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Controller Synthesis for a Class of Second-Order Nonlinear Systems

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Abstract—**This paper presents a constructive method to design stabilizing controllers for a class of second-order nonlinear systems whose input parts may vanish at the origin. A sufficient condition for the existence of control Lyapunov functions (CLFs) and stabilizing controllers is provided. In the case that the small control property does not hold, a formula for constructing continuous state feedback controllers to avoid extremely large control magnitude near the origin and to achieve ultimate boundedness is presented.**

I. INTRODUCTION

In general, finding Lyapunov functions for nonlinear systems is a difficult task unless the systems are in some specific forms. For nonlinear systems in feedback linearizable form, strict feedback form, and feedforward form, there are systematic ways to derive stabilizing controllers and the corresponding Lyapunov functions [3-6]. However, there are some limitations in the systems to guarantee the existence of stabilizing controllers. For demonstration, consider a second-order strict feedback system

$$\begin{aligned}\dot{x}_1 &= x_2 + \phi(x_1)\\ \dot{x}_2 &= f(x_1,x_2) + g(x_1,x_2)u\end{aligned}$$

To apply the backstepping approach to solve its stabilization problem, traditionally it is necessary that the function $g(x_1,x_2)$ (called input part) is nonzero in a neighborhood of the origin. If the input part $g(x_1,x_2)$ vanishes at the origin ($g(0,0)=0$), the backstepping approach cannot be used to solve their stabilization problem. For other class of second-order nonlinear systems [2], the input parts are also required to be nonzero at the origin.

In this paper, we consider the stabilization problem for a class of second-order nonlinear systems whose input parts vanish at the origin. A sufficient condition for the existence of control Lyapunov functions and stabilizing controllers of the considered system is first derived; then, globally and asymptotically stabilizing state feedback controllers can be constructing by using Sontag's formula [7]. The basic idea adopted in this paper is from the CLF introduced in [1, and 5-7]. We discuss both the cases that the *small control property* holds or not. If the *small control property* does not hold, we provide a new formula for constructing continuous feedback laws to avoid extremely large control magnitude near the origin and, in the mean time, to achieve ultimate boundedness.

II. PROBLEM FORMULATION

Consider a second-order nonlinear system:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,x_2)\\ \dot{x}_2 &= f_2(x_1,x_2) + g(x_1,x_2)u\end{aligned} \tag{1}$$

where $x=\begin{bmatrix}x_1 & x_2\end{bmatrix}^T \in R^2$ is the state, $u \in R$ is the control input, functions $f_1(.,.)$, $f_2(.,.)$, and $g(.,.)$ are smooth. Suppose $g(x_1,x_2)=0$ if and only if $x_2=\eta(x_1)$ for some smooth function $\eta(.)$ with $\eta(0)=0$.

We want is to find a function $p(.)$ such that the state feedback controller $u=p(x)$ to globally and asymptotically stabilize (1). If continuous *stabilizing* controllers cannot be found, then we want to derive a continuous controller u=$h(x)$ to achieve ultimate boundedness.

III. MAIN RESULTS

Since $g(x_1,x_2)$ vanishes at the origin, the feedback linearization and the backstepping approaches cannot be employed to solve the considered problem. The following theorem provides a sufficient condition for the existence of CLFs and stabilizing controllers to (1).

Theorem 1: Suppose $g(x_1,x_2)=0$ if and only if $x_2=\eta(x_1)$. If $x_1 f_1(x_1,\eta(x_1))<0$ for all $x_1 \neq 0$, then

$$V(x)=0.5x_1^2+0.5\left(x_2-\eta(x_1)\right)^2$$

is a control Lyapunov function for system (1) and

$$u=p(x)=\begin{cases}-\dfrac{a(x)+\sqrt{a^2(x)+b^4(x)}}{b(x)}, & \text{if } x_2 \neq \eta(x_1)\\ 0, & \text{if } x_2=\eta(x_1)\end{cases} \tag{2}$$

is a globally and asymptotically stabilizing controller for system (1), where

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$$a(x) = x_1 \cdot f_1(x_1, x_2) - (x_2 - \eta(x_1)) \cdot \eta_{x_1}(x_1)$$
$$\times f_1(x_1, x_2) + (x_2 - \eta(x_1)) \cdot f_2(x_1, x_2)$$
$$b(x) = (x_2 - \eta(x_1)) \cdot g(x_1, x_2).$$

Proof: Note that $V(x)$ is radially unbounded. It is clear that

$$\dot{V}(x) = a(x) + b(x)u. \qquad (3)$$

Since $b(x) = 0$ if and only if $x_2 = \eta(x_1)$, then

$$a(x)\big|_{b(x)=0} = x_1 \cdot f_1(x_1, \eta(x_1)) < 0$$

Therefore, $V(x)$ is a control Lyapunov function for the system (1). The fact that the controller (2) stabilizes the system (1) is adopted from [7]. ΔΔ

If $V(x)$ satisfies the *small control property* (see [7]), then the controller (2) is smooth in $R^2 \setminus \{0\}$ and continuous at x=0. However, if $V(x)$ does not satisfy the *small control property*, the controller (2) has extremely large control magnitude near the origin. In the following, for the case that the *small control property* does not hold, we provide a new formula for constructing continuous controllers. Note that we can find s>0 such that $\lim_{\|x\|\to 0} \|x\|^s \cdot \|p(x)\| = 0$ any smooth $p(x)$.

