2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Stable Gain-Scheduling on Endogenous Signals

Bryan P. Rasmussen, *Student Member, IEEE*, and Andrew G. Alleyne[1], *Senior Member, IEEE*

Abstract— **Gain-scheduling is possibly the most widely used nonlinear control design technique in industry. However, guaranteeing the stability of the nonlinear closed loop can be extremely challenging, specifically for endogenously scheduled controllers. Given a set of locally linear models and previously designed controllers, this paper addresses the problem of 1) guaranteeing internal stability of the nonlinear closed loop, and 2) determining the class of disturbances and reference changes that can be stably endured, despite arbitrarily fast changes in the scheduling parameter. A simple example illustrates the approach.**

I. Introduction

Many physical systems exhibit dynamics that vary appreciably over the typical operating regime; a single linear controller may fail to achieve acceptable performance throughout the envelope of conditions. One of the most popular solutions to this nonlinear control design problem is the use of gain-scheduling, in which a nonlinear controller constructed by interpolating a family of local controllers. Thus the nonlinear control design problem can be divided into several control problems where linear design tools are generally employed. This "divide-and-conquer" [1] methodology has found success in a variety of applications. The success of the gain-scheduled approach seems directly attributable to its simplicity in design and implementation. In the literature, a wide variety of approaches are termed "gain-scheduling." For the purpose of this paper, we consider a gain-scheduled controller as one that interpolates between local linear control laws as a function of scheduling variables, which capture the system nonlinearities.

Despite the overwhelming success in industrial practice, the principal difficulty of using gain-scheduled control is guaranteeing stability. By blending several linear controllers, the resulting global controller is nonlinear, and results in the addition of "hidden coupling terms" [2] or additional dynamics due to the interpolation functions. Thus, although the gain-scheduled controller may be stable at every fixed value of the scheduling variable, the true global closed loop may not be stable.

[1] A.G. Alleyne is a Professor at the University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (phone: 217-244-9993; fax:217-244-6534; email: alleyne@uiuc.edu).

B.P. Rasmussen is a Ph.D. candidate at the University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (email: brasmuss@uiuc.edu).

This work was supported by the Air Conditioning and Refrigeration Center at the University of Illinois at Urbana-Champaign, Project #163.

Stability of the closed loop system is generally addressed by one of the following three methods. First, proving that the gain-scheduled controller is stable for all fixed values of the scheduling variable, and then inferring overall stability by assuming slowly varying scheduling parameters. Although this is appropriate for exogenously scheduled systems, it may not be possible for endogenously scheduled systems where the scheduling parameters may vary arbitrarily fast. Second, stability for time-varying, but often rate-limited, scheduling parameters is guaranteed by construction of the gain-scheduled controller using LPV or LFT methods. This generally assumes an explicit model of the nonlinear system, which may not be available. Third, stability is verified through extensive simulation and experiments.

In this paper we address the following problem: Given a set of local linear models of the plant

$$\begin{bmatrix} \dot{x}_{pi} \\ y_{pi} \end{bmatrix} = \begin{bmatrix} A_{pi} & B_{pi} \\ C_{pi} & 0 \end{bmatrix} \begin{bmatrix} x_{pi} \\ e_1 \end{bmatrix} \tag{1}$$

and given *a priori* a set of local linear controllers

$$\begin{bmatrix} \dot{x}_{ki} \\ u_{ki} \end{bmatrix} = \begin{bmatrix} A_{ki} & B_{ki} \\ C_{ki} & 0 \end{bmatrix} \begin{bmatrix} x_{ki} \\ e_2 \end{bmatrix} \tag{2}$$

interpolate the local controllers to create a gain scheduled controller, $K_\alpha(s)$, such that the internal stability is guaranteed for arbitrarily fast variations in the scheduling variable. Furthermore, guarantee that for finite energy, $\|w\|_2$, or peak amplitude, $\|w\|_\infty$, disturbances, the maximum deviation in system outputs is bounded by $\|y\|_\infty < \gamma$. In the solution approach, the gain-scheduled controller is constructed by weighting the outputs of individual linear controllers; an approach termed a Local Controller Network. Similarly, the nonlinear model is assumed to be adequately represented as a Local Model Network. The disturbance-to-output bounds are guaranteed by the use of Lyapunov methods, which are formulated as Linear Matrix Inequalities that can be solved computationally efficiently.

