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Non-smooth Feedback Stabilizer for Strict-feedback Nonlinear Systems Not Even Linearizable at the Origin

S.G. Cheong, J. Back, H. Shim and J.H. Seo

***Abstract*— We present a continuous feedback stabilizer for nonlinear systems in the strict-feedback form, whose chained integrator part has the power of positive odd *rational* numbers. Since the power is not restricted to be larger than or equal to one, the linearization of the system at the origin may fail. Nevertheless, we will show that the closed-loop system is globally strongly stable with the proposed continuous (but, possibly not differentiable) feedback. We formulate a condition that enables our design by characterizing the powers of the given system. The condition also shows that our result is an extension of (Qian and Lin, *Systems and Control Letters*, 2001) where the power of odd positive integers has been considered.**

I. INTRODUCTION

In practice, there exist systems that do not have the first approximation at the origin, e.g., a leaky bucket whose dynamics is given by $\dot{h} = -C\sqrt{h}$ [1, p.41], or the hydraulic control systems [2]. Partly motivated by this fact, we construct a continuous (but possibly non-differentiable) state feedback stabilizer which globally stabilizes a single-input nonlinear system in the strict-feedback form given by

$$\begin{aligned}\dot{x}_1 &= x_2^{r_1} + \phi_1(x_1)\\ \dot{x}_2 &= x_3^{r_2} + \phi_2(x_1, x_2)\\ &\vdots\\ \dot{x}_n &= u^{r_n} + \phi_n(x_1, \cdots, x_n)\end{aligned} \tag{1}$$

where $\phi_i(x_1, \cdots, x_i)$, $i = 1, \cdots, n$, are C^1 functions vanishing at the origin and r_i's are *rational numbers* whose numerators and denominators are all positive odd integers (we will call such r_i a positive odd rational number). We stress that the linearization of the system at the origin may fail, that is, it may not exist, since r_i's can have a value less than 1.

This is a sharp contrast to the previous works [3]–[9] which have considered a system whose right-hand side is C^1 in the state x, or all r_i's are greater than or equal to 1 so that its linearization at the origin may be uncontrollable. In [3], [4], [6], they constructed a state feedback for the system (1) in which all r_i's are positive odd integers. Lin and Qian [3] explicitly constructed, using a tool called adding a power integrator, a globally stabilizing *smooth* feedback control law for system (1) under the condition that the odd integer powers r_i are in decreasing order (i.e., $r_1 \geq \cdots \geq r_n \geq 1$), and under a growth condition that $|\phi_i(x_1, \cdots, x_i)| \leq (|x_1|^{r_i} + \cdots + |x_i|^{r_i})\gamma_i(x_1, \cdots, x_i)$, $i = 1, \cdots, n$, where each $\gamma_i(\cdot)$ is a smooth nonnegative function. The decreasing assumption and the growth condition have been removed in [4], [7] while a continuous (instead of smooth) feedback is obtained in [4] and a smooth but time-varying feedback is designed in [7]. More generally, a triangular system

$$\begin{aligned}\dot{x}_i &= f_i(x_1, \cdots, x_{i+1}), \qquad i = 1, \cdots, n-1,\\ \dot{x}_n &= f_n(x_1, \cdots, x_n) + u,\end{aligned}$$

has been dealt with in [8], [9], but it is assumed that all $f_i(\cdot)$'s are C^∞ so that its linearization at the origin does exist.

The proposed feedback stabilizer for system (1) is continuous but may not be differentiable in the state because the linearization of the system may fail, or it may be unstable but uncontrollable recalling the Brockett's necessary condition [10] for smooth feedback. Also, uniqueness of the solution for system (1) is not guaranteed since the right-hand side of (1) may not be locally Lipschitz (again because some r_i can be less than 1 and/or the feedback control may also be just C^0). Therefore, the concept of stability meant in this paper is *global strong stability* (GSS) introduced in [11], which is understood that the origin is stable in the sense of Kurzweil and every solution of the system with any initial condition in $\mathbb{R}^n$ converges to zero as time goes to infinity (although a solution from a certain initial condition may not be unique). In fact, like in [4], we will design a C^0 feedback control as well as a corresponding C^1 (positive definite and proper) Lyapunov function. Then, by [11, pp. 23,24], GSS of the closed-loop system follows once the time derivative of the Lyapunov function is a negative definite function of the state.

The paper is organized as follows. In Section II-A, we state our main theorem whose proof is given in Section II-C where a constructive design procedure of the feedback stabilizer is presented. In Section II-B, the assumption proposed in the main theorem of Section II-A is discussed in detail, where some relation to the previous work [4] is also pointed out. We conclude the paper in Section IV after presenting a design example in Section III.

For convenience, let us define the set of all rational numbers whose numerators and denominators are all positive odd integers by Q_{odd}. Note that the set Q_{odd} is closed under multiplication, division and odd number of additions, but is not closed under even number of additions or subtraction, and that, furthermore, x^{a+b} or $x^{c(a+b)}$ for a, b, $c \in Q_{odd}$ is a positive function of x.

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All authors are with ASRI, School of Electrical Engineering and Computer Science, Seoul National Univ., Kwanak P.O.Box 34, Seoul, 151-600, Korea. Corres. author: J. Back (backhoon@acorn.snu.ac.kr)

II. Main Result

A. Statement of Main Theorem

We now state our main theorem.

