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Boundary Conditions and the Stability of a Class of 2D Continuous-discrete Linear Systems

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Abstract— **Differential linear repetitive processes are characterized by a series of sweeps, or passes, through a set of dynamics defined over a finite interval or duration with interaction between successive passes. They are distinct from other classes of 2D continuous-discrete linear systems due to the fact that information propagation in one of the two separate directions only occurs over a finite duration. Moreover, this is an intrinsic feature of the underlying dynamics as opposed to an assumption introduced for analysis purposes. This paper shows that the structure of the initial conditions at the start of each new pass of the process is critical to its stability properties.**

I. INTRODUCTION

The essential unique characteristic of a repetitive (termed multipass in the early literature) process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of sweeps, termed passes, of the processing tool. Assume that the pass length α (i.e. the duration of a pass of the processing tool), which is finite by definition, has a constant value for each pass. Then in a repetitive process the output vector, or pass profile, $y_k(t)$, $0 \leq t \leq \alpha$, (t being the independent spatial or temporal variable) produced on pass k acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \leq t \leq \alpha$, $k \geq 0$.

Industrial examples (see, for example, [2], [3]) include long-wall coal cutting and metal rolling operations. Also problem areas exist where adopting a repetitive process setting for analysis has clear advantages over alternatives. This is especially true for classes of iterative learning control schemes (see, for example, [1]) and of iterative solution algorithms for classes of dynamic nonlinear optimal control problems based on the maximum principle (see, for example, [6]). In the iterative learning control application, the stability theory for so-called differential and discrete linear repetitive processes is the basic starting point for a rigorous stability/convergence theory for a powerful class of such algorithms. For the optimal control algorithm, use of the repetitive process setting for analysis has (uniquely) led to the development of numerically reliable solution algorithms.

The basic unique control problem for repetitive processes is that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction (i.e. in the k-direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single-input single-output) repetitive processes and, in particular, long-wall coal cutting were based on first converting the system into an equivalent infinite-length single-pass process [3]. This, for example, resulted in a scalar differential/algebraic system to which standard scalar inverse-Nyquist stability criteria were then applied. In general, however, it was soon established that this approach to analysis (and controller design) would, except in a few very restrictive special cases, lead to incorrect conclusions [7]. The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature together with the effects of resetting the initial conditions before the start of each new pass.

This last fact has led to the development of a rigorous stability theory for linear repetitive processes. This theory [7] is based on an abstract model of the underlying dynamics in a Banach space setting which includes all processes with linear dynamics and a constant pass length as special cases. The theory shows that two distinct concepts of stability for these processes exist, termed asymptotic stability and stability along the pass respectively, where the former is a necessary condition for the latter.

In this paper, we consider the application of this theory to a very important sub-class known as differential linear repetitive processes. This sub-class is characterized by the fact that information propagation along the pass (t-direction) is described by a linear matrix differential equation and that from pass to pass (k-direction) by a linear matrix difference equation. Hence they can also be considered as a sub-class of so-called 2D differential-discrete linear systems which have received some attention in the literature, see, for example, [4]. Unlike such systems, however, they do have practical applications and the main point of this paper is to show that the structure of the initial conditions on each new pass is critical to the stability properties exhibited by a given example.

II. CASE 1 — SIMPLE BOUNDARY CONDITIONS

The state space model of a differential linear repetitive process has the following form over $0 \leq t \leq \alpha$, $k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t) \end{aligned} \quad (1)$$

Here on pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector and $u_k(t)$ is the $l \times 1$ vector of

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control inputs.

To complete the process description, it is necessary to specify the boundary conditions i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). In this section we consider the simplest possible choice for these, i.e.

$$\begin{aligned} x_{k+1}(0) &= d_{k+1},\ k \geq 0 \\ y_0(t) &= f(t) \end{aligned} \tag{2}$$

where d_{k+1} is $n \times 1$ vector with known constant entries and $f(t)$ is an $m \times 1$ vector whose entries are known functions of t over $0 \leq t \leq \alpha$. Note here that $x_{k+1}(0)$ is independent of the previous pass dynamics and hence this set of boundary conditions is termed 'simple'.

