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# Stabilization of Rigid Body Dynamics using the Serret-Andoyer Variables

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*Abstract*— **This paper develops a new controller for stabilization of rigid body dynamics. The state-space model is formulated using canonical elements, known as the Serret-Andoyer variables, thus far unused for engineering applications. The controllability of the problem is examined and a damping feedback is derived using the Jurdjevic-Quinn method. It is shown that the new feedback controller is an asymptotic smooth feedback stabilizer. The performance of the new controller is examined in a simulation, showing excellent dynamic closed-loop behavior.**

## I. Introduction

One of the classical problem of mechanics is that of a free motion of a rigid body, usually referred to as the *Euler-Poinsot* problem. The formulation and solution of this problem are usually performed by following two distinct steps: The *dynamics* of the rotation are represented by using differential equations written for the components of the body angular velocity, and then the *kinematic* equations are utilized to transform the body angular velocity into a spatial inertial frame. It is well known that the Euler-Poinsot problem admits a closed-form solution in terms of Jacobi's elliptic functions [1]. While this classical formulation is widespread among engineers, astronomers exhibit a marked preference for *canonical* variables, as they naturally accommodate modelling of disturbing torques, permitting convenient analysis of planetary librations.

The canonical variables for modelling rigid-body dynamics and kinematics are known as the *Serret-Andoyer* (SA) variables. The 19th century French mathematician Joseph Alfred Serret discovered these variables by solving the Hamilton-Jacobi equation written in terms of Eulerian coordinates [2]. Serret's treatment was later simplified by Radeau[3] and Tisserand [4]. However, this approach was widely popularized by Andoyer [5], who used spherical trigonometry to show that the Serret transformation was simply a change of Eulerian coordinates that depended upon the angular momentum components. Deprit [6] established the canonicity of the Serret transformation by using differential forms and without resorting to finding a generating function.

Probably the most distinguished feature of the SA variables is the reduction of the rotational dynamics to a single degree of freedom. In essence, the SA formulation reduces the dynamics by capturing the underlying symmetry of the free rigid body problem, stemming from conservation of energy and angular momentum. Hence, the SA formulation entails differential equations for an Eulerian angle and its conjugate momenta, which are readily integrable by quadrature. The single degree-of-freedom Hamiltonian yields a phase portrait which is similar to that of the simple pendulum, containing a *separatrix* confining the librational motion of the rigid body [6].

Remarkably, although the SA representation is ubiquitous in astronomical applications,[7], [8] most engineering textbooks dealing with classical mechanics and rigid body dynamics do not discuss the SA representation. This is true for both classical textbooks, such as Kane et al. [9], and more recent textbooks, such as Schaub and Junkins [10]. Hence, so far the SA variables have not been utilized in engineering applications, such as derivation of stabilizing controllers, a problem extensively dwelt upon in the literature [11], [12], [13], [14], [15]. The purpose of this paper, therefore, is to use the SA setup to derive a stabilizing controller for a (not necessarily symmetric) rigid satellite. We first present the SA formalism and develop a state-space model using the SA variables. We study controllability of the rigid-body dynamics and then develop a stabilizing Lyapunov-based controller based on the Jurdjevic-Quinn method [16]. We give closed-loop expressions for the stabilizing controller and show that it is smooth. A simulation of nadir stabilization is then carried out to illustrate the performance of the proposed controller.

## II. Eulerian Variables

Consider a rotation of a rigid body about its center of mass, $O$. The body frame, $\mathscr{B}$, is a Cartesian, dextral frame centered at $O$ defined by the unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, constituting the fundamental plane, and $\mathbf{b}_3 = \mathbf{b}_1 \times \mathbf{b}_2$. The attitude of $\mathscr{B}$ will be studied relative to an inertial, Cartesian, dextral frame, $\mathscr{I}$, defined by the the unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$ lying on the fundamental plane, and $\mathbf{s}_3 = \mathbf{s}_1 \times \mathbf{s}_2$. These frames are shown in Fig. 1. To rotate from $\mathscr{I}$ to $\mathscr{B}$, we shall utilize three consecutive rotations using the $3-1-3$ sequence. To this end, we define the line-of-nodes (LON) $OQ_2$, obtained from the intersection of the body fundamental plane and the inertial fundamental plane, as shown in Fig. 1. Let $\mathbf{l}$ be a unit vector coinciding with $OQ_2$. The rotation sequence can now be defined as follows:

- $R(\phi, \mathbf{s}_3)$, a rotation about $\mathbf{s}_3$ by $0 \le \phi \le 2\pi$, mapping $\mathbf{s}_1$ onto $\mathbf{l}$.
- $R(\theta, \mathbf{l})$, a rotation about $\mathbf{l}$ by $0 \le \theta \le \pi$, mapping $\mathbf{s}_3$ onto $\mathbf{b}_3$.
- $R(\psi, \mathbf{b}_3)$, a rotation about $\mathbf{b}_3$ by $0 \le \psi \le 2\pi$, mapping $\mathbf{l}$ onto $\mathbf{b}_1$.