Theorem 1: Suppose that, for the system (1), all $r_i \in Q_{odd}$, $i = 1, \cdots, n$. If there exist $\mu_0, \mu_1, \cdots, \mu_n \in Q_{odd}$ such that

$$\mu_0, \cdots, \mu_n \geq 1, \tag{2}$$

$$\frac{r_1}{\mu_1} \leq \frac{1}{\mu_0}, \frac{r_2}{\mu_2} \leq \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}\right\}, \cdots, \frac{r_n}{\mu_n} \leq \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}, \cdots, \frac{1}{\mu_{n-1}}\right\}, \tag{3}$$

$$0 \leq \frac{1}{\mu_0} - \frac{r_1}{\mu_1} \leq \frac{1}{\mu_1} - \frac{r_2}{\mu_2} \leq \cdots \leq \frac{1}{\mu_{n-1}} - \frac{r_n}{\mu_n}, \tag{4}$$

then there exists a C^0 feedback controller $u = u(x)$ with $u(0) = 0$ which renders the origin of the closed system globally strongly stable. In addition, if the assumption holds with all $\mu_i = 1$ $(i = 0, \cdots, n)$, then the feedback controller $u(\cdot)$ is smooth. ♢

Remark 1: From (4), the condition that $\mu_i = 1$ $(i = 0, \cdots, n)$ is equivalent to $1 \geq r_1 \geq \cdots \geq r_n$. Note also that, once a set of μ_i's satisfying the assumption is found, $(q\mu_i)$ with $q \in Q_{odd}$ also satisfies the assumption if (2) holds with them. On the other hand, the value of r_n does not restrict the existence of a set of μ_i's for the assumption (because a large μ_n can always be chosen), which is conceptually presumed since the input term u^{r_n} of the system (1) can be replaced by another control v. ♢

B. Discussions on the Assumption

The assumption of Theorem 1 results in a generalization of the previous works. To see this, we provide several cases when the assupmtion holds automatically in the following:

(a) the dimension of the system (1) is less than 3, and all $r_i \in Q_{odd}$,
(b) the dimension of the system (1) is 3 and all $r_i \in Q_{odd}$ are such that $r_i \leq 1$, $i = 1, 2$,
(c) all $r_i \in Q_{odd}$ are such that $r_i < 1/2$, $i = 1, \cdots, n-1$,
(d) all $r_i \in Q_{odd}$ are such that $1 \leq r_i$, $i = 1, \cdots, n-1$,
(e) all $r_i \in Q_{odd}$ are such that $1 \leq r_i$, $i = 1, \cdots, m$, and $r_i < 1/2$, $i = m+1, \cdots, n-1$, with some $1 \leq m \leq n-2$,
(f) all $r_i \in Q_{odd}$ are in decreasing order, i.e., $r_1 \geq \cdots \geq r_{n-1}$.

It is seen that the result of [4] is recovered as the case (d), and the case (e) implies a cascade connection of (d) and (c). Now let us prove the claims one by one.

Proof of (a): In case of $n = 1$, the selection of $\mu_0 = 1$ and $\mu_1 = \max\{1, r_1\}$ satisfies the assumption. When $n = 2$, the assumption holds with $\mu_0 = 1$, $\mu_1 = \max\{1, r_1\}$ and $\mu_2 = r_2 \max\{1, r_1, 1/r_1\}$.

Proof of (b): Pick μ_1, μ_2, μ_3 as

$$\frac{1}{\mu_1} = \frac{1}{r_1}\frac{1}{\mu_0}, \qquad \frac{1}{\mu_2} = \frac{1}{r_2}\frac{1}{\mu_0},$$
$$\frac{1}{\mu_3} = \frac{1}{r_3}\min\left\{1, 1 + \frac{1}{r_2} - \frac{1}{r_1}\right\}\frac{1}{\mu_0}.$$

The assumption (3) follows from these since $1/\mu_1 \geq 1/\mu_0$ and $1/\mu_2 \geq 1/\mu_0$ due to $r_i \leq 1$.

Now choose μ_0 as

$$\mu_0 = \max\left\{\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\min\left\{1, 1 + \frac{1}{r_2} - \frac{1}{r_1}\right\}\right\},$$

so that $1 \leq \mu_i \in Q_{odd}$, $i = 0, \cdots, 3$ (assumption (2)).

Then, the assumption (4) also holds. Indeed,

$$\frac{1}{\mu_0} - \frac{r_1}{\mu_1} = 0, \qquad \frac{1}{\mu_1} - \frac{r_2}{\mu_2} = \left(\frac{1}{r_1} - 1\right)\frac{1}{\mu_0} \geq 0,$$
$$\frac{1}{\mu_2} - \frac{r_3}{\mu_3} = \left(\frac{1}{r_2} - \min\left\{1, 1 + \frac{1}{r_2} - \frac{1}{r_1}\right\}\right)\frac{1}{\mu_0} \geq \left(\frac{1}{r_1} - 1\right)\frac{1}{\mu_0}.$$