The stability theory for linear constant pass length repetitive processes is based on the following abstract model of the underlying dynamics where E_α is a suitably chosen Banach space with norm $||.||$ and W_α is a linear subspace of E_α

$$y_{k+1} = L_\alpha\, y_k + b_{k+1},\ k \geq 0 \tag{3}$$

In this model $y_k \in E_\alpha$ is the pass profile on pass k, $b_{k+1} \in W_\alpha$, and L_α is a bounded linear operator mapping E_α into itself. The term $L_\alpha y_k$ represents the contribution from pass k to pass $k+1$ and b_{k+1} represents known initial conditions, disturbances and control input effects.

The linear repetitive process (3) is said to be asymptotically stable [7] if $\exists$ a real scalar $\delta > 0$ such that, given any initial profile y_0 and any disturbance sequence $\{b_k\}_{k\geq 1} \in W_\alpha$ bounded in norm (i.e. $||b_k|| \leq c_1$ for some constant $c_1 \geq 0$ and $\forall k \geq 1$), the output sequence generated by the perturbed process

$$y_{k+1} = (L_\alpha + \gamma) y_k + b_{k+1},\ k \geq 0 \tag{4}$$

is bounded in norm whenever $||\gamma|| \leq \delta$.

This definition is easily shown to be equivalent to the requirement that $\exists$ finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ such that

$$||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k,\ k \geq 0 \tag{5}$$

(where $||.||$ is also used to denote the induced operator norm). Necessary and sufficient conditions for this last condition are that

$$r(L_\alpha) < 1 \tag{6}$$

where $r(\cdot)$ denotes the spectral radius of its argument.

In the case of processes described by (1) and (2), it can be shown that asymptotic stability holds if, and only if, $r(D_0) < 1$. Also if this property holds and the control input sequence applied $\{u_k\}_k$ converges strongly to u_∞ as $k \to \infty$ then the resulting output pass profile sequence $\{y_k\}_k$ converges strongly to y_∞ — the so–called limit profile defined (with $D = 0$ for ease of presentation) over $0 \leq t \leq \alpha$ by

$$\begin{aligned} \dot{x}_\infty(t) &= (A + B_0(I_m - D_0)^{-1}C)x_\infty(t) + Bu_\infty(t) \\ y_\infty(t) &= (I_m - D_0)^{-1}Cx_\infty(t) \\ x_\infty(0) &= d_\infty \end{aligned} \tag{7}$$

where d_∞ is the strong limit of the pass sequence $\{d_k\}_{k\geq 1}$.

In effect, this result states that if a process is asymptotically stable then its repetitive dynamics can, after a 'sufficiently large' number of passes, be replaced by those of a 1D discrete linear system. Note, however, that this property does not guarantee that the limit profile is stable in the 1D differential linear systems sense, i.e. all eigenvalues of the matrix $(A + B_0(I_m - D_0)^{-1}C)$ have strictly negative real parts — a point which is easily illustrated by the example given below. Hence it is possible to converge in the pass-to-pass direction to a limit profile which is 'unstable along the pass'.

The reason why asymptotic stability does not guarantee a limit profile which is 'stable along the pass' is due to the finite pass length. In particular, asymptotic stability is easily shown to be bounded-input bounded-output (BIBO) stability with respect to the finite and fixed pass length. Also in cases where this feature is not acceptable, the stronger concept of stability along the pass must be used. In effect, for the abstract model (4), this requires that (5) holds uniformly with respect to the pass length α. One of several equivalent statements of this is the requirement that $\exists$ finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$ independent of α which satisfy

$$||L_\alpha^k|| \leq M_\infty \lambda_\infty^k,\ \forall\, \alpha > 0,\ \forall\, k \geq 0 \tag{8}$$

and it is clear that asymptotic stability is a necessary condition for stability along the pass.