The composite rotation, $R \in SO(3)$, transforming any inertial vector into the body frame, is given by

$$R(\phi, \theta, \psi) = R(\psi, \mathbf{b}_3) R(\theta, \mathbf{l}) R(\phi, \mathbf{s}_3) \tag{1}$$

Evaluating Eq. (1) gives

$$R(\phi,\theta,\psi) = \begin{bmatrix} c_\psi c_\phi - s_\psi c_\theta s_\phi & c_\psi s_\phi + s_\psi c_\theta c_\phi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\psi c_\theta s_\phi & -s_\psi s_\phi + c_\psi c_\theta c_\phi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix} \tag{2}$$

where we have used the compact notation $s_x = \sin(x), c_x = \cos(x)$.

To get the kinematic equations, we recall that the *body angular velocity* vector, $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ satisfies [9]

$$\widehat{\boldsymbol{\omega}} = -\dot{R} R^T \tag{3}$$

where

$$\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{4}$$

Substituting Eq. (2) into (4) yields the well-known relations

$$\omega_1 = \dot{\phi} s_\theta s_\psi + \dot{\theta} c_\psi \tag{5}$$

$$\omega_2 = \dot{\phi} c_\psi s_\theta - \dot{\theta} s_\psi \tag{6}$$

$$\omega_3 = \dot{\psi} + \dot{\phi} c_\theta \tag{7}$$

The attitude dynamics is usually formulated using the *Euler-Poinsot* equations

$$\mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega} = T\mathbf{u} \tag{8}$$

where $\mathbf{u}$ is a torque input in body axes, $T$ is the torque input matrix and $\mathbb{I}$ is the tensor of inertia. Assuming that the body axes coincide with the principal axes of inertia, we can write

$$\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3). \tag{9}$$

Alternatively, we can utilize a Hamiltonian formalism to obtain the equations for attitude dynamics. To that end, we shall assume now that the rotational motion is free, $\mathbf{u} = 0$, and later introduce $\mathbf{u}$ back into Hamilton's equations.

The Lagrangian of the free motion equals the rotational kinetic energy, i. e.

$$\mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega} \tag{10}$$

After substituting Eqs. (5)-(7) and (9) into (10), the conjugate momenta $\Phi, \Theta, \Psi$ are calculated using the configuration variables $(\phi, \theta, \psi)$ [17], [18]:

$$\begin{aligned} \Phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_1 s_\theta s_\psi (\dot{\phi} s_\psi s_\theta + \dot{\theta} c_\psi) + I_2 s_\theta c_\psi (\dot{\phi} c_\psi s_\theta - \dot{\theta} s_\psi) \\ &+ I_3 c_\theta (\dot{\phi} c\theta + \dot{\psi}) \end{aligned} \tag{11}$$

$$\Theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_1 c_\psi (\dot{\phi} s_\theta s_\psi + \dot{\theta} c_\psi) - I_2 s_\psi (\dot{\phi} c_\phi s_\theta - \dot{\theta} s_\psi) \tag{12}$$

$$\Psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3 (\dot{\phi} c_\theta + \dot{\psi}) \tag{13}$$

To study the relationships between the body rotational angular momentum, $\mathbf{G}$, and the conjugate momenta, let $\mathbf{G} = g_1 \mathbf{b}_1 + g_2 \mathbf{b}_2 + g_3 \mathbf{b}_3$ be the angular momentum written in the body frame and $\mathbf{G} = G_1 \mathbf{s}_1 + G_2 \mathbf{s}_2 + G_3 \mathbf{s}_3$ be the angular momentum in the inertial frame. Since

$$\mathbf{G} = \mathbb{I}\boldsymbol{\omega}, \tag{14}$$

one arrives at

$$g_1 = \frac{\Phi s_\psi + \Theta s_\theta c_\psi - \Psi s_\psi c_\theta}{s_\theta} \tag{15}$$

$$g_2 = \frac{\Phi c_\psi - \Theta s_\psi s_\theta - \Psi c_\psi c_\theta}{s_\theta} \tag{16}$$

$$g_3 = \Psi \tag{17}$$

$$G_1 = \frac{\Psi s_\phi + \Theta s_\theta c_\phi - \Phi s_\phi c_\theta}{s_\theta} \tag{18}$$

$$G_2 = \frac{\Phi c_\theta c_\phi - \Psi c_\phi + \Theta s_\phi s_\theta}{s_\theta} \tag{19}$$

$$G_3 = \Phi \tag{20}$$

Finally, to get the Hamiltonian we shall use the Legendre transformation

$$\mathcal{H} = \Phi\dot{\phi} + \Theta\dot{\theta} + \Psi\dot{\psi} - \mathcal{L} \tag{21}$$