Proof of (c): We first choose $\mu_0, \cdots, \mu_{n-1}$ to be proportional to μ_n which will be determined later. Let $r_{\min} \triangleq \min\{r_1, r_2, \cdots, r_n\}$ and choose μ_{n-1} as

$$\frac{1}{\mu_{n-1}} = \frac{r_n/r_{\min}}{\mu_n} \triangleq \frac{\tilde{r}_{n-1}}{\mu_n}, \qquad \mu_n \in Q_{odd} \tag{5}$$

so that $\frac{1}{\mu_{n-1}} - \frac{r_n}{\mu_n} = \frac{1}{\mu_n}(\tilde{r}_{n-1} - r_n) > 0$ from $r_{\min} < 1/2$. Similarly, determine $\mu_{n-2}, \cdots, \mu_0$ recursively as

$$\begin{aligned} \frac{1}{\mu_{n-2}} &= \frac{r_{n-1}}{\mu_{n-1}} + \frac{1}{\mu_n}(\tilde{r}_{n-1} - r_n) \\ \frac{1}{\mu_{n-3}} &= \frac{r_{n-2}}{\mu_{n-2}} + \frac{1}{\mu_n}(\tilde{r}_{n-1} - r_n) \\ &\vdots \\ \frac{1}{\mu_0} &= \frac{r_1}{\mu_1} + \frac{1}{\mu_n}(\tilde{r}_{n-1} - r_n), \end{aligned} \tag{6}$$

which are all in Q_{odd} since $\mu_n \in Q_{odd}$. Then, (4) follows with

$$0 < \frac{1}{\mu_0} - \frac{r_1}{\mu_1} = \frac{1}{\mu_1} - \frac{r_2}{\mu_2} = \cdots = \frac{1}{\mu_{n-1}} - \frac{r_n}{\mu_n}.$$

The above choice of μ_i's also results in

$$\begin{aligned} \frac{1}{\mu_{n-2}} &= \frac{1}{\mu_n}\left(r_{n-1}\tilde{r}_{n-1} + \tilde{r}_{n-1} - r_n\right) \triangleq \frac{\tilde{r}_{n-2}}{\mu_n} \\ \frac{1}{\mu_{n-3}} &= \frac{1}{\mu_n}\left(r_{n-2}\tilde{r}_{n-2} + \tilde{r}_{n-1} - r_n\right) \triangleq \frac{\tilde{r}_{n-3}}{\mu_n} \\ &\vdots \\ \frac{1}{\mu_0} &= \frac{1}{\mu_n}\left(r_1\tilde{r}_1 + \tilde{r}_{n-1} - r_n\right) \triangleq \frac{\tilde{r}_0}{\mu_n}. \end{aligned} \tag{7}$$

Therefore, by picking $\mu_n \in Q_{odd}$ such that

$$\mu_n \geq \max\{1, \tilde{r}_0, \cdots, \tilde{r}_{n-1}\},$$

the assumption (2) follows from (5) and (7).

Since $r_n = r_{\min}\tilde{r}_{n-1}$, we have from (7) that

$$\tilde{r}_{n-1} \le \tilde{r}_{n-2} = r_{n-1}\tilde{r}_{n-1} + \tilde{r}_{n-1} - r_{\min}\tilde{r}_{n-1} < 2\tilde{r}_{n-1}$$

because $r_{\min} \le r_{n-1} < 1/2$, and that

$$\tilde{r}_{n-1} \le \tilde{r}_{n-3} = r_{n-2}\tilde{r}_{n-2} + \tilde{r}_{n-1} - r_{\min}\tilde{r}_{n-1} < 2\tilde{r}_{n-1}$$

because $\tilde{r}_{n-1} \le \tilde{r}_{n-2}$ from above and $r_{n-2} + 1 - r_{\min} \ge 1$ for the left inequality, and because $\tilde{r}_{n-2} < 2\tilde{r}_{n-1}$ from above and $(2r_{n-2} + 1 - r_{\min}) < 2$. Likewise, we have that

$$\tilde{r}_{n-1} \le \tilde{r}_i = r_{i+1}\tilde{r}_{i+1} + \tilde{r}_{n-1} - r_{\min}\tilde{r}_{n-1} < 2\tilde{r}_{n-1} \quad (8)$$

for $i = 0, \cdots, n-1$, where the case $n-1$ is trivially true. From the left inequality of (8),

$$\frac{\tilde{r}_{n-1}}{\mu_n} = \frac{\tilde{r}_{n-1}}{\tilde{r}_i}\frac{\tilde{r}_i}{\mu_n} \le \frac{\tilde{r}_i}{\mu_n} = \frac{1}{\mu_i}, \quad i = 0, \cdots, n-1. \quad (9)$$

Finally, from (5), (7), the right inequality of (8), and since $2r_i < 1$,

$$\frac{r_i}{\mu_i} = \frac{r_i\tilde{r}_i}{\mu_n} \le \frac{2r_i\tilde{r}_{n-1}}{\mu_n} < \frac{\tilde{r}_{n-1}}{\mu_n},$$

for $i = 1, \cdots, n-1$. This inequality, combined with (9), leads to the assumption (3).

Proof of (d): Pick any $\mu_0 \in Q_{odd}$ such that $\mu_0 \ge 1$. Then, by taking μ_i's recursively as

$$\mu_1 = r_1\mu_0, \ \mu_2 = r_2\mu_1, \ \cdots, \ \mu_n = r_n\mu_{n-1},$$

the assumption is satisfied. In fact, (3) and (4) hold with equality (because $r_i \ge 1$, and thus, $\mu_0 \le \mu_1 \le \cdots \le \mu_n$), and clearly, (2) is also satisfied.

Proof of (e): First, we choose $\bar{\mu}_0, \cdots, \bar{\mu}_{n-m}$ through the algorithm in Proof of (c) for $r_{m+1}, \cdots, r_n$ (as if they were $r_1, \cdots, r_{n-m}$). And, let

$$\mu_{m+i} = r_1 \cdots r_m \bar{\mu}_i, \quad i = 0, \cdots, n-m.$$

Note that a multiplication of any value of Q_{odd} larger than 1 does not violate the assumption. For the rest of μ_i, we choose

$$\mu_{m-1} = \frac{\mu_m}{r_m} = r_1 \cdots r_{m-1}\bar{\mu}_0, \quad \cdots, \quad \mu_0 = \frac{\mu_1}{r_1} = \bar{\mu}_0.$$

Since $r_i \ge 1$ for $i = 0, \cdots, m$, this selection ensures (2).

