $N_{\bar{\Omega},k}^{\Phi} = 0$ and $\hat{N}_{\Omega,k}^{\Phi} = N_{\Omega,k}^{\Phi}$ a diagonal matrix in which all principal values are either zero or one and at least one value is one. Constraint (18) then becomes

$$\begin{aligned} &\bar{K}_\Omega' D_\Omega \bar{K}_\Omega + PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}} + a \cdot \bar{Q}_k \\ \geq\ &(I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega)' PB_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} (I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega) \\ &+PB_{\bar{\Omega}}\left(I - \Gamma_{\bar{\Omega}}^{\Phi}\right)\bar{K}_{\bar{\Omega}} B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega \\ &+\bar{P}B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega + \left(\bar{K}_\Omega' S_\Omega - \bar{P}B_\Omega\right) N_{\Omega,k}^{\Phi} \bar{K}_\Omega, \end{aligned}$$

which can be reformulated as

$$\begin{aligned} &a \cdot \bar{Q}_k - \bar{P}B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega \\ &+PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}}\left(I - B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega\right) \\ \geq\ &(I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega)' PB_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} (I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega) \\ &-\bar{P}B_\Omega N_{\Omega,k}^{\Phi} \bar{K}_\Omega - \bar{K}_\Omega' N_{\Omega,k}^{\Phi} S_\Omega (I - N_{\Omega,k}^{\Phi})\bar{K}_\Omega. \quad (29) \end{aligned}$$

Note that for the special case $\hat{N}_{\Omega,k}^{\Phi} = N_{\Omega,k}^{\Phi} = I$, (29) is always satisfied if one chooses $\Gamma_{\bar{\Omega}}^{\Phi} = 0$ and $\Gamma_\Omega^\Phi \leq I$.

Based on the above discussions, we conclude the following remark: the ATSLQRC may serve as a guaranteed cost control which achieves the performance bound (13) for actuators that may completely fail in either of the two sets $\bar{\Omega}$ and Ω (but of course not both) provided that the design parameters $\Gamma_{\bar{\Omega}}^{\Phi}$ and Γ_Ω^Φ both satisfy the constraints (28) and (29). Thus, we have the following Theorem.

Theorem 2. The state-feedback system obtained by applying the ATSLQRC (6)-(12) to system (1), which can achieve the performance bound (13) for actuators that may completely fail in either of the two sets $\bar{\Omega}$ and Ω provided that the design parameters $\Gamma_{\bar{\Omega}}^{\Phi}$ and Γ_Ω^Φ both satisfy the following two constraints:

$$(I) \qquad \Xi \geq 0 \qquad if \ \ \Theta \geq 0, \quad (30)$$

$$(II) \qquad \Xi \geq -\Theta \qquad if \ \ \Theta < 0, \quad (31)$$

where

$$\begin{aligned} \Xi \ =\ &a \cdot \bar{Q}_k - \bar{P}B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega - \Pi \\ &+PB_{\bar{\Omega}}(I - \Gamma_{\bar{\Omega}}^{\Phi})\bar{K}_{\bar{\Omega}}\left(I - B_\Omega \Gamma_\Omega^\Phi \bar{K}_\Omega\right), \quad (32) \\ \Theta \ =\ &\bar{P}B_\Omega N_{\Omega,k}^{\Phi} \bar{K}_\Omega + \bar{K}_\Omega' N_{\Omega,k}^{\Phi} S_\Omega (I - N_{\Omega,k}^{\Phi})\bar{K}_\Omega + \Pi \\ &-(I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega)' PB_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} (I - B_\Omega \hat{N}_{\Omega,k}^{\Phi} \bar{K}_\Omega), \quad (33) \\ \Pi \ =\ &(PB_{\bar{\Omega}} - \bar{K}_{\bar{\Omega}}' N_{\bar{\Omega},k}^{\Phi} S_{\bar{\Omega}})(I - N_{\bar{\Omega},k}^{\Phi})\bar{K}_{\bar{\Omega}}. \quad (34) \end{aligned}$$

Proof: It suffices to verify the following two conditions: 1) $\Theta \geq 0$ implies $\Xi \geq 0 \geq -\Theta$ and 2) $\Theta < 0$ implies $\Xi \geq -\Theta \geq 0$. This completes the proof. ∎

IV. A GUARANTEED COST CONTROL DESIGN VIA THE ATSLQRC

In this section, we will present an optimal guaranteed cost control design based on the two simplified constraints, i.e., (30) and (31). To simplify the design, the design parameters of the ATSLQRC, i.e., $\Gamma_{\bar{\Omega}}^{\Phi}$ and Γ_Ω^Φ, are taken as the following specific forms: $\Gamma_{\bar{\Omega}}^{\Phi} = \gamma_{\bar{\Omega}}^{\Phi} I$ and $\Gamma_\Omega^\Phi = \gamma_\Omega^\Phi I$.

