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Closed-Form Deflection-Limiting Commands

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Abstract— **Command signals that can move a flexible system without residual vibration and also limit the transient deflection are very useful. Unfortunately, these types of commands can be difficult to generate and have historically required intense numerical optimization. A new method for creating deflection-limiting commands is described and evaluated. The major advancement is that the problem is solvable in closed form, rather than via numerical optimization. Simulations of a mass-spring-mass system are presented as a function of deflection limit and move distance. The results compare favorably to previously presented optimized solutions.**

I. INTRODUCTION

An extensive array of control schemes have been developed to control unwanted vibration in flexible systems [1]-[6]. One approach involves designing a smooth reference command that leads to low levels of vibration [5], while others utilize command shaping [1],[2],[6]-[12]. One of the more successful approaches is to generate a reference command that drives the system to cancel out its own vibration. The earliest incarnation of this self-canceling command generation was developed in the 1950's by O.J.M. Smith [11]. His posicast control method involved breaking a command of certain magnitude into two smaller magnitude commands, one of which is delayed one-half period of vibration. Unfortunately, his technique was extremely sensitive to modeling errors [13]. The idea languished until 1990 when Singer and Seering developed reference commands that were robust enough to be effective on a wide range of systems [12]. Their technique is known as input shaping and many useful extensions have been made to this robust technique.

Input shaping is implemented by convolving a sequence of impulses, known as the input shaper, with a desired system command to produce a shaped input that is then used to drive the system. This process is demonstrated in Fig. 1. The amplitudes and time locations of the impulses are determined by solving a set of constraint equations that attempt to control the dynamic response of the system. The elimination of the unwanted vibration comes at the expense of the system's rise time. The rise time is delayed by the shaper's duration, Δ. Therefore, it is always desirable to create shapers with the shortest duration possible.

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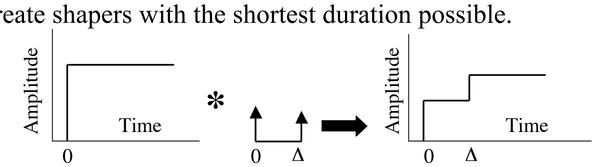

Fig. 1. Input Shaping Example.

The constraint on residual vibration amplitude can be expressed as the ratio of residual vibration amplitude with shaping to that without shaping. The percentage vibration can be determined by using the expression for residual vibration of a second-order harmonic oscillator of frequency ω and damping ratio ζ, which is given in [14]. The vibration from a series of impulses is divided by the vibration from a single impulse to get the percentage vibration:

$$V(\omega,\zeta) = e^{---t_n} \sqrt{[C(-,-)]^2 + [S(-,-)]^2}\,, \tag{1}$$

where,

$$C(\omega,\zeta) = \sum_{i=1}^{n} A_i e^{\zeta\zeta t_i} \cos\left(\zeta\sqrt{1--^2}\,t_i\right), \tag{2}$$

and

$$S(\omega,\zeta) = \sum_{i=1}^{n} A_i e^{\zeta\zeta t_i} \sin\left(\zeta\sqrt{1--^2}\,t_i\right). \tag{3}$$

If V(ζ ,ζ) is set equal to zero at the modeling parameters, (ζ_m, ζ_m), then a shaper that satisfies (1) is called a Zero Vibration (ZV) shaper. This is the type of solution proposed by Smith in the 1950's.

A ZV shaper will not work well on many systems because it will be sensitive to modeling errors. For input shaping to work well on most real systems, the constraint equations must ensure robustness to modeling errors [12]. Singer and Seering's robust input shaping was achieved by setting the derivative with respect to the frequency of the residual vibration equal to zero. The resulting shaper is called a Zero Vibration and Derivative (ZVD) shaper. The improved robustness can be seen by plotting sensitivity curves – amplitude of vibration *vs*. modeling error, as shown in Fig. 2.