Substitution yields

$$\begin{aligned} \mathcal{H}(\phi, \theta, \psi, \Phi, \Theta, \Psi) &= \frac{1}{2}\left(\frac{s_\psi^2}{I_1} + \frac{c_\psi^2}{I_2}\right)\left(\frac{\Phi - \Psi c_\theta}{s_\theta}\right)^2 \\ &+ \frac{\Psi^2}{2I_3} + \frac{1}{2}\left(\frac{c_\psi^2}{I_1} + \frac{s_\psi^2}{I_2}\right)\Theta^2 \\ &+ \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{\Phi - \Psi c_\theta}{s_\theta}\right)\Theta s_\psi c_\psi \end{aligned} \tag{22}$$

We note that in the Hamiltonian (22), the coordinate $\phi$ is *cyclic* (ignorable), so that the rotation is symmetric about the inertial axis $\mathbf{s}_3$ and therefore the momentum variable $\Phi$ is an integral of the motion. This symmetry implies a reduction to only two-degrees of freedom. based on this result, Serret [2] has raised the following question: Is there a canonical transformation that further reduces the system into a *single* degree of freedom? We shall dwell upon this issue in the following section.

## III. The Serret-Andoyer Transformation

The method originally utilized by Serret [2] for reduction of the Hamiltonian (22) was to find a *canonical transformation* of the form

$$(\phi, \theta, \psi, \Phi, \Theta, \Psi) \to (g, h, l, G, H, L) \tag{23}$$

by solving the resulting Hamilton-Jacobi equation for Hamilton's principle function, $\mathcal{S}$. However, to render the treatment simpler, we shall utilize here a differential form-based approach [17] to derive a canonical transformation.

We shall first use Andoyer's geometric insight to find simple relations between the Eulerian and the SA variables. To that end, we shall consider a transformation from the inertial to the body frame via an *intermediate* frame defined by the angular momentum vector and the invariable plane, as shown in Fig. (1).

Referring to Fig. (1), let $OQ_1$ and $OQ_3$ denote the LON's obtained from the intersection of the invariable plane with the inertial and body planes, respectively. Let $\mathbf{i}$ be a unit vector along the direction of $OQ_1$ and $\mathbf{j}$ be a unit vector along $OQ_3$. Define a $3-1-3-1-3$ rotation sequence as follows:

- $R(h, \mathbf{s}_3)$, a rotation about $\mathbf{s}_3$ by $0 \le h < 2\pi$, mapping $\mathbf{s}_1$ onto $\mathbf{i}$.
- $R(\sigma, \mathbf{i})$, a rotation about $\mathbf{i}$ by $0 \le \sigma < \pi$, mapping $\mathbf{s}_3$ onto the angular momentum vector, $\mathbf{G}$.
- $R(g, \mathbf{G}/G)$, a rotation about a unit vector pointing in the direction of the angular momentum by $0 \le g < 2\pi$, mapping $\mathbf{i}$ onto $\mathbf{j}$.
- $R(\beta, \mathbf{j})$, a rotation about $\mathbf{j}$ by $0 < \beta < \pi$, mapping $\mathbf{G}$ onto $\mathbf{b}_3$.
- $R(l, \mathbf{b}_3)$, a rotation about $\mathbf{b}_3$ by $0 \le l \le 2\pi$, mapping $\mathbf{j}$ onto $\mathbf{b}_1$.

The composite rotation, $R \in SO(3) \times SO(2)$, may be written as

$$R(h, \sigma, g, \beta, l) = R(l, \mathbf{b}_3) R(\beta, \mathbf{j}) R(g, \mathbf{G}/G) R(\sigma, \mathbf{i}) R(h, \mathbf{s}_3) \tag{24}$$

A sufficient condition for the transformation (23) to be canonical, given that the Hamiltonian is time-independent, can be formulated using perfect differentials as follows [10]:

$$\Phi d\phi + \Theta d\theta + \Psi d\psi = L dl + G dg + H dh \tag{25}$$

Let us first evaluate the left-hand side of (25). The perfect differential of $R(\phi, \theta, \psi)$ is readily found to be [17]

$$dR(\phi, \theta, \psi) = \mathbf{s}_3 d\phi + \mathbf{l} d\theta + \mathbf{b}_3 d\psi \tag{26}$$