The remainder of this paper is organized as follows. Section 2 reviews past research efforts concerning gain-scheduling. Section 3 presents two frameworks for the interpolation of the local controllers and models. The necessary LMI tools used for guaranteeing the disturbance-to-output bounds are given in Section 4, and an illustrative example is presented in Section 5.

II. Background

A. Gain-Scheduling Overview

In 2000, two separate survey articles appeared detailing the history and scope of gain-scheduling [2],[1]. The acknowledged motivation of gain-scheduled controllers is the ability to solve a nonlinear control problem by using available linear control tools. Application areas have included flight control, engine control, power plants, vehicle control, and many others.

Rugh and Shamma discuss gain-scheduling in terms of a four-step process [2]. First, determine a linear parameter-varying model of the plant either from first principals, or by interpolating identified models. Second, use linear design methods to design controllers. Third, determine the method of interpolating the linear controllers as a function of the scheduling variable(s). Two common rules of thumb for selecting the scheduling variables is that they should 1) capture the plant nonlinearities, and 2) they should vary slowly [3]. Fourth, assess the stability and performance of the overall system, often by extensive simulations or experiments.

B. Stability Guarantees

The stability analysis of a gain-scheduled controller in closed loop with a nonlinear plant can pose a challenging problem. Most analytical methods for guaranteeing stability focus on achieving *frozen-parameter stability*, or closed loop stability for any fixed value of the scheduling parameter (i.e. no scheduling dynamics are considered). Although individual controllers may be stabilizing at the design points, the interpolated controller may not be stabilizing at intermediate points. A simple example of this phenomenon can be given as following. Let a plant and two stabilizing controllers be defined as in Equation 3. An interpolated controller could be defined as in Equation 4 where $\alpha \in [0,1]$. Although both K_1 and K_2 stabilize the plant, the blended controller K_b destabilizes the plant for $\alpha \in [0.25,\ 1]$.

$$P(s) = \frac{1}{s+1},\ K_1(s) = \frac{-1}{s+0.5},\ K_2(s) = \frac{-1}{s-0.5} \tag{3}$$

$$K_b(s) = \alpha K_1 + (1-\alpha) K_2 \tag{4}$$

Interpolation methods that guarantee frozen parameter stability have been termed "stability-preserving" interpolation methods [4]. Recent research has focused on guaranteeing this level of stability by design [5],[6]. Then by assuming a slowly varying scheduling variable, the stability of the true system is inferred. Alternatively, internal stability of the gain-scheduled system can be guaranteed by construction, by designing an LPV controller that guarantees that a common or parameter dependant Lyapunov function can be found to prove stability of the nonlinear closed loop. This requires that the controllers be designed en mass and have a common structure, and in general depend on a bound on the rate of change of the scheduling variable. Examples of these types of approaches can be found in [7],[5],[8].

While these types of approaches may be valid for a class of systems where the controller is scheduled using an exogenous signal, with known limits on the rate of change, the nonlinear behavior of many systems is more appropriately captured by scheduling parameters that are functions of plant outputs or controlled inputs. Guaranteeing the stability of these endogenously scheduled systems is a more challenging problem, because a bound on the rate of change may not be known *a priori*. Furthermore, practical stability necessitates that despite reference changes or disturbances, the scheduling variables and system outputs remain within prescribed bounds.

C. Interpolation Methods

Different methods for interpolation are reported in these survey articles, including the interpolation of poles, zeros, and gains, interpolation of $\mathcal{H}_\infty$ controllers by interpolating Riccati equations, interpolation of balanced state space matrix coefficients, interpolation of state and observer gains, and the interpolation of eigenvalue placement state feedback gains (for a list of references, see [2]). Alternatively, several authors have reported the use of controller blending, where the system signals of each of the linear controllers is blended as a function of operating condition to form a global nonlinear controller [9],[10],[11].