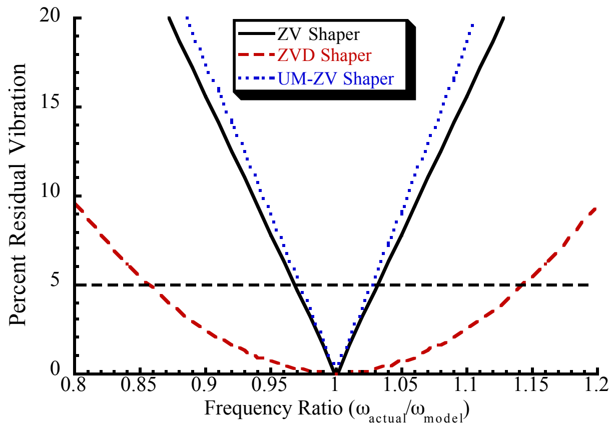

Fig. 2. Sensitivity Curve.

Most shapers are derived using the constraint of positive impulse amplitudes to keep the amplitudes from going to positive and negative infinity. An alternative constraint requires that the impulses have unity magnitude [15]. A Unity Magnitude, Zero Vibration (UM-ZV) shaper is defined when the new amplitude constraint is combined with the zero vibration requirement. In addition to having a shorter duration than the standard ZV shaper, the UM-ZV creates commands that are compatible with On-Off actuators. Figure 3 shows a step command shaped with a UM-ZV shaper. Unlike the shaped command in Fig. 1, the UM-ZV shaped command can easily be realized using on-off actuation.

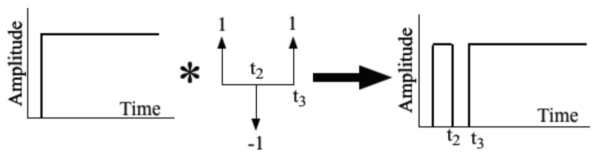

Fig. 3. Unity Magnitude, Zero Vibration Command Shaper.

Solving the input shaping constraint equations often requires a nonlinear optimization. However, input shapers can be designed in the digital domain, rather than the continuous domain, to eliminate the need for a nonlinear optimization [10], [16], [17]. This characteristic of digital shaping is extremely useful. Eliminating the need for the nonlinear optimization greatly simplifies the process used to create the commands. However, even the linear optimization requires significant computational effort for some of the more complicated shaper applications.

II. REST-TO-REST MOVES FOR FLEXIBLE SYSTEMS

There has been much work done to develop reference commands for moving flexible systems. One significant area of work has been the generation of On-Off command profiles for reaction jets [1], [2], [9], [18]. Initial efforts created commands that eliminated residual vibration at the end of the move. While these commands successfully eliminated vibration, they could create large deflections during the slew. In an effort to avoid internal stresses that result from these deflections, commands have been developed that place a limit on the transient deflection [19], [20]. For input shaping to work, only estimates of the system's natural frequency and damping ratio are needed, so simple models like the one shown in Fig. 4 can be used.

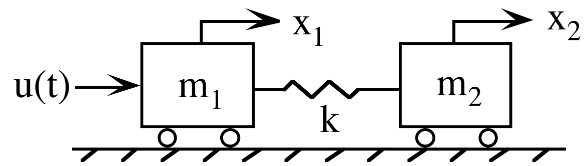

Fig. 4. Mass-Spring Model

The amplitudes and time locations of the impulses in an input shaper are determined by satisfying a set of constraint equations while minimizing the maneuver duration. Typical constraints are: 1) Residual vibration constraints limit the oscillations at the end of the move, 2) Robustness constraints, such as the zero derivative discussed in the introduction, increase the ability of the command to perform well in the presence of modeling errors, 3) The requirement of time optimality is needed because of the transcendental nature of equations (2) and (3) there will always be multiple solutions to the constraint equations, 4) Rigid-body constraints are used to ensure the system's mass center will move the desired amount, and 5) Impulse amplitude constraints are used to ensure that the convolved command does not saturate or exceed actuator limits (these constraints limit the magnitude of the impulses to below some value).

Solution of the above constraints will lead to commands that eliminate residual vibration and have some level of robustness to modeling errors.