Several equivalent sets of necessary and sufficient conditions for stability along the pass of processes described by (1) and (2) are known [7] but here it is the following set which will be required.

Theorem 1: Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then a differential linear repetitive process described by (1) and (2) is stable along the pass if, and only if, $r(D_0) < 1$, $r(A) < 1$ and all eigenvalues of the transfer function matrix

$$G(s) = C(sI_n - A)^{-1}B_0 + D_0 \tag{9}$$

have modulus strictly less than unity $\forall\, s = \imath\omega,\ \omega \geq 0$.

The first condition here (i.e. $r(D_0) < 1$) is asymptotic stability and the second (i.e. $r(A) < 1$) can be interpreted physically as the requirement that the first pass profile is uniformly bounded with respect to the pass length. Note, however, that these conditions are not strong enough for stability along the pass as the following simple example demonstrates.

Consider the single-input single-output case when $A = -1,\ B = 0, B_0 = 1 + \beta,\ C = 1,\ D = 0,\ D_0$. Then in this case the limit profile (7) is unstable as 1D linear system if $\beta > 0$. Also $G(s) = \frac{1+\beta}{s+1}$ and hence stability along the pass requires that $\beta < 0$. In physical terms, this means that each frequency component of the initial profile must be attenuated from pass-to-pass.

The conditions for stability along the pass in this case are easy to test using, in effect, well established 1D linear systems stability tests. In the next section, we will show that this situation is not true if the pass initial vector part of the boundary conditions become a function of the previous pass profile.

III. Case 2 — Dynamic Boundary Conditions

The boundary conditions of the previous section are the simplest possible and cases exist where they are simply not strong enough to adequately model the underlying process dynamics (even for initial simulation and/or control analysis). Instead, it is necessary to consider a state initial vector sequence which is an explicit function of the previous pass profile. One possible form is

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{N} K_j y_k(t_j) \tag{10}$$

where d_{k+1} is as in (2), $0 \le t_1 < t_2 < \cdots < t_N \le \alpha$, are N sample points along the previous pass, and $K_j,\ 1 \le j \le N$, is an $n \times m$ matrix with constant entries. One possible physical application of such boundary conditions is in the coal cutting example where at the end of each pass the cutting machine is hauled back in reverse at high speed to the start and then the machine plus supporting infrastructure is 'snaked forward' by hydraulic rams such that it now rests on the previous pass profile. The weight of the machine (up to 5 tonnes) means that the initial conditions at the start of the new pass will definitely not be independent of the previous pass profile. Moreover, as discussed further below, suitable choice of the parameters in (10) leads to links with other classes of linear systems. This set of boundary conditions is termed 'dynamic' (to reflect their pass-to-pass dependence).

It is routine to show that processes described by (1) and (10) can be written as a special case of the abstract model (and hence the stability theory can be applied). In the case of asymptotic stability, we have the following result where for the remainder of this paper we assume without loss of generality that $D_0 = 0$ in (1).

Theorem 2: Suppose that the pair $\{A, B_0\}$ is controllable and $D_0 = 0$ for simplicity of presentation. Then a differential linear repetitive process described by (1) and (10) is asymptotically stable if, and only if, $r(L_\alpha) < 1$ where

$$r(L_\alpha) = \max\{0, \sup\{|z| : z \neq 0 \ \&\ \det(\mathrm{zI_n} - \mathrm{M(z)}) = 0\}\}$$

$$M(z) := \sum_{j=1}^{N} K_j C e^{\hat{A}(z) t_j}$$

and

$$\hat{A}(z) = A + z^{-1} B_0 C,\ z \neq 0$$

Proof: (See also [5]) Here is required to compute $r(L_\alpha)$ and no general rules exist for this task, other than the obvious necessity to compute the spectral values of L_α and hence their moduli. Note also that severe difficulties could arise if the space E_α and/or the operator L_α have a complex structure. As shown below, however, this task is possible for the processes considered in this (and the next) section where the approach to the spectral calculations used is to consider the equation

$$(zI - L_\alpha) y = \eta \tag{11}$$

and construct necessary and sufficient conditions on the complex scalar z to ensure that (i) a solution exists $\forall\, \eta \in E_\alpha$, and (ii) that this solution is bounded in the sense that $||y|| \le K_0 ||\eta||$ for some real scalar $K_0 > 0$ and $\forall\, \eta \in E_\alpha$.