Also, $\mu_0, \cdots, \mu_m$ satisfy (4) with equality (recall that their selection is the same as in Proof of (d)). In fact, the assumption (4) holds with all μ_i's because $\mu_m, \cdots, \mu_n$ have been constructed as in Proof of (c), and it hold that $1/\mu_{m-1} - r_m/\mu_m \le 1/\mu_m - r_{m+1}/\mu_{m+1}$ because the left hand side is zero.

While $\mu_0, \cdots, \mu_m$ again satisfy (3) as in Proof of (d), the assumption also holds with $\mu_{m+1}, \cdots, \mu_n$. Indeed,

$$\frac{r_{m+1}}{\mu_{m+1}} \le \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}, \cdots, \frac{1}{\mu_m}\right\}, \quad \cdots,$$
$$\frac{r_n}{\mu_n} \le \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}, \cdots, \frac{1}{\mu_{n-1}}\right\}$$

because $1/\mu_m \le \cdots \le 1/\mu_0$, and thus, the above holds by the construction of $\mu_m, \cdots, \mu_n$.

Proof of (f): If $r_1 \ge \cdots \ge r_m \ge 1 > r_{m+1} \ge \cdots \ge r_{n-1}$ with a certain $1 \le m \le n-2$, pick μ_i as

$$\mu_0 = 1, \ \mu_1 = r_1\mu_0, \ \cdots, \ \mu_m = r_m\mu_{m-1},$$
$$\mu_{m+i} = \mu_m, \quad (i = 1, \cdots, n-m-1,)$$
$$\mu_n = \mu_m \frac{r_n}{r_{\min}},$$

where $r_{\min} \triangleq \min\{r_1, \cdots, r_n\}$. Then, (2) holds in an obvious way, and $\mu_0, \cdots, \mu_m$ satisfy (3) and (4) with equality by the same reason as in the proof for (d), while, with $\mu_{m+1}, \cdots, \mu_n$, (3) and (4) become

$$\frac{r_{m+1}}{\mu_{m+1}} = \frac{r_{m+1}}{\mu_m} \le \frac{1}{\mu_m}, \quad \frac{r_{m+2}}{\mu_{m+2}} = \frac{r_{m+2}}{\mu_m} \le \frac{1}{\mu_m}, \quad \cdots,$$

and

$$0 = \frac{1}{\mu_{m-1}} - \frac{r_m}{\mu_m} \le \frac{1 - r_{m+1}}{\mu_m} \le \frac{1 - r_{m+2}}{\mu_m} \le \cdots$$
$$\le \frac{1}{\mu_m} - \frac{r_n}{\mu_n} = \frac{1}{\mu_m} - \frac{r_{\min}}{\mu_m},$$

which are true.

If $m = n-1$, this is a special case of (d), and if $1 > r_1$, then the assumption holds similarly with $\mu_0 = \cdots = \mu_{n-1} = 1$ and $\mu_n = r_n/r_{\min}$.

Remark 2: Other than the cases of items (a)–(f) in the above, the existence of μ_i's satisfying the assumption depends on the given set of r_i's[1]. However, there does exist an example of r_i's that do not fit into any cases (a)–(f), for which it is impossible to find appropriate μ_i's. Indeed, consider $r_1 = \frac{1}{3}$, $r_2 = 3$, $r_3 = 1$ and suppose that there exist $\mu_0, \cdots, \mu_3$ satisfying the assumption. Then, by (4),

$$0 \le -\frac{1}{\mu_0} + \frac{4}{3}\frac{1}{\mu_1} - \frac{3}{\mu_2}, \quad 0 \le -\frac{1}{\mu_1} + \frac{4}{\mu_2} - \frac{1}{\mu_3},$$

and by the second inequality of (3),

$$\frac{3}{\mu_2} \le \frac{1}{\mu_0}.$$

These three inequalities are compactly written as

$$\begin{bmatrix} -1 & \frac{4}{3} & -3 & 0 \\ 0 & -1 & 4 & -1 \\ 1 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\mu_0} \\ \frac{1}{\mu_1} \\ \frac{1}{\mu_2} \\ \frac{1}{\mu_3} \end{bmatrix} \succeq 0$$

where $\succeq$ implies the component-wise comparison. Now, by pre-multiplying a row vector $[2, 3, 2]$, we obtain

$$-(1/3)/\mu_1 - 3/\mu_3 \ge 0,$$

which is a contradiction of (2). ◊

[1]Since the conditions (2), (3) and (4) can be easily converted into the LMI (Linear Matrix Inequality) form, it can also be solved numerically.

C. Constructive Proof of the Main Theorem

In order to prove Theorem 1, we construct a feedback stabilizer through a modified backstepping procedure. The novelty of this procedure resides in the construction of the corresponding control Lyapunov function at each step. Unlike the conventional backstepping [12] or the construction of [4], the control Lyapunov function needs to be chosen considering the design of later steps to come. To enable this, we have formulated by the assumption of Theorem 1 a key property necessary for the selection of Lyapunov functions at each step. (Recall that the selection of μ_i's is affected by the set of whole r_i's in the assumption.) This implies that we will use the values of μ_i's, which is obtained by the assumption, in the backstepping procedure.

Furthermore, we will frequently employ the following inequalities borrowed[2] from [4].