On-Off commands for a vibration free rest-to-rest move can be created analytically [21]. As shown in Figure 5, these commands consist of a transition from rest to acceleration (transition 1), a transition from acceleration to deceleration (transition 2) and finally a transition from deceleration to rest (transition 3). For example, if a UM-ZV shaper is used for the one-unit transition (transitions 1 & 3; a transition to or from rest) and a ZV shaper is used for the two-unit transition (transition 2; a transition from acceleration to deceleration), the command shaper will be of the form:

$$\begin{bmatrix} a_i \\ t_i \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \end{bmatrix} \quad (4)$$

Selecting these transitions insures that the resulting command will have only positive pulses during the acceleration portion of the slew and negative pulses during the deceleration portion. The time locations of the impulses in the individual shapers are determined by the constraint on residual vibration. The time spacing between transitions 1 & 2 and transitions 2 & 3 are determined by the constraints on the rigid body motion. The shaper described by (4) can then be convolved with a step to create an on-off command as shown in Fig. 6.

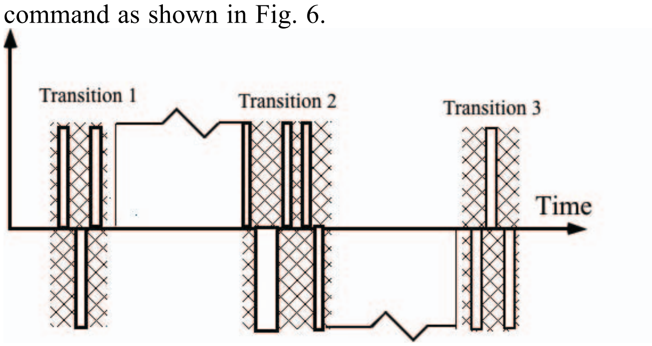

Fig. 5. Analytic On-Off Commands.

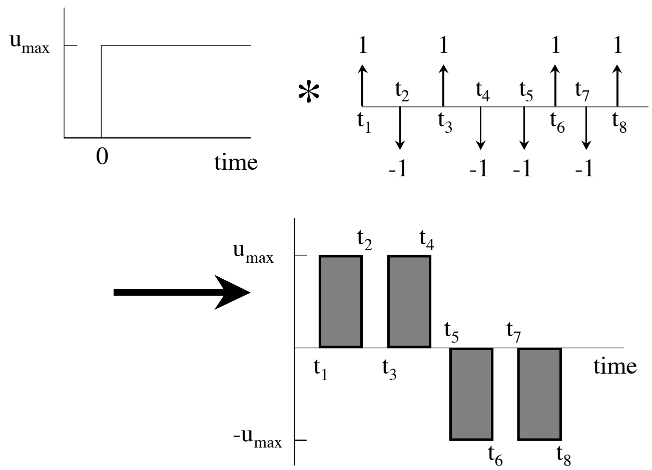

Fig. 6. Creating On-Off Commands using Input Shaping.

The input command and deflection for a zero residual vibration rest-to-rest slew of five units (m_1=m_2=k=1) are shown in Figure 7. While there is no vibration at the conclusion of the slew, the deflection reached a level of 0.5 during the slew. Depending on the system and application, this level of transient deflection may be undesirable or unacceptable. If the deflection is large, the system may be damaged, or the endpoint may deviate considerably from an intended trajectory. In order to control the level of deflection during the slew, an expression for the deflection as a function of the input shaper must be obtained. The desired function can be generated using superposition of deflections from individual step inputs.