To evaluate $r(L_\alpha)$ we examine the problem of solving (11) for $y \in E_\alpha$ when $\eta \in E_\alpha$. In particular, we construct conditions on z such that the map $\eta \mapsto y$ is defined and bounded in E_α. Note also that if $f = L_\alpha y$ then $zy - f = \eta \Rightarrow y = z^{-1}(f + \eta)$ and the relationship between f and η is described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_0 y(t) \\ f(t) &= Cx(t),\ x(0) = \sum_{j=1}^{N} K_j y(t_j) \end{aligned}$$

Using $y = z^{-1}(f + \eta)$ we now have that

$$\begin{aligned} x(t) &= e^{\hat{A}(z)t} \left\{ \sum_{j=1}^{N} z^{-1} K_j \left[Cx(t_j) + \eta(t_j) \right] \right. \\ &+ \left. \int_0^t e^{\hat{A}(z)(t-\tau)} z^{-1} B_0 \eta(\tau)\, d\tau \right\} \end{aligned} \tag{12}$$

The existence of a solution y to $(zI - L_\alpha)y = \eta$ is equivalent to the consistency of (12) for $x(t)$ at $t_1, t_2, \cdots, t_N$. (Note: Matching $y(t)$ at $t_1, t_2, \cdots, t_N$ ensures that $zy - f = \eta$ between these points.) Also a solution exists if, and only if,

$$\begin{aligned} zx(t_h) &- \sum_{j=1}^{N} e^{\hat{A}(z)t_h} K_j C x(t_j) = \sum_{j=1}^{N} e^{\hat{A}(z)t_h} K_j \eta(t_j) \\ &+ \int_0^{t_h} e^{\hat{A}(z)(t_h - \tau)} B_0 \eta(\tau)\, d\tau \end{aligned} \tag{13}$$

Now let $g_h(z)$ denote the right-hand side of this last equation and define

$$g(z) := \begin{bmatrix} g_1^T(z) & \cdots & g_N^T(z) \end{bmatrix}^T$$

Also let $\hat{M}(z)$ denote the $nN \times nN$ matrix whose block entries are $n \times n$ matrices $\hat{M}_{hj}(z)$ where

$$\hat{M}_{hj}(z) = e^{\hat{A}(z)t_h} K_j C$$

Hence (13) can be written as

$$\left[zI_{nN} - \hat{M}(z) \right] X_s = g(z) \tag{14}$$

where $X_s = \begin{bmatrix} x^T(t_1) & \cdots & x^T(t_N) \end{bmatrix}^T$.

It is now routine to show that a sufficient condition for the existence of a unique solution X_s of (14) is that $\det(zI_{nN} - \hat{M}(z)) \neq 0$. In this situation, $z \notin \sigma(L_\alpha)$. Also it is easy to show that this non-singularity condition is necessary by using the controllability assumption to construct an f such that there is no solution to $\left[zI_{nN} - \hat{M}(z)\right] X_s = g(z)$, and hence no solution to $(zI - L_\alpha)y = f$ for that choice of f.

Finally, write

$$\hat{M}(z) = \begin{bmatrix} e^{\hat{A}(z)t_1} \\ \vdots \\ e^{\hat{A}(z)t_N} \end{bmatrix} \begin{bmatrix} K_1C & \cdots & K_NC \end{bmatrix}$$

and note that $\det(zI_{nN} - \hat{M}(z)) = z^{n(N-1)}\det(zI_n - M(z))$. The result is now proved by noting that

$$\sigma(L_\alpha) = \{z : z \neq 0,\ \det(\mathrm{zI_n - M(z)}) = 0\} \cup \{0\}$$

■

Further simplification (reduction in dimension) is possible in some special cases, e.g. the following.