Hence, the discrete-time algebraic Riccati equation (DARE) by (10) is rewritten as follows:

$$P = A'P(I - \gamma_{\bar{\Omega}}^{\Phi} B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}})(I - \gamma_{\Omega}^{\Phi} B_{\Omega} \bar{K}_{\Omega})A + Q. \quad (35)$$

In order to have a stabilizing solution of the DARE (35), the parameters $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} are constrained as follows: $0 \leq \gamma_{\bar{\Omega}}^{\Phi} \leq 1$ and $0 \leq \gamma_{\Omega}^{\Phi} \leq 1$.

Next, we note that if the estimates of actuator failures can be accurately obtained for $T = 1$ (see [14] for details), then one can always choose the initial estimates of actuator failures as large as possible such that $\bar{Q}_0 \rightarrow 0$ and $x_1'Px_1 \rightarrow x_0'A'PAx_0$. Thus, the upper performance bound (2) is given by $x_0'(Q + A'PA)x_0$. Furthermore, we assume the following more conservative design:

$$N_{\bar{\Omega},k}^{\Phi} = I, \quad \hat{N}_{\Omega,k}^{\Phi} = N_{\Omega,k}^{\Phi} = I, \quad (36)$$

which means that actuator outages only exist in either of the two sets $\bar{\Omega}$ and Ω but not both. Hence, using (30)-(34), (36), and $a = 0$ (we refer this kind of gain margin determination problem to guaranteed cost control design problem), we have the performance gain margins (PGM) of the ATSLQRC in the following Lemma (the proof is straightforward and omitted).

Lemma 1. The state-feedback system obtained by applying the ATSLQRC (6)-(12) to system (1), which can achieve the performance bound

$$J \leq x_0'(Q + A'PA)x_0, \quad (37)$$

and accommodate the following gain perturbations:

$$0 \leq n_{\bar{\Omega}}^{i} < \infty, \quad 0 \leq n_{\Omega}^{j} < \infty, \quad (38)$$

provided that $0 \leq \gamma_{\bar{\Omega}}^{\Phi} \leq 1$ and $0 \leq \gamma_{\Omega}^{\Phi} \leq 1$ both satisfy the following two guaranteed cost gain margin constraints:

$$(I) \quad \Xi \geq 0 \quad if \;\; \Theta \geq 0, \quad (39)$$

$$(II) \quad \Xi \geq -\Theta \quad if \;\; \Theta < 0, \quad (40)$$

where

$$\begin{aligned} \Xi &= \left(1 - \gamma_{\bar{\Omega}}^{\Phi}\right) PB_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \left(I - \gamma_{\Omega}^{\Phi} B_{\Omega} \bar{K}_{\Omega}\right) \\ &\quad -\gamma_{\Omega}^{\Phi} \bar{P} B_{\Omega} \bar{K}_{\Omega}, \quad (41) \\ \Theta &= \bar{P} B_{\Omega} \bar{K}_{\Omega} \\ &\quad -(I - B_{\Omega}\bar{K}_{\Omega})' P B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} (I - B_{\Omega} \bar{K}_{\Omega}). \quad (42) \end{aligned}$$

■

Then, we consider the practical issue how to solve (39) and (40). We first note that matrix Θ in (42) is a key fault indicator that shows which actuator failure model is the more influential one. If $\Theta \geq 0$, then one obtains that the true of constraint (39) will always promise the constraint in (40). In other words, the outage in the set Ω will yield the worst system's performance. We call the set Ω in this case by the dominant set. Thus, considering that the actuator outage may occur in the whole set, we only need to check the requirement: $\Xi \geq 0$, which is then named as the dominant constraint, to guarantee the whole system's performance cost. In such a case, we search for the range $0 \leq \gamma_{\bar{\Omega}}^{\Phi} \leq 1$, and then find the minimum and the maximum values of γ_{Ω}^{Φ}, which are denoted by $\gamma_{\Omega,min}^{\Phi}$ and $\gamma_{\Omega,max}^{\Phi}$, respectively, that will make matrix Ξ in (41) be a positive semidefinite one. On the other hand, if $\Theta < 0$, then one obtains that the dominant set is $\bar{\Omega}$ and the outage in the set $\bar{\Omega}$ will yield the worst system's performance. In this case, we search for the range $0 \leq \gamma_{\Omega}^{\Phi} \leq 1$, and then find the minimum and the maximum values of $\gamma_{\bar{\Omega}}^{\Phi}$ that will make matrix $[\Xi + \Theta]$ be a positive semidefinite one. We name the above obtained values of γ_{Ω}^{Φ} and $\gamma_{\bar{\Omega}}^{\Phi}$ as the feasible solutions of the ATSLQRC.