In this paper, the latter type of gain-scheduling, termed "output-blending," is concerned. This approach is similar to the Tagaki-Sugeno models/controllers in the field of fuzzy logic [12]. The principal benefit of this approach is the ability to simply gain-schedule between controllers of different sizes and structures; no restrictions are placed on the design of the individual controllers.

III. Gain-Scheduled Networks

A. Local Controller and Local Model Networks

This output blending approach generally assumes the configuration shown in Figure 1. This diagram illustrates the closed loop dynamics in the standard framework. The input disturbance w_1, and the control output signals u, are inputs to the nonlinear plant model, whose output y and the reference disturbance w_2, are inputs to the each of the linear controllers. The nonlinear controller is formed by weighting the individual outputs of several linear controllers. These weighting or blending functions are a function of a scheduling variable ρ, as $\alpha = h(\rho)$. Common assumptions include $\alpha_i \in [0,1]$ and $\sum \alpha_i = 1$, with the magnitude based on the relative distance to the respective design point in the scheduling space.

This approach is sometimes termed a Local Controller Network (LCN) and is attractive because of the simplicity of the controller implementation. A controller is constructed using standard linear techniques and a linear approximation of the nonlinear model, either using a linearized first principles model or a data-driven identified model. This process is repeated for several key operating conditions, and the resulting controllers are computed in parallel, while a weighted sum of their outputs is applied to the nonlinear plant. For the purposes of this paper, the local controllers are assumed to be given a priori and without common size or structure. Although this approach is simply and frequently implemented, guarantees of closed loop stability are not currently available.

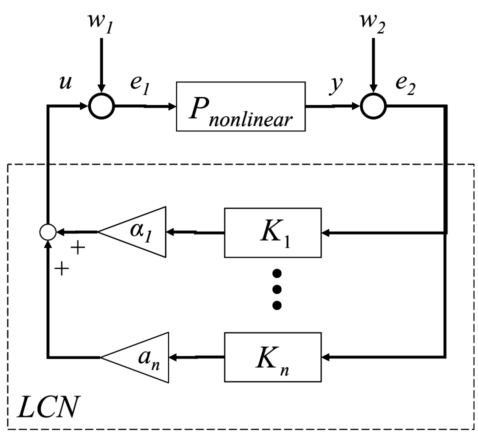

Figure 1: Output Blending Based on a Local Controller Network (LCN)

For analysis purposes, the nonlinear plant is often assumed to be adequately represented by a Local Model Network (LMN), where the local linear representations of the plant are the result of the linearization of a nonlinear model, or simply data-driven empirical models. We will denote the LMN as $P_\alpha(s) = \sum \alpha_i P_i(s)$, and similarly the LCN as $K_\alpha(s) = \sum \alpha_i K_i(s)$. For the purposes of this study we assume that each linear controller and plant model can be represented as in Equations1 and 2.

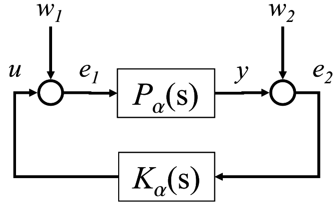

Figure 2: Closed Loop System with LMN/LCN

The closed loop stability analysis of the LMN/LCN poses a significant challenge, including the possibility that individual plants/controllers may not be open loop stable. Because of the nature of output blending, the LCN $K_\alpha(s)$ can be written in state space form as shown in Equation 5, and similarly for the LMN $P_\alpha(s)$. Thus the Local Model Network and Local Controller Network are Linear Parameter Varying (LPV) systems that are affine in the bounded parameters $\alpha_i \in [0,1]$. The closed loop system (Fig. 2) from $[w_1 \quad w_2]^T$ to $[u \quad y]^T$ is given in state-space form in Equation 6, and denoted $G_\alpha(s)$. Note that this system can be represented as a system affinely parameterized in α with constraint $\sum \alpha_i = 1$. Alternatively, the system can be recognized as a polytopic model formed from a convex set of individual models