An expression for the deflection of the system shown in Fig. 4 is easily derived. The result is applicable to other systems with one flexible mode and a rigid-body mode. The derivation begins with the equations of motion:

$$\mathbf{F}(s) = \left(m_1 s^2 + k\right)\mathbf{x}_1(s) - k\mathbf{x}_2(s) \tag{5}$$

$$0 = \left(m_2 s^2 + k\right)\mathbf{x}_2(s) - k\mathbf{x}_1(s) \tag{6}$$

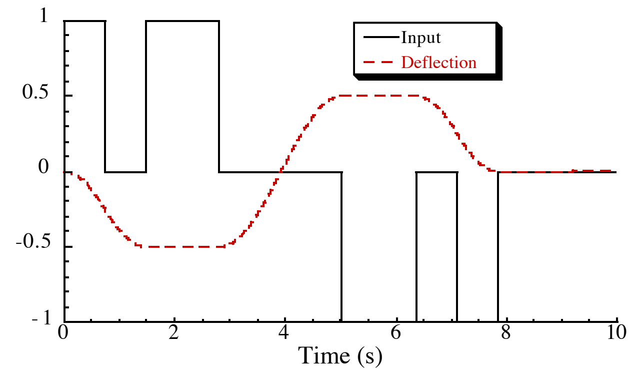

Fig. 7. Five Unit Rest-To-Rest Move Of Benchmark System.

Defining the deflection to be $\mathbf{x}_2(s)$-$\mathbf{x}_1(s)$, and assuming the input is a step of magnitude u_{max} with zero initial conditions gives the deflection from a step input as a function of time. We can express the deflection as:

$$D(t) = \left(\frac{D_{max}}{2}\right)\left[\cos(\omega t) - 1\right] \tag{7}$$

where ω is the natural frequency of oscillation

$$\omega = \sqrt{\left(\frac{m_1 + m_2}{m_1 m_2}\right)k} \tag{8}$$

and the maximum deflection magnitude, D_{max}, is given by:

$$D_{max} = \frac{2u_{max}m_2}{k(m_1 + m_2)} \tag{9}$$

The coefficient in (7) is written as $D_{max}/2$ because the quantity enclosed in the brackets has a maximum magnitude of two.

Multiple versions of (7) can be used to generate a function that describes the deflection throughout a slew containing many step inputs (On-Off commands are just a series of positive and negative steps). Assuming that the command profile consists of a series of pulses, then the deflection throughout the slew is given by:

$$D(t) = D_{(m)-(m-1)}(t) = \sum_{i=1}^{m}\left(\frac{D_{max}}{2}\right)a_i\left[\cos\left(w\left(t - t_i\right)\right) - 1\right]. \tag{10}$$

while,

$$t_m \le t < t_{m+1}, m = 1,\ldots,n \tag{11}$$

It is important to note the restriction presented by the qualifier $t_m \le t < t_{m+1}$ in (11). The deflection that occurs between the first and second impulses of the input, $D_{1\text{-}2}(t)$, (the period during the first pulse) is given by (10) when m

= 1. The deflection, $D_{2\text{-}3}(t)$, between the second and third impulses is given by (10) when m = 2. Equation 10 amounts to a piecewise-continuous function composed of n finite length segments; each of the segments has a limited range of applicability. Note that the magnitude of deflection caused by a series of pulses can exceed D_{max} if the deflection components from individual pulses interfere constructively.

III. Analytic Deflection Limiting Commands

When $D_{max}/2$ is greater than the deflection limit, the analytic on-off commands previously reported cannot be used. For the constraints to be met,

$$\frac{D_{\max}}{2}\sum_{i=1}^{m} a_i\left[\cos\left(w\left(t-t_i\right)\right)-1\right] \le Deflim \tag{12}$$

where Deflim is the deflection limit. This must hold during the times between transition 1 and 2 and between transition 2 and 3. This can be acccomplished by modifying the impulse amplitude constraint to be

$$\sum a_i = 2\left(\frac{Deflim}{D_{\max}}\right) \tag{13}$$

creating the new shaper shown in Figure 8. Combining the impulse amplitudes shown in Figure 8 and (13) we can solve for a_3 as

$$a_3 = 2\left(\frac{Deflim}{D_{\max}}\right) \tag{14}$$

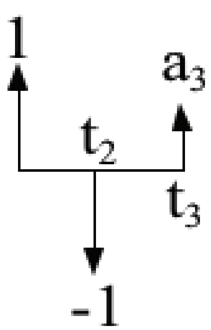

Fig. 8. Deflection-Limiting Transition Shaper.