Finally, the proposed optimal guaranteed cost control is obtained via finding the optimal γ_{Ω}^{Φ} and $\gamma_{\bar{\Omega}}^{\Phi}$ of the above obtained feasible solutions that will achieve either 1) the minimal uppermost-performance cost denoted by $\bar{J}_{min}$ (the cost associated with the worst-case condition) or 2) the minimal lowermost-performance cost denoted by $\underline{J}_{min}$ (the cost associated with the nominal condition).

V. AN ILLUSTRATIVE EXAMPLE

To illustrate the proposed guaranteed cost control design method, the author considered the system in [12], which is given as follows:

$$A = \begin{bmatrix} 1.00 & 0.05 & 0.05 & 0.10 \\ -0.05 & 0.95 & 0.05 & 0.00 \\ 0.10 & 0.10 & 1.01 & 0.06 \\ 0.00 & 0.05 & 0.00 & 1.00 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.00 & 0.00 \\ 0.09 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.05 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, P_0 = I_4.$$

The actuator failure model is described as follows:

$$0 \leq n_{\bar{\Omega}} \leq 30, \quad 0 \leq n_{\Omega} \leq 30. \quad (43)$$

The diagnostic filter of the ATSLQRC, which estimates the actuator faults, is given as follows:

$$\begin{aligned} \begin{bmatrix} \hat{n}_{\bar{\Omega},k} \\ \hat{n}_{\Omega,k} \end{bmatrix} &= \bar{B}_k^{+}(x_k - Ax_{k-1}), \\ \bar{B}_k &= \begin{bmatrix} B_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} \hat{U}_k A x_k & B_{\Omega} \bar{K}_{\Omega} A x_k \end{bmatrix}, \quad (44) \\ \hat{U}_k &= I - \hat{n}_{\Omega,k-1} B_{\Omega} \bar{K}_{\Omega}. \end{aligned}$$

It should be mentioned that $\bar{B}_k$ given by (44) is of full column rank for this simulation example, which guarantees the faults estimates can be accurately obtained. To simplify the discussions, the actuator failures considered in this simulation example are focused on unknown fixed constants which are in the range of the specified failure model (43).

Applying the searching procedure given in the last section to the considered system, we obtained the feasible solutions

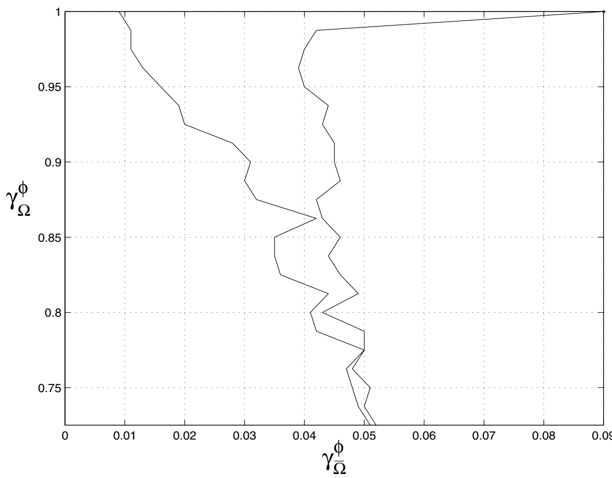

Fig. 1. The feasible solutions of $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} of the ATSLQRC

TABLE I

TWO OPTIMAL PERFORMANCE COSTS OF THE ATSLQRC

	$\gamma_{\bar{\Omega}}^{\Phi}$	γ_{Ω}^{Φ}	$n_{\bar{\Omega}}$	n_{Ω}	J	J^*
$\bar{J}_{min}$	0.038	0.875	0	30	353.50	3205.45
			30	0	326.37	
			30	30	259.84	
$\underline{J}_{min}$	0.09	1	0	30	460.48	524.67
			30	0	231.27	
			30	30	189.36	

of $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} in Fig. 1, from which it is clear that the dominant set in this example is $\bar{\Omega}$. From Fig. 1 one could roughly say that the smaller the value of $\gamma_{\bar{\Omega}}^{\Phi}$ is chosen the narrower the range of feasible γ_{Ω}^{Φ} may have. Since the feasible solutions do not include the case $\gamma_{\bar{\Omega}}^{\Phi}=0$, it is clear that the conventional TSLQRC control structure [11]-[14] can not be applied to the general fault tolerant control where any actuator may completely fail. Further, the conventional optimal control, which is obtained by choosing $\gamma_{\bar{\Omega}}^{\Phi}=1$ and $\gamma_{\Omega}^{\Phi}=1$, is also not applicable to the present addressed problem. Also note that the choice: $\gamma_{\bar{\Omega}}^{\Phi}=0$ and $\gamma_{\Omega}^{\Phi}=1$, which corresponds to the more conservative design suggested by Veillette [3], is still not applicable.