Corollary 1: Consider a differential linear repetitive process described by (1) and (10) in the special case when $K_j = KT_j,\ 1 \leq j \leq N$, where K is an $n \times m$ matrix with constant entries and $T_j,\ 1 \leq j \leq N$, are $m \times m$ matrices with constant entries. Then in this case asymptotic stability holds if, and only if, all solutions of

$$\det\left(\mathrm{zI_m} - \sum_{\mathrm{j=1}}^{\mathrm{N}} \mathrm{T_j C e^{\hat{A}(z)t_j} K}\right) = 0$$

have modulus strictly less than unity.

In general, Theorem 2 shows that the property of asymptotic stability for differential linear repetitive processes is critically dependent on the structure of $x_{k+1}(0),\ k \geq 0$. Suppose also that this sequence is incorrectly modelled as in (2) instead of as a special case of (10). Then the process under consideration could well be interpreted as asymptotically stable when in actual fact it is asymptotically unstable (and hence unstable along the pass).

The limit profile in the case when the condition in Theorem 2 holds is given by

$$\begin{aligned} \dot{x}_\infty(t) &= (A + B_0C)x_\infty(t) + Bu_\infty(t) \\ y_\infty(t) &= Cx_\infty(t) \\ x_\infty(0) &= (I_n - M(1))^{-1}d_\infty \end{aligned}$$

where the matrix inverse involved exists by asymptotic stability.

Consider now the delay-differential systems in $\mathbb{R}^n$ modelled by the state space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_0x(t-\alpha) + Bu(t),\ t \geq 0 \\ x(t-\alpha) &:= x_0(t),\ 0 \leq t \leq \alpha \end{aligned} \qquad (15)$$

If the delay α is interpreted as a pass length then it is obvious that these systems have certain structural similarities to the repetitive processes considered here. In particular, introduce the following change of variables over $0 \leq t \leq \alpha,\ k \geq 0$,

$$\begin{aligned} u_{k+1}(t) &= u(k\alpha + t) \\ x_k(t) &= x((k-1)\alpha + t) \end{aligned}$$

and define the pass profile as $y_k = x_k,\ k \geq 0$. Then the defining equation (15) can be written as a differential unit memory linear repetitive process with state initial vector defined by $x_{k+1}(0) = x_k(\alpha),\ k \geq 0$, i.e. as a special case of (10).

The conditions for asymptotic stability follow as a special case of Theorem 2, i.e. when $N = 1$ and $t_1 = \alpha$. In which case we have asymptotic stability if, and only if, all roots of

$$\det\left(\mathrm{zI_n - e^{(A+z^{-1}B_0)\alpha}}\right) = 0 \qquad (16)$$

have modulus strictly less than unity. Also set $z = e^{s\alpha}$. Then (16) is equivalent to the requirement that

$$\det\left(\mathrm{sI_n - A - B_0e^{-s\alpha}}\right) \neq 0,\ \mathrm{Re(s)} \geq 0 \qquad (17)$$

i.e. the repetitive process concept of asymptotic stability coincides with the normal condition for stability of this class of delay differential systems. For an in-depth analysis of this particular case see [2].

Turning now to stability along the pass, we require the following result [7] for the abstract model of (3).

Theorem 3: A linear repetitive process described by (3) is stable along the pass if, and only if,

(a)

$$r_\infty := \sup_{\alpha > 0} r(L_\alpha) < 1 \qquad (18)$$

and

(b)

$$M_0 := \sup_{\alpha > 0} \sup_{|z| \geq \lambda} ||(zI - L_\alpha)^{-1}|| < \infty \qquad (19)$$

for some real number $\lambda \in (r_\infty, 1)$.