Multiplying both sides of (26) by $\mathbf{G}$ while taking advantage of the identities (cf. (34)-(20))

$$\mathbf{G} \cdot \mathbf{s}_3 = \Phi, \quad \mathbf{G} \cdot \mathbf{l} = \mathbf{G} \cdot (\mathbf{s}_1 \mathrm{c}_\phi + \mathbf{s}_2 \mathrm{s}_\phi) = \Theta, \quad \mathbf{G} \cdot \mathbf{b}_3 = \Psi, \tag{27}$$

yields the symplectic structure

$$\mathbf{G} \cdot dR = \Phi d\phi + \Theta d\theta + \Psi d\psi \tag{28}$$

We shall now repeat the above procedure for the right-hand side of (25). The perfect differential of $R(h, \sigma, g, \beta, l)$ is evaluated similarly to (26) so as to get

$$dR(h, \sigma, g, \beta, l) = \mathbf{s}_3 dh + \mathbf{i} d\sigma + \mathbf{G}/G dg + \mathbf{j} d\beta + \mathbf{b}_3 dl \tag{29}$$

Since

$$\mathbf{G} \cdot \mathbf{s}_3 = \Phi, \mathbf{G} \cdot \mathbf{i} = 0, \mathbf{G} \cdot \mathbf{G}/G = G, \mathbf{G} \cdot \mathbf{j} = 0, \mathbf{G} \cdot \mathbf{b}_3 = \Psi, \tag{30}$$

multiplying both sides of (29) by $\mathbf{G}$ gives

$$\mathbf{G} \cdot dR = \Phi dh + G dg + \Psi dl \tag{31}$$

Hence, choosing

$$\Phi = H, \quad \Psi = L \tag{32}$$

will render the transformation (23) canonical.

It can be shown that the resulting single degree-of-freedom Hamiltonian is

$$\mathcal{H}(g, h, l, G, H, L) = \frac{1}{2} \left( \frac{\mathrm{s}_l^2}{I_1} + \frac{\mathrm{c}_l^2}{I_2} \right) (G^2 - L^2) + \frac{L^2}{2I_3} \tag{33}$$

Alternatively, we could have utilized the relations [6]

$$\begin{aligned} g_1 &= I_1 \omega_1 = G \mathrm{s}_\beta \mathrm{s}_l = \sqrt{G^2 - L^2} \mathrm{s}_l && (34) \\ g_1 &= I_2 \omega_2 = G \mathrm{s}_\beta \mathrm{c}_l = \sqrt{G^2 - L^2} \mathrm{c}_l && (35) \\ g_3 &= I_3 \omega_3 = L && (36) \end{aligned}$$

to obtain the same result.

We note that in the new Hamiltonian the coordinates $g, h$ are cyclic, and hence the momenta $G, H$ are integrals of the motion. Also, since $G \ge L$,

$$\mathcal{H} \ge 0 \tag{37}$$

To get the canonical equations of motion, denote the generalized coordinates by $\mathbf{q} = [g, h, l]^T$ and the conjugate momenta by $\mathbf{p} = [G, H, L]^T$. Hamilton's equations in the absence of external torques are

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}} && (38) \\ \dot{\mathbf{p}} &= -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} && (39) \end{aligned}$$

Evaluating Hamilton's equations for (33), yields the canonical equations of free rotational motion:

$$\begin{aligned} \dot{g} &= \frac{\partial \mathcal{H}}{\partial G} = G \left( \frac{\mathrm{s}_l^2}{I_1} + \frac{\mathrm{c}_l^2}{I_2} \right) && (40) \\ \dot{h} &= \frac{\partial \mathcal{H}}{\partial H} = 0 && (41) \\ \dot{l} &= \frac{\partial \mathcal{H}}{\partial L} = L \left( \frac{1}{I_3} - \frac{\mathrm{s}_l^2}{I_1} - \frac{\mathrm{c}_l^2}{I_2} \right) && (42) \\ \dot{G} &= -\frac{\partial \mathcal{H}}{\partial g} = 0 && (43) \\ \dot{H} &= -\frac{\partial \mathcal{H}}{\partial h} = 0 && (44) \\ \dot{L} &= -\frac{\partial \mathcal{H}}{\partial l} = (L^2 - G^2) \mathrm{s}_l \mathrm{c}_l \left( \frac{1}{I_1} - \frac{1}{I_2} \right) && (45) \end{aligned}$$

Eqs. (42), (45) are separable differential equations, and hence can be solved in *closed form* utilizing the fact that $\mathcal{H}$ is constant (implying conservation of energy) [17].