As a final step, we search for the optimal $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} that will achieve the optimal criterion: $\bar{J}_{min}$ or $\underline{J}_{min}$. Two optimal performance costs of the ATSLQRC, which corresponds to $\bar{J}_{min}$ and $\underline{J}_{min}$, are illustrated by Tab. 1. Also listed in Tab. 1 are their corresponding actual performance costs denoted by J and upper performance bounds denoted by J^*.

From Tab. 1, we observed the following: 1) the larger the values of $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} are chosen, the smaller the upper performance bound may result, and 2) the smaller the upper performance bound is obtained, the larger the worst performance cost (corresponding to the case $n_{\bar{\Omega}}=0$ and $n_{\Omega}=30$) and the smaller the nominal performance cost (corresponding to the case $n_{\bar{\Omega}}=30$ and $n_{\Omega}=30$) will result. The reason behind these observations is mainly due to the fact that the larger the values of $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} are chosen the smaller the stabilizing solution of the DARE (35). And hence the smaller performance bound and less reliability will be achieved. Similar results can also be found in [11] and [13].

VI. APPLICATION TO A PASSIVE FAULT TOLERANT CONTROL DESIGN

In this section, we shall show that the unified gain margin constraint of the ATSLQRC given in Theorem 1 can be applied to solve a passive fault tolerant control design problem [8], which is stated as below. Consider a system of the form

$$x_{k+1}=Ax_k+B_1u_k^1+B_2u_k^2+\cdots+B_mu_k^m. \tag{45}$$

Assume that each of the pairs $(A,B_i), i=1,\cdots,m$, is stabilizable. Then, the reliable control design problem is to derive a fixed controller K_kx_k (assuming full state feedback) such that the nominal control law

$$u_k=\begin{bmatrix} u_k^1 & u_k^2 & \cdots & u_k^m \end{bmatrix}'=K_kx_k,$$

as well as each of the m control laws

$$u_k=\begin{bmatrix} 0 \\ u_k^2 \\ \vdots \\ u_k^m \end{bmatrix}, u_k=\begin{bmatrix} u_k^1 \\ 0 \\ \vdots \\ u_k^m \end{bmatrix}, \cdots, u_k=\begin{bmatrix} u_k^1 \\ u_k^2 \\ \vdots \\ 0 \end{bmatrix}, \tag{46}$$

internally stabilizes system (45).

To apply the ATSLQRC to the above fault tolerant control design problem, we first define the sets $\bar{\Omega}$ and Ω as follows: $\bar{\Omega}=\{1,2,\cdots,p\}$ and $\Omega=\{p+1,p+2,\cdots,m\}$, where p is a suitable chosen number. Since the actuator fault is in the form of outage, we only need to consider the following two special actuator failures: 1) $N_{\bar{\Omega}}=I$ & $N_{\Omega}=0$ and 2) $N_{\bar{\Omega}}=0$ & $N_{\Omega}=I$. In order to derive a fixed structure of the controller, the estimated faults are chosen by $\hat{N}_{\bar{\Omega},k}=I$ and $\hat{N}_{\Omega,k}=I$. Accordingly, the dedicated control is given as follows:

$$u_k=-\begin{bmatrix} \bar{K}_{\bar{\Omega}}(I-B_{\Omega}\bar{K}_{\Omega}) \\ \bar{K}_{\Omega} \end{bmatrix}Ax_k. \tag{47}$$

The problem remains to find $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} which satisfy the unified gain margin constraint (18), in which a is chosen by $a=1$ (we refer this kind of gain margins determination problem to reliable control design problem). The relationship between $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} is then given in the following Lemma (the proof is straightforward and is omitted).