- For $x, y \in \mathbb{R}$ and $1 \le q \in Q_{odd}$, we have
$$|x+y|^q \quad \le \quad 2^{q-1}|x^q + y^q|. \qquad (10)$$
- For $0 < c, d \in \mathbb{R}$ and $\rho > 0$,
$$|x|^c|y|^d \le \frac{c}{c+d}\rho|x|^{c+d} + \frac{d}{c+d}\rho^{-\frac{c}{d}}|y|^{c+d}. \qquad (11)$$
- For $a, b, c \in \mathbb{R}$, if $0 < a \le b \le c$, we have
$$|x|^b \le |x|^a + |x|^c = |x|^a(1+|x|^{c-a}), \quad x \in \mathbb{R}, \qquad (12)$$
because $|x|^b \le |x|^a \le |x|^a + |x|^c$ for $|x| \le 1$ and $|x|^b \le |x|^c \le |x|^a + |x|^c$ for $|x| > 1$.
- Let $\phi_i : \mathbb{R}^i \to \mathbb{R}$ be a C^1 function with $\phi_i(0) = 0$. Then, there exists a smooth non-negative function $\gamma_i(x_1, x_2, \cdots, x_i)$ such that
$$|\phi_i(x_1, \cdots, x_i)| \le (|x_1| + \cdots + |x_i|)\gamma_i(x_1, \cdots, x_i). \qquad (13)$$

In addition to those $\mu_0, \cdots, \mu_n$ selected through the assumption, we choose $v_0, \cdots, v_{n-1} \in Q_{odd}$ satisfying
$$\begin{aligned} &v_0, v_1, \cdots, v_{n-1} \ge 1, \\ &v_0 + \frac{r_1}{\mu_1} = v_1 + \frac{r_2}{\mu_2} = \cdots = v_{n-1} + \frac{r_n}{\mu_n}, \end{aligned} \qquad (14)$$
which will also be used through the procedure.

The proof is given in an induction and the structure of the proof is similar to [4]. For the first step of the induction, we choose a Lyapunov function as
$$V_1(x_1) = \int_{x_1^*}^{x_1} \left(\sigma^{\mu_0} - x_1^{*\mu_0}\right)^{v_0} \, d\sigma$$
where $x_1^* \equiv 0$ for convenience. Note that $V_1(\cdot)$ is C^1, positive definite and proper. Then, by (13),
$$\begin{aligned} &\dot{V}_1(x_1) = \left(x_1^{\mu_0}\right)^{v_0} \left(x_2^{r_1} + \phi_1(x_1)\right) \\ &\le \left(x_1^{\mu_0}\right)^{v_0} \left(x_2^{r_1} - x_2^{*r_1} + x_2^{*r_1}\right) + |x_1|^{v_0\mu_0}|x_1|\gamma_1(x_1). \end{aligned}$$

[2]Inequalities (10) and (11) are proved (with slight extension) by [4, Lemma 2.3] and [4, Lemma 2.4], respectively, while inequality (13) is quite standard.

Since $\mu_0 r_1/\mu_1 \le 1$ by (3), we have, using (12),
$$|x_1| \le |x_1|^{\mu_0\frac{r_1}{\mu_1}}(1 + |x_1|^{\left(c - \mu_0\frac{r_1}{\mu_1}\right)})$$
where $c \ge 1$. Then, by taking a smooth positive function $\tilde{\gamma}_1$ such that $(1 + |x_1|^{(c-\mu_0 r_1/\mu_1)})\gamma_1(x_1) \le \tilde{\gamma}_1(x_1)$, we obtain
$$\begin{aligned} \dot{V}_1(x_1) \le &\left(x_1^{\mu_0}\right)^{v_0} \left(x_2^{r_1} - x_2^{*r_1}\right) \\ &+ \left(x_1^{\mu_0}\right)^{v_0} x_2^{*r_1} + \left(x_1^{\mu_0}\right)^{v_0 + \frac{r_1}{\mu_1}} \tilde{\gamma}_1(x_1). \end{aligned}$$
If we take the virtual control x_2^* such that
$$x_2^{*r_1} = -\left(x_1^{\mu_0}\right)^{\frac{r_1}{\mu_1}}(n + \tilde{\gamma}_1(x_1)) := -\left(x_1^{\mu_0}\right)^{\frac{r_1}{\mu_1}} \alpha_1^{\frac{r_1}{\mu_1}}(x_1), \qquad (15)$$
then we have
$$\dot{V}_1(x_1) \le -n\left(x_1^{\mu_0}\right)^{v_0 + \frac{r_1}{\mu_1}} + \left(x_1^{\mu_0}\right)^{v_0} \left(x_2^{r_1} - x_2^{*r_1}\right).$$
It should be noted that x_2^* and $x_2^{*r_1}$ may not be C^1, while $x_2^{*\mu_1}$ is (because $\mu_0 \ge 1$).