## IV. Controlling the Rigid Body using the Serret-Andoyer Setup

In order to introduce control torques into the SA setup, we shall first obtain an expression for $\dot{\boldsymbol{\omega}}$ based upon

Eqs. (34)-(36):

$$I_1\dot{\omega}_1 = \frac{\mathrm{s}_l(G\dot{G}-L\dot{L})}{\sqrt{G^2-L^2}} + \dot{l}\mathrm{c}_l\sqrt{G^2-L^2} \quad (46)$$

$$I_2\dot{\omega}_2 = \frac{\mathrm{c}_l(G\dot{G}-L\dot{L})}{\sqrt{G^2-L^2}} - \dot{l}\mathrm{s}_l\sqrt{G^2-L^2} \quad (47)$$

$$I_3\dot{\omega}_3 = \dot{L} \quad (48)$$

Substituting (34)-(36) and (46)-(48) into the Euler-Poinsot equations (8) and solving for $\dot{l}, \dot{G}, \dot{L}$, will yield modified expressions for these derivatives reflecting the effect of a control input:

$$\dot{l} = L\left(\frac{1}{I_3} - \frac{\mathrm{s}_l^2}{I_1} - \frac{\mathrm{c}_l^2}{I_2}\right) + \mathbf{w}_1^T T\mathbf{u} \quad (49)$$

$$\dot{G} = \mathbf{w}_2^T T\mathbf{u} \quad (50)$$

$$\dot{L} = (L^2-G^2)\mathrm{s}_l\mathrm{c}_l\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + \mathbf{w}_3^T T\mathbf{u} \quad (51)$$

where the vector fields $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are given by

$$\mathbf{w}_1 = \frac{1}{\sqrt{G^2-L^2}}\begin{bmatrix} \mathrm{c}_l \\ -\mathrm{s}_l \\ 0 \end{bmatrix}$$

$$\mathbf{w}_2 = \frac{1}{G}\begin{bmatrix} \mathrm{s}_l\sqrt{G^2-L^2} \\ \mathrm{c}_l\sqrt{G^2-L^2} \\ L \end{bmatrix}$$

$$\mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (52)$$

The state-space for the state $\mathbf{x} = [l, G, L]^T$ is the manifold $\mathcal{X} = \mathbb{S}^1 \times S$, where $S$ is the foliation $S = \{(g_1, g_2, g_3)|g_1^2 + g_2^2 + g_3^2 = G^2\}$. The state-space dynamics can be now written as

$$\Sigma(\mathbf{x}, \mathbf{u}) : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + WT\mathbf{u} \quad (53)$$

where $\mathbf{f} : \mathcal{X} \to \mathbb{R}^3$ is the vector field

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} L\left(\frac{1}{I_3} - \frac{\mathrm{s}_l^2}{I_1} - \frac{\mathrm{c}_l^2}{I_2}\right) \\ 0 \\ (L^2-G^2)\mathrm{s}_l\mathrm{c}_l\left(\frac{1}{I_1} - \frac{1}{I_2}\right) \end{bmatrix} \quad (54)$$

and

$$W = \begin{bmatrix} \frac{\mathrm{c}_l}{\sqrt{G^2-L^2}} & -\frac{\mathrm{s}_l}{\sqrt{G^2-L^2}} & 0 \\ \frac{\mathrm{s}_l\sqrt{G^2-L^2}}{G} & \frac{\mathrm{c}_l\sqrt{G^2-L^2}}{G} & \frac{L}{G} \\ 0 & 0 & 1 \end{bmatrix} \quad (55)$$

The equilibria of $\Sigma$ are

$$\{L_{eq} = 0, l_{eq} = 0, \frac{\pi}{2}, \pi, G_{eq} \neq 0\}, \{G_{eq} = L_{eq} = 0, l_{eq} \neq 0\} \quad (56)$$

In this work we shall be primarily interested in deriving a globally stabilizing controller for system $\Sigma$. We shall do so by first showing that $\Sigma$ is *accessible*, find a Lyapunov function for $\Sigma(\mathbf{x}, 0)$ and then utilize the *Jurdjevic-Quinn* formalism [16] to derive the feedback controller. To that end, let us revisit some nonlinear controllability and stabilization results pertinent to our the investigation.

The first step in synthesizing a controller for any dynamical system is to determine whether the system is controllable. In order to examine controllability properties of the nonlinear systems dwelt upon in the sequel, we shall adopt the notion of *accessibility* [19], a weaker form of controllability, defined as follows.