Lemma 2. The state-feedback system obtained by applying the fixed controller (47) to system (1), which remains stable and can accommodate the faults in (46) provided that $0\leq\gamma_{\bar{\Omega}}^{\Phi}\leq 1$ and $0\leq\gamma_{\Omega}^{\Phi}\leq 1$ both satisfy the following two guaranteed cost gain margin constraints:

$$(I) \qquad \Xi\geq-\Psi \qquad\qquad if\ \ \Delta\geq 0, \tag{48}$$

$$(II) \qquad \Xi\geq-\Psi-\Delta \qquad\qquad if\ \ \Delta<0, \tag{49}$$

where Ξ is given by (41) and

$$\Psi = (I - B_\Omega \bar{K}_\Omega)' \bar{K}'_{\bar{\Omega}} R_{\bar{\Omega}} \bar{K}_{\bar{\Omega}} (I - B_\Omega \bar{K}_\Omega), \quad (50)$$
$$\Delta = \Theta + \bar{K}'_\Omega R_\Omega \bar{K}_\Omega - \Psi, \quad (51)$$

in which Θ is given by (42). ■

From Lemma 2, it is clear that if $\gamma_{\bar{\Omega}}^{\Phi}$ and γ_{Ω}^{Φ} both satisfy (48) and (49), then the fixed controller (47) tends to stabilize both the nominal system as well as the faulty system in which the actuators in either of (not both) the two sets $\bar{\Omega}$ and Ω may completely fail.

Then, we want to show that the above results can be directly applied to the faults (46). Without loss of generality, we only consider the following two special cases. The first one is illustrated by the fault: $u_k^1 = 0$ and $u_k^i \neq 0$, where $i = 2, \cdots, m$, and the other $u_k^{p+1} = 0$ and $u_k^j \neq 0$, where $j = 1, \cdots, p, p+2, \cdots, m$. In the first case, one has

$$N_{\Omega,k}^{\Phi} = I, \ N_{\bar{\Omega},k}^{\Phi} = N_1 = diag\{0_{1\times 1}, I_{(p-1)\times(p-1)}\}. \quad (52)$$

Using (18), (41), and (50)-(52), one has

$$\begin{aligned} & \Xi + \Psi + \Delta \\ \geq\ & -\hat{U}'_k P B_{\bar{\Omega}} N_1 S_{\bar{\Omega}}^{-1} B'_{\bar{\Omega}} P \hat{U}_k \\ & -\hat{U}'_k \bar{K}'_{\bar{\Omega}} N_1 \left(R_{\bar{\Omega}} + B'_{\bar{\Omega}} P B_{\bar{\Omega}} (I - N_1) \right) \bar{K}_{\bar{\Omega}} \hat{U}_k. \quad (53) \end{aligned}$$

Since the RHS of (53) is negative semidefinite, it is clear that (49) implies (53), which verifies the first case. On the other hand, one has

$$N_{\bar{\Omega},k}^{\Phi} = I, \quad N_{\Omega,k}^{\Phi} = N_1, \quad (54)$$

for the second case. Using (18), (41), (50), and (52), one has

$$\begin{aligned} & \Xi + \Psi \\ \geq\ & -\bar{P} B_\Omega N_1 S_\Omega^{-1} B'_\Omega \bar{P} \\ & -\bar{K}'_\Omega N_1 \left(R_\Omega + B'_\Omega \bar{P} B_\Omega (I - N_1) \right) \bar{K}_\Omega. \quad (55) \end{aligned}$$

Since the RHS of (55) is negative semidefinite, it is clear that (48) implies (55). This has verified the second case.

The results of this section suggest that the passive reliable control design problem considered here may be solved by using a more simple and compact method, which is just like a modified optimal control, as compared to that given by [8], where the dynamical order of the resulting controller for some systems may have to be considerably large, or by [10], where the controller order is reduced by a time-scheduling switch of multiple models. Also, the guaranteed cost performance of this new passive control can be easily addressed (the same as that discussed in the last section). Moreover, the proposed two-stage design method may also suggest a possible approach to solve the fault tolerant control problem in which several actuators can fail simultaneously, which appears to be a subject of future research raised by [8].

VII. CONCLUSIONS

A new fault tolerant control which can stabilize the system not only during its nominal condition but also in the case that several actuators may completely fail simultaneously is proposed. The existence of this controller is promised through verifying a new proposed guaranteed cost gain margin constraint. A guaranteed cost control design method and a passive fault tolerant control design method are presented to show the applications of the proposed results. This research suggests that the proposed control serves as an effective method to derive a practical fault tolerant control system where any one of the actuators might fail. And, the proposed two-stage design method may also suggest a possible approach to solve the fault tolerant control problem in which several actuators can fail simultaneously.

VIII. ACKNOWLEDGMENTS

This research was supported by the National Science Council, Taiwan, R.O.C. under Grant NSC 93-2213-E-233-006. The author also gratefully acknowledges the contribution of reviewers' comments.

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