The impulse amplitudes set by Figure 8 and (14) can be inserted into (2) and (3) to yield:

$$1-\cos(\omega t_2)+a_3\cos(\omega t_3)=0 \tag{15}$$

$$-\sin(\omega t_2)+a_3\sin(\omega t_3)=0 \tag{16}$$

These equations can be solved for t_2 and t_3 by

$$t_2=\cos^{-1}[(0.5*(2-2\frac{Deflim}{D_{\max}})^2)]/\omega \tag{17}$$

$$t_3=\cos^{-1}(\frac{-Deflim}{D_{\max}})/\omega \tag{18}$$

The above equations are used to create one-unit transitions that accelerate the system to its deflection limit without residual vibration. The two-unit transition is created by using a modified ZV shaper; that is, two impulses of magnitude 2(Deflection Limit/D_{max}) separated by T/2. Therefore, $t_5=t_4+T/2$ where T is the period of vibration. Figure 9 shows the command resulting from this formulation.

All that is left to be determined is t_4, the duration of the acceleration portion of the move. This is done be examining the rigid-body constraints. Looking at only the first half of the move for an undamped system (the second half can be found from symmetry), we know that we must be at the midpoint of the move at the midpoint of the command, or

$$x(t_{mid})=\frac{x_d}{2} \tag{19}$$

where x_d is the desired move distance and t_{mid} is described by

$$t_{mid}=t_4+\frac{T}{4} \tag{20}$$

By integrating the rigid-body equation of motion with respect to time, an expression for the mass center as a function of switch times is obtained

$$t_4=\frac{(-b+\sqrt{b^2-4ac})}{2a} \tag{21}$$

where

$$a=\alpha * Deflim \tag{22}$$

$$b=\alpha*\left(t_2-2*Deflim*t_3+2*Deflim*\frac{1}{4f}\right) \tag{23}$$

$$c=-\frac{\alpha}{2}t_2^2+\alpha*Deflim*t_3^2+\alpha\left(\frac{1}{4f}\right)t_2$$
$$|2*/*Deflim\left|\frac{1}{4f}\right|*t_3\,|\,\frac{x_d}{2} \tag{24}$$

and α is the force-to-mass ratio. Now the complete command shaper can by described by

$$\begin{bmatrix} a_i \\ t_i \end{bmatrix} = \begin{bmatrix} 1 & -1 & a_3 & -1 & -1 & a_3 & -1 & 1 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \end{bmatrix} \tag{25}$$

with a_3, t_2, t_3 and t_4 defined by (13), (17), (18) and (20), respectively, and t_5, t_6, t_7 and t_8 determined from symmetry. These equations show that a_3 is proportional to the deflection limit, while the impulse time locations are functions of the deflection limit and the system parameters.

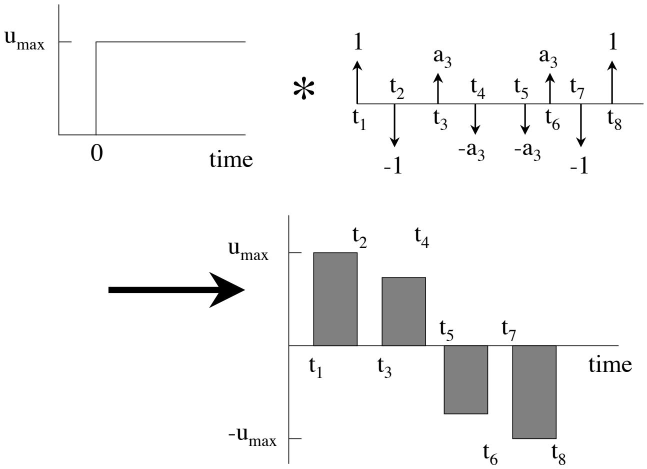

Fig. 9. Analytic Deflection-Limiting Command.