*Definition 1:* Let $\mathcal{R}_T(\mathbf{x}_0)$ denote the set of states reachable from the initial state $\mathbf{x}_0$ in a finite time $t_f$ using some admissible control $\mathbf{u} \in U$. The system $\Sigma$ is said to be *accessible* from $\mathbf{x}_0$, if $\mathcal{R}_T(\mathbf{x}_0)$ has a nonempty interior in $\mathcal{X}$.

In order to provide sufficient conditions for accessibility, we review a few basics of differential geometry. First, recall that the Lie bracket is a binary operation which associates to an (ordered) pair of vector fields, $\mathbf{w}_0,\ \mathbf{w}_1$, the vector field

$$[\mathbf{w}_0,\ \mathbf{w}_1] = D\mathbf{w}_1 \cdot \mathbf{w}_0 - D\mathbf{w}_0 \cdot \mathbf{w}_1 \quad (57)$$

where $D\mathbf{w}_i$ denotes the Jacobian matrix of $\mathbf{w}_i$. The "*ad*" operator is iteratively defined by

$$ad^1_{\mathbf{w}_0}\mathbf{w}_i = [\mathbf{w}_0,\ \mathbf{w}_i],\ ad^{k+1}_{\mathbf{w}_0}\mathbf{w}_1 = [\mathbf{w}_0,\ ad^k_{\mathbf{w}_0}\mathbf{w}_i] \quad (58)$$

We can now write the smooth (differential geometric) distribution

$$\Delta(\mathbf{x}) = \mathrm{span}\{\mathbf{w}_i(\mathbf{x}),\ ad^{k+1}_{\mathbf{w}_0}\mathbf{w}_i(\mathbf{x}),\ i = 0, ...m\}. \quad (59)$$

The following theorem provides a sufficient condition for accessibility [19]:

*Theorem 1 (accessibility rank condition):* System $\Sigma$ is accessible from $\mathbf{x}_0$ if $\dim\Delta(\mathbf{x}_0) = n$.

We shall determine accessibility of $\Sigma$ by considering the distribution

$$\Delta = \{\mathbf{w}_0,\ \mathbf{w}_1,\ \mathbf{w}_2,\ [\mathbf{w}_0, \mathbf{w}_1], [\mathbf{w}_0, \mathbf{w}_2], [\mathbf{w}_0, \mathbf{w}_3]\} \quad (60)$$

where

$$[\mathbf{w}_0, \mathbf{w}_1] = \begin{bmatrix} \frac{(a_1-a_3)L\mathrm{s}_l}{\sqrt{G^2-L^2}} \\ \frac{(a_3-a_2)L\mathrm{c}_l\sqrt{G^2-L^2}}{G} \\ (a_1-a_2)\mathrm{c}_l\sqrt{G^2-L^2} \end{bmatrix}$$

$$[\mathbf{w}_0, \mathbf{w}_2] = \begin{bmatrix} \frac{(a_2-a_3)L\mathrm{c}_l}{\sqrt{G^2-L^2}} \\ \frac{(a_1-a_3)L\mathrm{s}_l\sqrt{G^2-L^2}}{G} \\ (a_1-a_2)\mathrm{s}_l\sqrt{G^2-L^2} \end{bmatrix}$$

$$[\mathbf{w}_0, \mathbf{w}_3] = \begin{bmatrix} (a_1-a_3) + (a_2-a_1)\mathrm{c}_l^2 \\ \frac{(a_2-a_1)(G^2-L^2)\mathrm{s}_{2l}}{2G} \\ 0 \end{bmatrix}, \quad (61)$$

and

$$a_i = 1/I_i,\ i = 1, 2, 3. \quad (62)$$

A calculation of rank($\Delta$) will determine accessibility. To that end, we shall use the identity rank$\Delta$ = rank$\Delta\Delta^T$ and examine for which values of the state variables $\det(\Delta\Delta^T) = 0$. Performing the symbolic calculation shows that the solution set is

$$Z = \{l = 0, G = l\} \tag{63}$$

Hence, the system is accessible $\forall \mathbf{x} \in \mathcal{X}\backslash Z$.

A concept related (but, in the nonlinear case, not necessarily identical) to accessibility is that of feedback stabilizability. The *feedback stabilization problem* is usually stated as follows: Given some set-point $\mathbf{x}_d \in \mathcal{X}$, find a feedback control law $u \in \mathcal{U}$ that renders $\mathbf{x}_d$ an asymptotically stable equilibrium (i.e. Lyapunov stable and attractive) of $\Sigma$. If such a feedback exits, it is called an *internal asymptotic feedback stabilizer* of $\Sigma$. If $K(\mathbf{x})$ is smooth, it is called a *smooth internal asymptotic feedback stabilizer*. We also distinguish between local and global feedback stabilizers.