There are two possible cases which need to be considered for processes described by (1) and . The first of these is that as $\alpha \to \infty$ we keep N and t_j fixed and the second is that as $\alpha \to \infty$ we allow $N \to \infty$ and $t_j \to \infty$. Of these, the first is more generally relevant and the task now is to examine (11), i.e. the equation $(zI - L_\alpha)y = \eta$, in E_α where $\alpha \geq t_N$ and $\eta \in E_\alpha$. We require that

(a) $\forall\ \alpha \geq t_N,\ r(L_\alpha) < 1$, which, due to the assumptions on N and t_j holds if, and only if, Theorem 2 holds, and

(b) $\exists\, \lambda \in (r_\infty, 1)$ such that the map $f \mapsto y$ is defined and uniformly bounded with respect to $\alpha \geq t_N$ and $|z| \geq \lambda$.

We now have the following result.

Theorem 4: (See also [5].) Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then a differential linear repetitive process described by (1) and (10) is stable along the pass if, and only if,

(a) the condition of Theorem 2 holds,

(b) all eigenvalues of the matrix A have strictly negative real parts, and

(c)
$$\sup_{\omega \geq 0} r(G(\imath\omega)) < 1 \quad (20)$$
where
$$G(s) = C(sI_n - A)^{-1}B_0$$

Proof: The necessity of (a) follows as $r(L_\alpha) \leq r_\infty < 1$ requires asymptotic stability on finite intervals. To prove the necessity of (b) it is necessary to consider unbounded intervals. Write $[0,\infty) = [0,\alpha_0] \cup [\alpha_0,\infty)$ with $\alpha_0 > t_N$ and let $f \in E_\infty$. Then it can be shown that (6) implies that $y \in E_\infty$. Also by the controllability assumption we can choose η such that $x(\alpha_0)$ in (12) for this case is arbitrary and the resultant response (i.e. $y(t)$ generated by the solution of (12)) on $[\alpha_0,\infty)$ is then given by $y(t) = Ce^{\hat{A}(z)(t-\alpha_0)}x(\alpha_0)$. The uniform boundedness of the response, and the observability assumption, now establish that $\hat{A}(z)$ is Hurwitz for all choices of complex z in the range $\{z : |z| \geq \lambda\}$. The necessity of (b) now follows from this fact, the multivariable Nyquist criterion, and the matrix identity

$$\det(sI_n - \hat{A}(z)) = \frac{\det(sI_n - A)\det(zI_m - G(s))}{z^m}$$

To prove sufficiency, first note that condition (a) and the fact that the spectrum of L_α is independent of α ensures that $r_\infty < 1$ for any $\alpha > t_N$. Now consider the equation $(zI - L_\alpha)y = \eta$. Condition (a) then ensures that the map $(zI - L_\alpha)^{-1} : \eta \mapsto y$ in $L_2^m[0,\alpha_0] \cap L_\infty[0,\alpha_0]$ is uniformly bounded on any interval $[0,\alpha_0]$ with $\alpha_0 > t_N$. Condition (b) shows that the corresponding matrix $\hat{A}(z)$ is bounded and stable for all complex z in the set $\{z : |z| \geq \lambda\}$ where λ is any point in the non-empty set $(\sup_{\omega\geq 0} \; r(G_1(\imath\omega)), 1)$. Hence for these values of z $\exists \tilde{M} > 0$ and $\epsilon > 0$ such that $||e^{\hat{A}(z)t}|| \leq \tilde{M}e^{-\epsilon t}$. The map $\eta \mapsto y$ in $L_2^m[\alpha_0,\alpha) \cap L_\infty[\alpha_0,\alpha)$ is hence also uniformly bounded over the infinite range $\alpha \geq \alpha_0$ and $\{z : |z| \geq \lambda\}$. Condition (b) of Theorem 3 follows by combining these results. ■

Suppose now that as $\alpha \to \infty$, $N \to \infty$ and $t_j \to \infty$. These are termed 'drifting' boundary conditions and the following analysis is possible.