$$\begin{bmatrix} \dot{x}_{k1} \\ \vdots \\ \dot{x}_{kn} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{k1} & & 0 \\ & \ddots & \\ 0 & & A_{kn} \end{bmatrix}}_{A_K} \underbrace{\begin{bmatrix} x_{k1} \\ \vdots \\ x_{kn} \end{bmatrix}}_{x_K} + \underbrace{\begin{bmatrix} B_{k1} \\ \vdots \\ B_{kn} \end{bmatrix}}_{B_K} e_2 \tag{5}$$

$$u = \underbrace{\begin{bmatrix} \alpha_1 C_{k1} & \cdots & \alpha_n C_{kn} \end{bmatrix}}_{C_K} \begin{bmatrix} x_{k1} \\ \vdots \\ x_{kn} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_P \\ \dot{x}_K \end{bmatrix} = \underbrace{\begin{bmatrix} A_P & B_P C_K \\ B_K C_P & A_K \end{bmatrix}}_{\tilde{A}(\alpha)} \begin{bmatrix} x_P \\ x_K \end{bmatrix} + \underbrace{\begin{bmatrix} B_P & 0 \\ 0 & B_K \end{bmatrix}}_{\tilde{B}} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C_P \\ C_K & 0 \end{bmatrix}}_{\tilde{C}(\alpha)} \begin{bmatrix} x_P \\ x_K \end{bmatrix}$$

B. Local Q-Networks and Local S-Networks

While the Local Controller Network is a straightforward and intuitive framework for creating a gain-scheduled controller, the blended controller may not be stabilizing for fixed intermediate values of the scheduling variable. Furthermore, the framework requires that the each of the local plant and controller models be open-loop stable. However, an alternative framework for blending the controller efforts results in the recovery of the original local controllers at the design points, but increases the stability of the gain-scheduled system. This framework has been termed J-Q interpolation [4], or blending of the Youla parameters [6]. Past researchers have proposed this framework as a means of guaranteeing frozen parameter stability at intermediate design points, while recovering the original local controllers at the design points.

At this point we introduce coprime factors and the associated Youla parameterization (see [13] for more details). A controller may be decomposed into coprime factors $K(s) = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ where $U, \tilde{U}, V, \tilde{V} \in RH_\infty$. Similarly we may decompose a plant model as $P(s) = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. Assuming that K_0 stabilizes P_0, these coprime factors satisfy the Bezout identity (Equation 7), and all stabilizing controllers may be parameterized as in Equation 8 where $Q \in RH_\infty$ and is termed the Youla parameter. Thus when implementing a blended controller for a plant P_0, the blended controller may be constructed as in Equation 9 where each Q_i is given in Equation 10 and K_0 is any controller that stabilizes the plant. Thus at $\alpha_i = 1$ the original local controller K_i is recovered. Moreover, because the $K(Q)$ is stabilizing for any $Q \in RH_\infty$, then $K(Q_\alpha)$ is stabilizing for every frozen value of α, since $Q_i \in RH_\infty$ and thus $\sum \alpha_i Q_i \in RH_\infty$.

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

$$K(Q) = (U_0 - M_0 Q)(V_0 - N_0 Q)^{-1} \tag{8}$$

$$K(Q_\alpha) = \left[U_0 - M_0\left(\sum \alpha_i Q_i\right)\right]\left[V_0 - N_0\left(\sum \alpha_i Q_i\right)\right]^{-1} \tag{9}$$

$$Q_i = \tilde{U}_i V_0 - \tilde{V}_i U_0 \tag{10}$$

These ideas can be carried further by defining a similar notion for the LMN using the dual Youla parameter $S \in RH_\infty$. All plants that are stabilized by the controller K_0 may be parameterized as in Equation 11, where $P_0 = N_0 M_0^{-1}$ is any nominal plant stabilized by K_0. Similar to a controller network formed by the interpolation of the Youla Parameter Q, a model network may be formed as an interpolation of the dual Youla parameter S (Equations 12 and 13).