IV. RESULTS

Simulations were conducted on the benchmark system shown in Fig. 4. All the parameters were set to unity, yielding a system with a force-to-mass ratio of 0.5 and a natural frequency of 0.2251 Hz. Commands were generated to move the system 5 units with a deflection limit of 0.4. Figure 10 shows the input and deflection response from the analytic commands developed here and the digitally designed deflection-limiting commands previously reported [21]. Digital deflection-limiting commands are commands that are created in the digital domain via a linear optimization.

The input commands and the deflection responses are remarkably similar for both sets of commands. In fact, the digital commands tend toward the analytic solution as the digital time spacing is decreased. The commands and deflection responses shown in Figure 11 are for a move distance of 6 units and a deflection limit of 0.3. The similarities between the analytic and digital commands are repeated.

The digital commands were shown previously to be within one digital time spacing of the optimal solution [21]. Figure 12 shows a comparison of command durations for a range of deflection limits given a 5-unit move distance. The analytic command durations are equal to or less than those of the digital commands.

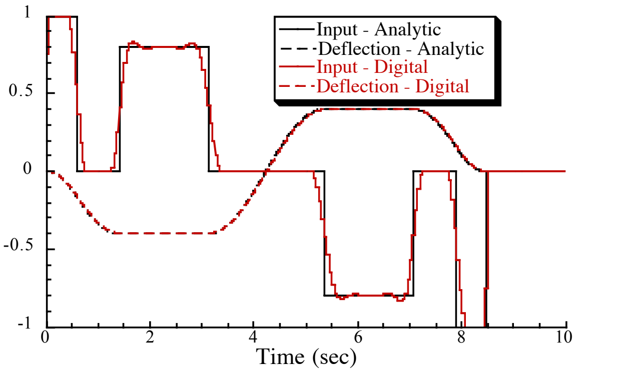

Fig. 10. Move distance = 5 units, deflection limit = 0.4.

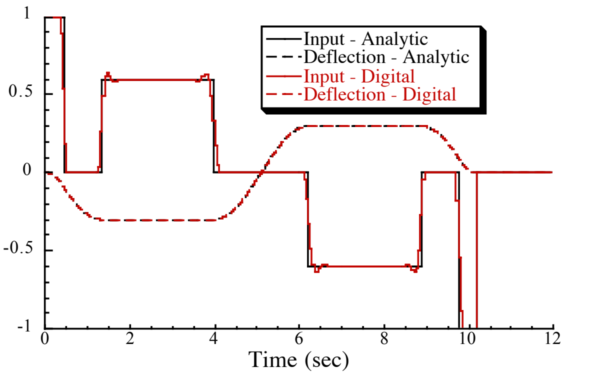

Fig. 11. Move distance = 6 units, deflection limit = 0.3.

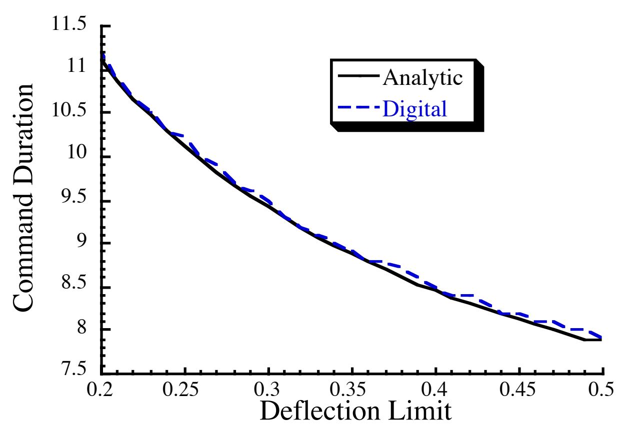

Fig. 11. Command Duration Comparison.

V. CONCLUSION

A new procedure has been presented for easily creating deflection-limiting commands in closed form. Commands created analytically give similar performance to those that are created with the more difficult techniques previously reported. This result provides a real-time solution to the problem of moving flexible systems without residual vibration and a with limited amount of transient deflection.

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