To find a feedback control law for the attitude dynamics, we shall utilize a Lyapunov-based approach, summarized by the following theorem [16]:

*Theorem 2 (Jurdjevic-Quinn):* Assume that $\Sigma(\mathbf{x}_0, \mathbf{u})$ is accessible $\forall \mathbf{x}_0 \in \mathcal{X}$. Consider the unforced system $\Sigma(\mathbf{x}, \mathbf{0})$. If $\exists V : \mathcal{X} \to \mathbb{R}_{>0}$ for $\Sigma(\mathbf{x}, \mathbf{0})$ satisfying

$$V \in \mathcal{C}^\infty \tag{64}$$

$$V(\mathbf{x}_d) = \mathbf{0} \tag{65}$$

$$V > 0 \; \forall \mathbf{x} \in \mathcal{X}\backslash\{\mathbf{x}_d\} \tag{66}$$

$$\dot{V} \leq 0 \;\; \forall \mathbf{x} \in \mathcal{X}\backslash\{\mathbf{x}_d\} \tag{67}$$

for some desired equilibrium $\mathbf{x}_d$, then $V$ is said to be a Lyapunov function for $\Sigma(\mathbf{x}, \mathbf{0})$, and the *damping feedback*

$$u = -\eta[\nabla V \cdot WT]^T, \;\; \eta > 0 \tag{68}$$

renders $\Sigma(\mathbf{x}, \mathbf{u})$ Lyapunov stable.

Theorem 2 constitutes a powerful result, since it provides a simple formula for closed-loop smooth stabilization based on the existence of a (weak) Lyapunov function for the unforced system under the assumption that the forced system is accessible. The remaining challenge is to find a suitable (weak) Lyapunov function for $\Sigma(\mathbf{x}, \mathbf{0})$, and upgrade the result of Theorem 2 to asymptotic stability using LaSalle's invariance principle.

*Lemma 1:* Assume, without loss of generality, that $I_3 > I_2 > I_1$. If $L > L_d$ and $L(l - l_d) > 0$ then the quadratic form

$$V(\mathbf{x}, \mathbf{x}_d) = \frac{1}{2}(\mathbf{x} - \mathbf{x}_d)^T P(\mathbf{x} - \mathbf{x}_d), \quad P \in \mathbb{R}^{5\times 5}, \; P \succ 0 \tag{69}$$

is a Lyapunov function for $\Sigma(\mathbf{x}, \mathbf{0})$ and the resulting damping feedback

$$\mathbf{u} = -\eta(\mathbf{x}^T PWT)^T \tag{70}$$

constitutes a feedback stabilizer for $\Sigma$.

*Proof:* Conditions (64)-(66) are satisfied trivially. The remaining conditions, Eq. (67) is examined by calculating $\dot{V}$ along the trajectories of $\Sigma(\mathbf{x}, \mathbf{0})$:

$$\begin{aligned} \dot{V} &= (\mathbf{x} - \mathbf{x}_d)^T P\mathbf{f}(\mathbf{x}) \\ &= L(a_3 - a_1 \sin^2 l - a_2 \cos^2 l)(l - l_d)p_1 \\ &+ \frac{1}{2}(L^2 - G^2)\sin(2l)(a_1 - a_2)(L - L_d)p_3 \end{aligned} \tag{71}$$

Since $G \geq L$ and $l > 0$ by definition, the assumptions of the Lemma guarantee that $\dot{V} \leq 0$.

The upgrade to asymptotic feedback stabilization is provided by means of the following Lemma.

*Lemma 2:* The control law (70) is an internal asymptotic feedback stabilizer for $l_d \neq 0$.

*Proof:* Let us examine the limit set of the problem. Solving $\dot{V} = 0$ yields the limit set $M$,

$$\begin{aligned} M &= \{\mathbf{x} \in \mathcal{X} | \dot{V} = 0\} \\ &= \{l_d \neq 0, L_d \neq 0, L = 0, l = 0, \frac{\pi}{2}\} \\ &\cup \;\; \{l_d = 0, L_d = 0, l = 0, L \in \mathbb{R}\} \end{aligned} \tag{72}$$

The largest invariant set $N \subset M$ is

$$N = \{l_d = 0, L_d = 0, l = 0, L \in \mathbb{R}\}. \tag{73}$$

Hence, by LaSalle's invariance principle [20], the feedback controller (70) in an asymptotic feedback stabilizer $\forall \mathbf{x} \in \mathcal{X}\backslash N$.