Remark 3: When all $\mu_i = 1$, a smooth (i.e., C^∞) feedback stabilizer can be obtained. For this, instead of α_1 in (15), design a smooth α_1 such that
$$(n + \tilde{\gamma}_1(x_1))^{\frac{1}{r_1}} \le \alpha_1(x_1).$$
Then, by letting $x_2^{*r_1} = -\left(x_1^{\mu_0}\right)^{\frac{r_1}{\mu_1}} \alpha_1^{\frac{r_1}{\mu_1}}(x_1)$ instead of (15), all the above arguments still hold, but $x_2^* (= x_2^{*\mu_1})$ is smooth. The same technique can be used for subsequent steps. ♢

Inductive assumption: Suppose at $(k-1)$-th step, there are a C^1, proper and positive definite Lyapunov function $V_{k-1}(x_1, \cdots, x_{k-1})$ and a set of C^0 virtual controllers x_1^*, $\cdots$, x_k^* defined by
$$\begin{aligned} x_1^* &= 0, & \eta_1 &= x_1^{\mu_0} - x_1^{*\mu_0} \\ x_2^{*\mu_1} &= -\eta_1\alpha_1(x_1), & \eta_2 &= x_2^{\mu_1} - x_2^{*\mu_1} \\ &\vdots & &\vdots \\ x_k^{*\mu_{k-1}} &= -\eta_{k-1}\alpha_{k-1}(x_1, \cdots, x_{k-1}), & \eta_k &= x_k^{\mu_{k-1}} - x_k^{*\mu_{k-1}} \end{aligned}$$
where $\alpha_i^{\frac{r_i}{\mu_i}}(x_1, \cdots, x_i)$, $1 \le i \le k-1$, are smooth positive functions, such that
$$\begin{aligned} &\dot{V}_{k-1}(x_1, \cdots, x_{k-1}) \le -(n-k+2) \times \\ &\left(\eta_1^{v_0 + \frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2} + \frac{r_{k-1}}{\mu_{k-1}}}\right) + \eta_{k-1}^{v_{k-2}} \left(x_k^{r_{k-1}} - x_k^{*r_{k-1}}\right). \end{aligned}$$

To complete the induction, we consider at k-th step a Lyapunov function defined by
$$\begin{aligned} V_k(x_1, \cdots, x_k) &= V_{k-1}(x_1, \cdots, x_{k-1}) + U_k(x_1, \cdots, x_k), \\ U_k(x_1, \cdots, x_k) &\triangleq \int_{x_k^*}^{x_k} \left(\sigma^{\mu_{k-1}} - x_k^{*\mu_{k-1}}\right)^{v_{k-1}} d\sigma. \end{aligned}$$

The function V_k can be shown to be C^1, proper and positive definite with the following property: for $i = 1, \cdots, k-1$,
$$\frac{\partial U_k}{\partial x_i} = -\int_{x_k^*}^{x_k} v_{k-1}\left(\sigma^{\mu_{k-1}} - x_k^{*\mu_{k-1}}\right)^{v_{k-1}-1} d\sigma \frac{\partial\left(x_k^{*\mu_{k-1}}\right)}{\partial x_i}, \qquad (16)$$

and

$$\frac{\partial U_k}{\partial x_k} = \left(x_k^{\mu_{k-1}} - x_k^{*\mu_{k-1}}\right)^{v_{k-1}} = \eta_k^{v_{k-1}}. \quad (17)$$

Proofs of these properties proceed just in the same way as in the proofs for [4, Prop. 1 and 2] where the set of positive odd integers is considered instead of our Q_{odd}. To save the space from the lengthy proofs, we simply refer to the proofs of [4, Prop. 1 and 2] for the claims above (the last equality (17) is, however, trivial).

With these properties, we obtain

$$\begin{aligned}
&\dot{V}_k(x_1, \cdots, x_k) \\
&\le -(n-k+2)\left(\eta_1^{v_0+\frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}}\right) \\
&\quad + \eta_{k-1}^{v_{k-2}}\left(x_k^{r_{k-1}} - x_k^{*r_{k-1}}\right) \\
&\quad + \eta_k^{v_{k-1}}\left(x_{k+1}^{r_k} + \phi_k(x_1, \cdots, x_k)\right) + \sum_{i=1}^{k-1} \frac{\partial U_k}{\partial x_i}\dot{x}_i. \quad (18)
\end{aligned}$$

We investigate the terms in (18) one by one. For the second term, we first note that it is true that

$$\left|x_k^{r_{k-1}} - x_k^{*r_{k-1}}\right|^{\frac{\mu_{k-1}}{r_{k-1}}} \le 2^{\frac{\mu_{k-1}}{r_{k-1}}-1}\left|x_k^{\mu_{k-1}} - x_k^{*\mu_{k-1}}\right|$$

from (10) since $\mu_{k-1}/r_{k-1} \ge 1$ due to (2) and (3). Then, it follows that

$$\left|x_k^{r_{k-1}} - x_k^{*r_{k-1}}\right| \le 2^{1-\frac{r_{k-1}}{\mu_{k-1}}}\left|x_k^{\mu_{k-1}} - x_k^{*\mu_{k-1}}\right|^{\frac{r_{k-1}}{\mu_{k-1}}}.$$

Hence we obtain

$$\begin{aligned}
&\left|\eta_{k-1}^{v_{k-2}}\left(x_k^{r_{k-1}} - x_k^{*r_{k-1}}\right)\right| \le |\eta_{k-1}|^{v_{k-2}} 2^{1-\frac{r_{k-1}}{\mu_{k-1}}} |\eta_k|^{\frac{r_{k-1}}{\mu_{k-1}}} \\
&\quad \le \frac{1}{3}\eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}} + \eta_k^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}} c_k \\
&\quad \le \frac{1}{3}\left(\eta_1^{v_0+\frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}}\right) + \eta_k^{v_{k-1}+\frac{r_k}{\mu_k}} c_k
\end{aligned} \quad (19)$$

where c_k is some positive constant obtained by employing (11) with a certain constant ρ leading to the $\frac{1}{3}$ term in the second inequality, and the last inequality follows from (14) and the fact that $x^{a+b} \ge 0$ for all $x \in \mathbb{R}$ with $a, b \in Q_{odd}$.