$$P(S) = (N_0 - V_0 S)(M_0 - U_0 S)^{-1} \tag{11}$$

$$P(S_\alpha) = \left[N_0 - V_0\left(\sum \alpha_i S_i\right)\right]\left[M_0 - U_0\left(\sum \alpha_i S_i\right)\right]^{-1} \tag{12}$$

$$S_i = \tilde{N}_i M_0 - \tilde{M}_i N_0 \tag{13}$$

The application of these ideas to gain-scheduling follows naturally. First a nominal plant P_0 and controller K_0 are selected, and assuming that each controller is stabilizing for each plant, the parameters Q_i and S_i are determined such that at $\alpha_i = 1$ the original local plant P_i and controller K_i are recovered. Then in place of a LCN, a network of the parameters Q_i is used. Similarly, the model network is formed by a network of the S_i parameters. The resulting interconnected system is depicted in Fig. 3. In keeping with the terminology, we refer the blended controller as a Local Q-Network (LQN) and the blended model as a Local S-Network (LSN). The interconnection of matrices J_K and J_P (given in Equations 14 and 15), can be verified as given in Equation 16. Thus the system of Fig. 3a is internally stable if and only if the system in Fig. 3b is stable. The interconnection in Fig. 3b is given in state-space form in Equation 17, and is denoted $H_\alpha(s)$.

$$J_K(s) = \begin{bmatrix} U_0 V_0^{-1} & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1} N_0 \end{bmatrix} \tag{14}$$

$$J_P(s) = \begin{bmatrix} N_0 M_0^{-1} & \tilde{M}_0^{-1} \\ M_0^{-1} & -M_0^{-1} U_0 \end{bmatrix} \tag{15}$$

$$F_l(J_P, J_K) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \dot{x}_S \\ \dot{x}_Q \end{bmatrix} = \underbrace{\begin{bmatrix} A_S & B_S C_Q \\ B_Q C_S & A_Q \end{bmatrix}}_{\tilde{A}_{QS}(\alpha)} \begin{bmatrix} x_S \\ x_Q \end{bmatrix} + \underbrace{\begin{bmatrix} B_S & 0 \\ 0 & B_Q \end{bmatrix}}_{\tilde{B}_{QS}} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$\begin{bmatrix} r \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C_S \\ C_Q & 0 \end{bmatrix}}_{\tilde{C}_{QS}(\alpha)} \begin{bmatrix} x_S \\ x_Q \end{bmatrix} \tag{17}$$

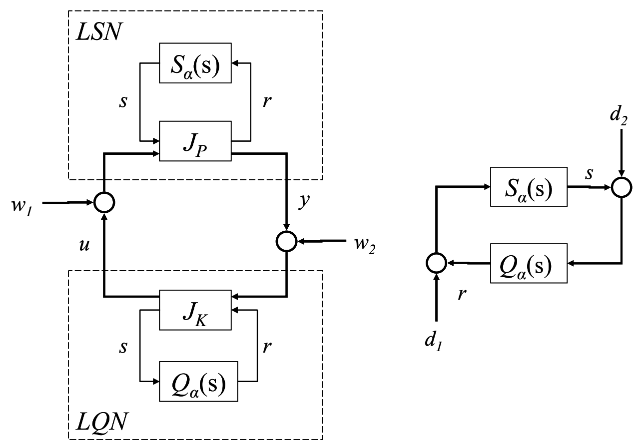

Figure 3: Closed Loop System with LQN/LSN and Simplified Loop

This framework allows stable scheduling of plants and controllers that may be open-loop unstable. Furthermore, this approach reveals a design freedom in choosing K_0 and P_0. That is, although the local dynamics are recovered at each design point, the nonlinear closed loop may differ at intermediate points, and depends on the choice of K_0 and P_0. Note that the relationship between $[w_1 \quad w_2]^T$ and $[u \quad y]^T$ for the modified framework is given by Equation 18, where T_1, T_2, and T_3 depend only on the choice of K_0 and P_0. In general, the choice of P_0 can be made such that the LSN adequately represents the nonlinear plant. However, the choice of K_0 is somewhat arbitrary, and can be chosen such that the class of acceptable disturbances is maximized.