The final step is to inquire the smoothness of the proposed controller. Let us first write the closed-form, analytic expressions for the attitude controller by substituting the Lyapunov function (69) into the Jurdjevic-Quinn formula (70):

$$\begin{aligned} u_1 &= -\eta\left[\frac{p_1(l - l_d)\mathrm{c}_l}{\sqrt{G^2 - L^2}} - \frac{p_2(G - G_d)\mathrm{s}_l\sqrt{G^2 - L^2}}{G}\right] \\ u_2 &= -\eta\left[\frac{p_1(l - l_d)\mathrm{s}_l}{\sqrt{G^2 - L^2}} - \frac{p_2(G - G_d)\mathrm{c}_l\sqrt{G^2 - L^2}}{G}\right] \\ u_3 &= -\eta\left[\frac{p_2(G - G_d)L}{G} - p_3(L - L_d)\right] \end{aligned}$$

## V. Simulation

To illustrate the performance of this feedback controller, consider the problem of nadir stabilization on a circular orbit. In this case, $\omega_3$ is required to match the orbital angular velocity, $\omega_0$, defined by

$$\omega_0 = \sqrt{\frac{\mu}{R_0^3}} \tag{74}$$

where $\mu = 3.986 \cdot 10^5\,\mathrm{km}^2/\mathrm{s}^3$ is the gravitational constant of the Earth, and $R_0$ is the orbital radius. The attitude of the satellite in SA variables is then determined by the angles $\beta$ and $l$. Let $R_0 = 7000\,\mathrm{km}$, so that $\omega_0 = 0.001078\,\mathrm{rad/s}$. In addition, assume that $I_3 = 1000\,\mathrm{kgm}^2, I_2 = 400\,\mathrm{kgm}^2, I_3 = 200\,\mathrm{kgm}^2$. Under these conditions, $L_d = I_3\omega_0 = 0.6468\,\mathrm{Ns}$. Choosing $\beta_d = 5$ deg then yields $G_d = 2.28\,\mathrm{Ns}$. We also set $l_d = 10\,\mathrm{deg}$. The initial conditions chosen were $L_0 = 0.8\,\mathrm{Ns}$, $G_0 = 1\,\mathrm{Ns}$, $l_0 = 50\,\mathrm{deg}$. The closed-loop simulation results are depicted in Figs. 2-3. Fig. 2 shows the time history of the states $l, G, L$. A smooth convergence to the desired set-points is achieved after 2 seconds. Fig. 3 depicts the control torque components. We see that the maximum torque required is about $10\,\mathrm{Nm}$. The usual trade-off of performance vs. control effort may be used to trade settling time for control effort.

## VI. Conclusions

We have shown that the Serret-Andoyer formalism is very useful for global stabilization of the rigid body dynamics. This formalism permits a straightforward derivation of a stabilizing controller and naturally supports practical problems such as nadir stabilization. Future research will look into optimal control using the SA setup and will dwell upon the under-actuated (failure configuration) problem.

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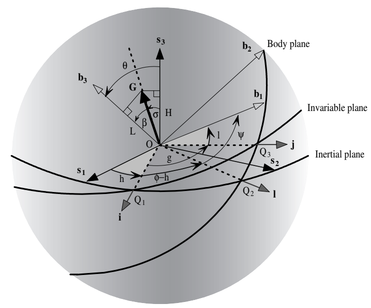


Fig. 1. Inertial and body-fixed coordinates

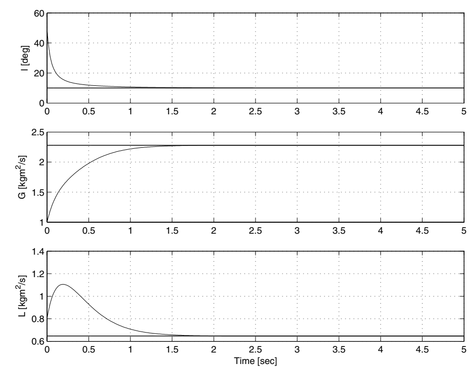


Fig. 2. Time history of the state variables

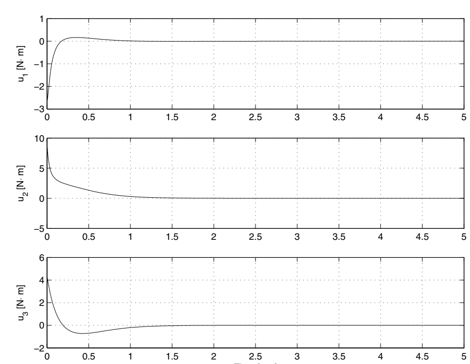


Fig. 3. Control torques