Theorem 2: Consider the system (1). Suppose that $g(x_1, x_2) = 0$ if and only if $x_2 = \eta(x_1)$, and that $x_1 f_1(x_1, \eta(x_1)) < 0$ for all $x_1 \neq 0$. Moreover, suppose the CLF $V(x) = 0.5x_1^2 + 0.5(x_2 - \eta(x_1))^2$ does not satisfy the *small control property*. Let s>0 is such that $\lim_{\|x\|\to 0} \|x\|^s \cdot \|p(x)\| = 0$, then

$$u = h(x) = \begin{cases} p(x), & \text{if } \|x\| > \varepsilon \\ (\|x\| / \varepsilon)^s \cdot p(x), & \text{if } \|x\| \le \varepsilon \end{cases}$$

is continuous in R^2 and is able to achieve ultimate boundedness (that is, $\lim_{t\to\infty} \|x(t)\| < \varepsilon$).

Proof: Substituting u=$h(x)$ into (3) yields

$$\dot{V}(x) = a(x) + b(x) \cdot h(x).$$

If $\|x\| \ge \varepsilon$, $h(x)$=$p(x)$ and therefore

$$\dot{V}(x) = a(x) + b(x) \cdot p(x) < 0.$$

This implies that $\lim_{t\to\infty} \|x(t)\| < \varepsilon$.

Now prove the continuity of $h(x)$. Since $p(x)$ is continuous in $R^2 \setminus \{0\}$, $h(x)$ is continuous in $\{x \in R^n \,\|\, \|x\| \ge \varepsilon\}$. Moreover, it can be verify that $(\|x\| / \varepsilon)^s\, p(x)$ is continuous in R^2, and that $(\|x\| / \varepsilon)^s\, p(x) = p(x)$ on the boundary $\|x\| = \varepsilon$. Hence, $h(x)$ is continuous in R^2. ΔΔ

IV. Example

Consider the system (1) with $f_1(x_1, x_2) = x_1 x_2 + 0.1 \cdot x_1^3 x_2^2$, $f_2(x_1, x_2) = \cos(x_2) \cdot x_1^2 x_2 + x_1$, and $g(x_1, x_2) = (x_1^2 + 2x_2) \cdot (1 + x_2^2)$. Note that $g(0,0) = 0$ and $g(x_1, x_2) = 0$ if and only if $x_2 = \eta(x_1) \equiv -0.5x_1^2$. Moreover, $x_1 f_1(x_1, \eta(x_1)) < 0$ for all $x_1 \neq 0$. Therefore,

$$V(x) = 0.5x_1^2 + 0.5(x_2 - \eta(x_1))^2 = 0.5x_1^2 + 0.5(x_2 + 0.5x_1^2)^2$$

is a CLF for the considered system. However, it does not satisfy the *small control property*. Let $a(x)$, $b(x)$, and $p(x)$ be defined as in Theorem 1. We can see that $\lim_{\|x\|\to 0} \|x\| \cdot \|p(x)\| = 0$. Let $h(x)$ be defined as in Theorem 2 with s=1 and $\varepsilon = 0.2$.

Fig. 1 shows the trajectories and control inputs of the considered system with u=$p(x)$ and u=$h(x)$, respectively. Note that in the case of u=$h(x)$, the state trajectory almost the same as that of the case u=$p(x)$ in the first 200 seconds. However, its control magnitude near the origin is very small comparing with that of u=$p(x)$.

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References

[1] Z. Artstein, "Stabilization with relaxed control," *Nonl. Anal.*, TMA 7, pp. 1163-1173, 1983.

[2] W. Dayawansa and C. Martin, "Asymptotic stabilization of two dimensional real analytic systems," *Systems & Control Letters*, vol. 12, pp. 205-211, 1989.

[3] K. Ezal, Z. Pan, and P. V. Kokotovic, "Locally optimal and robust backstepping Design," *IEEE Trans. on Automatic Control*, vol. 45, pp. 260-271, 2000.

[4] A. Isidori, *Nonlinear Control Systems*, Third Edition, Springer-Verlag, 1995.

[5] P. P. Kokotovic and M. Arcak, "Constructive nonlinear control: a historical perspective," *Automatica*, Vol. 37, 637-662, 2001.

[6] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*, Wiley-Interscience, 1995.

[7] E. D. Sontag, "A "universal" constructive of Artstein's theorem on nonlinear stabilization," *Systems & Control Letters*, Vol. 12, pp. 542-550, 1989.

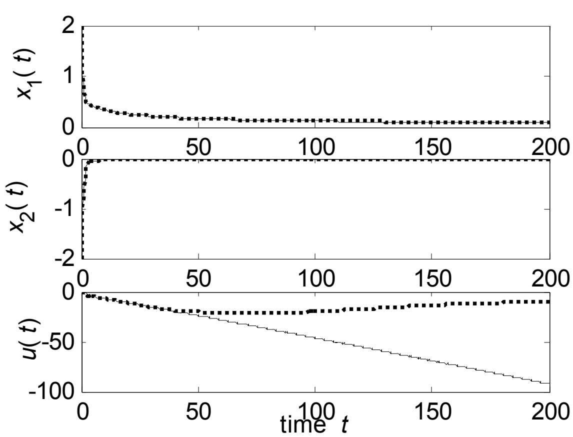

Fig. 1. The state trajectories and control inputs of the considered system with controllers u=$p(x)$ (solid line) and u=$h(x)$ (dotted line).