Lemma 1: Suppose that the pair $\{A, B_0\}$ is controllable and consider a differential linear repetitive process described by (1) and (10) in the case when as $\alpha \to \infty$, $N \to \infty$ and $t_j \to \infty$. Then (a) of Theorem 3 for stability along the pass holds in this case if, and only if, $r_\infty < 1$ where

$$r_\infty = \sup_{N\geq 1} \; \max\{0, \sup\{|z| : z \neq 0 \,\&|zI - M(z)| = 0\}\} \quad (21)$$

To examine (b) of Theorem 3, let η denote the projection of $\eta_\infty \in E_\infty$ into E_α via the natural projection $\eta(t) = \eta_\infty(t)$, $t \in [0,\alpha]$. Consider again the equation $zy - f = \eta$, $f = L_\alpha y$, in E_α but in the case when $\alpha \geq t_N$ and $\eta \in E_\alpha$. Then for $\lambda \in (r_\infty, 1)$ it follows that a solution of this equation exists on any interval $[0,\alpha]$ and can be computed from

$$\begin{aligned} y(t) &= z^{-1}[f(t) + \eta(t)] \\ \dot{x}(t) &= Ax(t) + B_0 y(t) \\ f(t) &= Cx(t) \\ x(0) &= \sum_{j=1}^{N_\alpha} K_j y(t_j) \end{aligned} \quad (22)$$

where N_α is the largest integer such that $t_j \leq \alpha$. Also for (b) of Theorem 3 to hold we require that (i) solutions of (22) exist $\forall \alpha$, and (ii) under the usual controllability and observability conditions, the solution y for any $\eta_\alpha \in E_\alpha$ should satisfy $||y|| \leq \tilde{M}||f_\alpha||$ for some constant $\tilde{M} > 0$ which is independent of α.

The first of these conditions is equivalent to Lemma 1 and for the second assume, without loss of generality, that $||\eta_\infty|| = 1$. Then this condition is just the requirement that the family $\{||\eta_\alpha||\}_{\alpha\geq 0}$ is uniformly bounded. Suppose also that the pair $\{C, A\}$ is observable. Then it follows immediately that the conditions of Theorem 1 are necessary for stability along the pass in this case.

Return now to the equation $zy - f = \eta$. Then for $|z| \geq \lambda > r_\infty$, $\lambda \in (r_\infty, 1)$, we have that $||y|| \leq r_\infty^{-1}[1 + ||f||]$. Hence it is sufficient to prove the uniform boundedness of f on $[0,\infty)$ or, since $f(t) = Cx(t)$ (the pair $\{C, A\}$ is assumed to be observable), that x is uniformly bounded on $[0,\infty)$.

Now return to (13) for this case and note that

$$\sum_{j=1}^{N_\alpha} K_j\left[Cx(t_j) + \eta(t_j)\right] = \left[\begin{array}{ccc} K_1C & \cdots & K_{N_\alpha}C \end{array}\right]$$

$$\times \left[\begin{array}{c} x(t_1) \\ \vdots \\ x(t_{N_\alpha}) \end{array}\right] + \sum_{j=1}^{N_\alpha} K_j\eta(t_j)$$

$$\begin{aligned} &= \left[\begin{array}{ccc} K_1C & \cdots & K_{N_\alpha}C \end{array}\right] H^{-1}g(z) + \sum_{j=1}^{N_\alpha} K_j\eta(t_j) \\ &= (zI - \hat{M}(z))^{-1}\left[\begin{array}{ccc} K_1C & \cdots & K_{N_\alpha}C \end{array}\right] g(z) \\ &\quad + \sum_{j=1}^{N_\alpha} K_j\eta(t_j) \\ &= (zI - \hat{M}(z))^{-1}\left\{\sum_{h=1}^{N_\alpha}\sum_{j=1}^{N_\alpha} K_hCe^{\hat{A}(z)t_h}K_j\eta(t_j)\right. \\ &\quad \left. + \sum_{h=1}^{N_\alpha} K_hC\int_0^{t_h} e^{\hat{A}(z)(t_h-\tau)}B_0\eta(\tau)\,d\tau\right\} \\ &\quad + \sum_{j=1}^{N_\alpha} K_j\eta(t_j) \end{aligned}$$

where

$$H = zI - \begin{bmatrix} e^{\hat{A}(z)(t_1)} \\ \vdots \\ e^{\hat{A}(z)(t_{N_\alpha})} \end{bmatrix} \begin{bmatrix} K_1C & \cdots & K_{N_\alpha} \end{bmatrix}$$

and $(zI - \hat{M}(z))^{-1}$ exists for $|z| \geq \lambda,\ \lambda \in (r_\infty, 1)$.