To handle the third term of (18), we note that

$$\begin{aligned}
&\left|\eta_k^{v_{k-1}}\phi_k(x_1, \cdots, x_k)\right| \\
&\le |\eta_k|^{v_{k-1}}(|x_1| + \cdots + |x_k|)\gamma_k(x_1, \cdots, x_k) \quad \text{by (13)} \\
&\le |\eta_k|^{v_{k-1}}\{(|x_1| + \cdots + |x_k - x_k^*|) \\
&\qquad + (|x_2^*| \cdots + |x_k^*|)\}\gamma_k(x_1, \cdots, x_k)
\end{aligned}$$

(since $|x_{i+1} - x_{i+1}^*|^{\mu_i} \le 2^{\mu_i - 1}|x_{i+1}^{\mu_i} - x_{i+1}^{*\mu_i}|$ for $i = 0, \cdots, k-1$ by (10), and since $x_{i+1}^{*\mu_i} = -\eta_i\alpha_i$)

$$\begin{aligned}
&\le |\eta_k|^{v_{k-1}}\left(\sum_{j=0}^{k-1}|\eta_{j+1}|^{\frac{1}{\mu_j}} 2^{1-\frac{1}{\mu_j}} + \sum_{j=1}^{k-1}|\eta_j|^{\frac{1}{\mu_j}}\alpha_j^{\frac{1}{\mu_j}}\right) \\
&\qquad \times \gamma_k(x_1, \cdots, x_k) \quad (20)
\end{aligned}$$

(since $r_k/\mu_k \le 1/\mu_0, 1/\mu_1, \cdots, 1/\mu_{k-1} \le 1$ by (2) and (3), by applying (12) to each term in the summation of (20))

$$\begin{aligned}
&\le |\eta_k|^{v_{k-1}}\left(\sum_{j=0}^{k-1}|\eta_{j+1}|^{\frac{r_k}{\mu_k}}\left(1 + |\eta_{j+1}|^{1-\frac{r_k}{\mu_k}}\right)2^{1-\frac{1}{\mu_j}}\right. \\
&\qquad \left. + \sum_{j=1}^{k-1}|\eta_j|^{\frac{r_k}{\mu_k}}\left(1 + |\eta_j|^{1-\frac{r_k}{\mu_k}}\right)\alpha_j^{\frac{1}{\mu_j}}\right)\gamma_k(x_1, \cdots, x_k).
\end{aligned}$$

Define a smooth positive function $\tilde{\gamma}_k(x_1, \cdots, x_k)$ such that

$$\tilde{\gamma}_k(x_1, \cdots, x_k) \ge (1 + |\eta_{j+1}|^{1-\frac{r_k}{\mu_k}})2^{1-\frac{1}{\mu_j}}\gamma_k(x_1, \cdots, x_k)$$

$$\tilde{\gamma}_k(x_1, \cdots, x_k) \ge (1 + |\eta_j|^{1-\frac{r_k}{\mu_k}})\alpha_j^{\frac{1}{\mu_j}}\gamma_k(x_1, \cdots, x_k)$$

for $j = 0, \cdots, k-1$ (with $\alpha_0 = 0$ for convenience). Then,

$$\begin{aligned}
&\left|\eta_k^{v_{k-1}}\phi_k(x_1, \cdots, x_k)\right| \\
&\le \sum_{j=1}^{k}|\eta_j|^{\frac{r_k}{\mu_k}}\left(|\eta_k|^{v_{k-1}}\tilde{\gamma}_k(x_1, \cdots, x_k)\right) \\
&\le \sum_{j=1}^{k}\left(\frac{1}{3}|\eta_j|^{v_{k-1}+\frac{r_k}{\mu_k}} + |\eta_k|^{v_{k-1}+\frac{r_k}{\mu_k}}\tilde{\zeta}_k(x_1, \cdots, x_k)\right)
\end{aligned}$$

(which follows from (11) with appropriate $\tilde{\zeta}_k$)

$$\begin{aligned}
&= \frac{1}{3}\left(\eta_1^{v_0+\frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}}\right) \\
&\qquad + \eta_k^{v_{k-1}+\frac{r_k}{\mu_k}}\zeta_k(x_1, \cdots, x_k)
\end{aligned} \quad (21)$$

where $\zeta_k(x_1, \cdots, x_k) = 1/3 + k\tilde{\zeta}_k$, due to (14).

Finally, about the last term of (18), there exist smooth positive functions $\beta_k(x_1, \cdots, x_k)$ such that

$$\begin{aligned}
\sum_{i=1}^{k-1}\frac{\partial U_k}{\partial x_i}\dot{x}_i &\le \frac{1}{3}\left(\eta_1^{v_0+\frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}}\right) \\
&\qquad + \eta_k^{v_{k-1}+\frac{r_k}{\mu_k}}\beta_k(x_1, \cdots, x_k). \quad (22)
\end{aligned}$$

The proof of this claim is lengthy, but can be found in the Appendix.