Return now to condition (b) of Theorem 3 in this case and suppose that $r_\infty < 1$ and the conditions of Theorem 4 hold. Then these ensure that $\exists$ real scalars $M_a > 0$, and $\epsilon > 0$, such that

$$||e^{\hat{A}(z)t}|| \leq M_a e^{-\epsilon t},\ \forall t \in [0, \infty),\ \forall\, |z| \geq \lambda,\ \lambda \in (r_\infty, 1)$$

Suppose also that $h := \inf_{j \geq 1}\,(t_{j+1} - t_j) > 0$, and that

$$\sum_{j=1}^{\infty} ||K_j|| < \infty \tag{23}$$

and hence

$$\hat{M}_\infty(z) = \sum_{j=1}^{\infty} K_j C e^{\hat{A}(z)t_j}$$

is uniformly absolutely convergent, i.e. $(zI - \hat{M}(z))^{-1}$ has a uniform bound (in norm) for $|z| \geq \lambda,\ \lambda \in (r_\infty, 1)$.

We are now in a position to establish the following result.

Theorem 5: Suppose that the pair $\{A, B_0\}$ is controllable, the pair $\{C, A\}$ is observable, and the conditions of Theorem 4 hold. Suppose also that $h = \inf_{j \geq 1}\,(t_{j+1} - t_j) > 0$. Then a differential linear repetitive process described by (1) and (10) and as $\alpha \to \infty$, $N \to \infty$, and $t_j \to \infty$. Then stability along the pass holds if (21) and (23) hold.

Proof: Given the assumptions and the analysis above, it remains to prove that the convolution terms in (12) applied to this case have uniform bounds. This follows immediately on noting that

$$t \mapsto \int_0^t e^{\hat{A}(z)(t-\tau)} z^{-1} B_0 \eta(\tau)\, d\tau$$

is uniformly bounded for $t \in [0, \infty),\ |z| \geq \lambda,\ \lambda \in (r_\infty, 1)$,

$$z \mapsto \sum_{h=1}^{\infty} K_h C \int_0^t e^{\hat{A}(z)(t_h - \tau)} B_0 \eta(\tau)\, d\tau$$

is absolutely convergent $\forall\ |z| \geq \lambda,\ \lambda \in (r_\infty, 1)$, and uniformly bounded in this range, and

$$\begin{aligned} z \quad \mapsto \quad & \sum_{h=1}^{N_\alpha} \sum_{j=1}^{N_\alpha} K_h C e^{\hat{A}(z)t_h} K_j \eta(t_j) \\ = \quad & \sum_{j=1}^{N_\alpha} \left[\sum_{h=1}^{N_\alpha} K_h C e^{\hat{A}(z)t_h} \right] K_j \eta(t_j) \\ = \quad & \sum_{j=1}^{N_\alpha} \hat{M}(z) K_j \eta(t_j) \end{aligned}$$

which is absolutely convergent and bounded in N_α for $|z| \geq \lambda,\ \lambda \in (r_\infty, 1)$, since $\hat{M}(z)$ has this property and $||\eta_\infty|| = 1$. ∎

IV. Conclusions

In this paper the stability analysis of an applicable class of linear repetitive processes has been considered. The major focus has been on the influence of the structure of the boundary conditions on stability of so-called differential linear repetitive processes which have immediate use in terms of both the development of systems theory and its applications. By considering three possible types (in increasing order of complexity) it can be concluded that failure to adequately model these conditions and, in particular, the state initial conditions on each pass can lead to a completely incorrect analysis of these processes.

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