Now, substituting (19), (21) and (22) into (18) yields

$$\begin{aligned}
&\dot{V}_k(x_1, \cdots, x_k) \le \\
&\quad -(n-k+1)\left(\eta_1^{v_0+\frac{r_1}{\mu_1}} + \cdots + \eta_{k-1}^{v_{k-2}+\frac{r_{k-1}}{\mu_{k-1}}}\right) \\
&\quad + \eta_k^{v_{k-1}}\left(x_{k+1}^{r_k} - x_{k+1}^{*r_k}\right) + \eta_k^{v_{k-1}}x_{k+1}^{*r_k} \\
&\quad + \eta_k^{v_{k-1}+\frac{r_k}{\mu_k}}\left(c_k + \zeta_k(x_1, \cdots, x_k) + \beta_k(x_1, \cdots, x_k)\right).
\end{aligned}$$

Therefore, by taking the virtual control x_{k+1}^* such that

$$\begin{aligned}
x_{k+1}^{*r_k} &= -\eta_k^{\frac{r_k}{\mu_k}}\{(n-k+1) \\
&\qquad + (c_k + \zeta_k(x_1, \cdots, x_k) + \beta_k(x_1, \cdots, x_k))\} \\
&\triangleq -\eta_k^{\frac{r_k}{\mu_k}}\alpha_k^{\frac{r_k}{\mu_k}}(x_1, \cdots, x_k), \quad (23)
\end{aligned}$$

we obtain

$$\begin{aligned}&\dot{V}_k(x_1,\cdots,x_k)\\&\le -(n-k+1)\left(\eta_1^{v_0+\frac{r_1}{\mu_1}}+\cdots+\eta_k^{v_{k-1}+\frac{r_k}{\mu_k}}\right)\\&\quad+\eta_k^{v_{k-1}}\left(x_{k+1}^{r_k}-x_{k+1}^{*r_k}\right),\end{aligned}$$

which proves the inductive argument. Note again that x_{k+1}^* and $x_{k+1}^{*r_k}$ may not be C^1, but $x_{k+1}^{*\mu_k}$ is.

At n-th step, applying the feedback control

$$u^{r_n}=x_{n+1}^{*r_n}=-\eta_n^{\frac{r_n}{\mu_n}}\alpha_n^{\frac{r_n}{\mu_n}}(x_1,\cdots,x_n) \tag{24}$$

with the C^1 proper and positive definite Lyapunov function $V_n(x_1,\cdots,x_n)$ constructed via the inductive procedure, we arrive at

$$\begin{aligned}\dot{V}_n(x_1,\cdots,x_n)&\le-\left(\eta_1^{v_0+\frac{r_1}{\mu_1}}+\cdots+\eta_n^{v_{n-1}+\frac{r_n}{\mu_n}}\right)\\&=-\big\{(x_1^{\mu_0}-x_1^{*\mu_0})^{v_0+\frac{r_1}{\mu_1}}+\cdots\\&\quad+(x_n^{\mu_{n-1}}-x_n^{*\mu_{n-1}})^{v_{n-1}+\frac{r_n}{\mu_n}}\big\}.\end{aligned} \tag{25}$$

Realizing the right hand side of (25) is negative definite and applying the Kurzweil's theorem [11, pp. 23,24], we conclude that the origin of the closed-loop system is globally strongly stable.

Finally, when all $\mu_i=1$, the technique of Remark 3 is applied to each step when we take α_i in (15), (23), (24), so that the resulting control $u(x)$ is smooth. Hence, the proof is completed.

III. An Example

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1&=x_2^{\frac{1}{3}}+x_1e^{x_1}\\\dot{x}_2&=u^{\frac{1}{5}}\end{aligned} \tag{26}$$

which is not linearizable at the origin. With $\mu_0=\mu_1=\mu_2=1$ (from Proof of (e)) and $v_0=1$, $v_1=v_0+r_1-r_2=\frac{17}{15}$ (from (14)), we choose $V_1(x_1)=\frac{1}{2}x_1^2$ which is C^1, proper and positive definite. Following the proof, we can compute the virtual control x_2^* as

$$x_2^*=-x_1\left(0.4+e^{x_1}(x_1^2+1)\right)^3$$

to obtain $\dot{V}_1(x_1)\le-0.4x_1^{\frac{4}{3}}+x_1(x_2^{\frac{1}{3}}-x_2^{*\frac{1}{3}})$. Next, we consider a C^1 proper positive definite function

$$V_2(x_1,x_2)=V_1(x_1)+10\int_{x_2^*}^{x_2}(\sigma-x_2^*)^{\frac{17}{15}}\,d\sigma.$$

The proof of Theorem 1 yields a smooth feedback control u, which is defined by

$$\begin{aligned}u=&-(x_2-x_2^*)\times\\&\left\{900+3(1+x_1^2)^2(0.4+e^{x_1}(2+x_2^2))^2\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\right.\\&\left.\quad+16\left(1+(x_2-x_2^*)^2\right)\frac{\partial x_2^*}{\partial x_1}\right\}^5,\end{aligned}$$

and the time derivative of V_2 becomes

$$\dot{V}_2(x_1,x_2)\le-\frac{0.4}{11}\left(x_1^{\frac{4}{3}}+(x_2-x_2^*)^{\frac{4}{3}}\right)$$

which implies that the origin of the closed system is GSS.

IV. Conclusion

In this paper, we have developed a C^0 state feedback design method for the powers of the integrators perturbed by C^1 lower triangular vector fields. Inspired by [4], we have extended the result of [4] in the sense that the powers are not restricted to positive odd integers. As a result, the system considered here may not have the first order approximation around the origin, which has rarely been studied in the literature. In spite of this difficulty, the stability result obtained in this paper is the global strong stability (GSS) which has been frequently studied in the literature.

A technical contribution of the paper is the selection of the powers in the control Lyapunov functions designed in the backstepping procedure since the appropriate powers should be selected reflecting the future design steps to come. To solve this problem, we extracted a necessary power sequence as a condition, with which the backstepping procedure has been enabled.

For the future work, the problem of the global stabilization under the condition that the powers of the integrators are odd rational between $1/2$ and 1, is left. And then, a continuous output feedback control for global stabilization of the systems will be of interest.

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