$$\begin{bmatrix} u \\ y \end{bmatrix} = (T_1 + T_2 H_\alpha T_3)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{18}$$

$$T_1 = \begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N}_0 & \tilde{M}_0 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{V}_0 & 0 \\ 0 & \tilde{M}_0 \end{bmatrix} - I \quad T_2 = \begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N}_0 & \tilde{M}_0 \end{bmatrix}^{-1} \quad T_3 = \begin{bmatrix} \tilde{V}_0 & \tilde{U}_0 \\ \tilde{N}_0 & \tilde{M}_0 \end{bmatrix} \tag{19}$$

IV. Stability Analysis using LMIs

With the closed loop system being represented as polytopic system, stability can be determined using Lyapunov based approaches. Specifically, the internal stability of the interconnected LMN/LCN system can be guaranteed for arbitrarily fast variations in the scheduling parameter if a common Lyapunov function can be found for each of the $\tilde{A}(\alpha_i)$ of the closed loop system. For *exogenously* gain-scheduled systems, this condition would be sufficient for guaranteeing global stability throughout the operating envelope, although no indications are given regarding performance. However, for *endogenously* scheduled systems, the above conditions are still insufficient. For practical stability, it is necessary to ensure the scheduling parameter remains within acceptable bounds for an assumed class of disturbances.

For endogenously scheduled systems, $\alpha = h(\rho)$ and $\rho = f(u, y)$. Although the existence of a common Lyapunov function guarantees the boundedness of u and y, large values of u and y could drive the system outside of the region for which the LMN, and hence the analysis, is valid. To quantify the worst case amplification from disturbances to u and y requires more analysis.

To this end, we consider two norm measures: finite energy to peak deviation $\|G_\alpha(s)\|_{L_2 \to L_\infty}$, and peak-to-peak deviation $\|G_\alpha(s)\|_{L_\infty \to L_\infty}$. For polytopic systems such as the interconnected LMN/LCN, upper bounds on these norms can be formulated as Linear Matrix Inequalities (LMIs).

Generalized H_2 Performance

The first case is often referred to as the generalized H_2 norm, and a bound can be efficiently determined by solving a set of LMI conditions that is equivalent to seeking the existence of a quadratic Lyapunov function of the form $V(x) = x^T P x$, which jointly satisfies Equations 20 and 21. For a polytopic system, these conditions are easily formed as the LMIs in Equation 22. Given this condition, we say that $\|G_\alpha(s)\|_{L_2 \to L_\infty} < \gamma$. This approach is, in essence, searching for an invariant ellipsoid that contains the set of all reachable states with finite energy, and then determining the set of peak outputs given a set of possible states. A more complete discussion and the accompanying proofs can be found in [14] and [15].

$$PA + A^T P + \tfrac{1}{\gamma} PBB^T P < 0 \tag{20}$$

$$P - \tfrac{1}{\gamma} C^T C > 0 \tag{21}$$

$$\begin{bmatrix} I & 0 \\ \tilde{A}_i & \tilde{B} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{A}_i & \tilde{B} \\ 0 & I \end{bmatrix} < 0 \qquad \begin{bmatrix} P & \tilde{C}_i^T \\ \tilde{C}_i & \gamma I \end{bmatrix} > 0 \tag{22}$$

Peak-to-Peak Performance

The solution to bounding the peak-to-peak norm is similar to the previous case, and is based on seeking for an invariant ellipsoid that contains the set of all reachable states with peak input, and then the peak output given the set of possible states. Similarly, for a polytopic system, these conditions are easily formed as LMIs in Equations 23 and 24. Given this condition, we say that $\|G_\alpha(s)\|_{L_\infty \to L_\infty} < \gamma$. Although this is not a strict set of LMIs, this challenge may be overcome by solving the LMIs for fixed δ, and doing a line search over $\delta \in [0, -2 \cdot \max(\mathrm{Re}(\lambda_j(\tilde{A}_i)))]$. Again we refer the reader to [14] and [15] for more information.

$$\begin{bmatrix} I & 0 \\ \tilde{A}_i & \tilde{B} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \delta P & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -\mu I \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{A}_i & \tilde{B} \\ 0 & I \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} \delta P & 0 & \tilde{C}_i^T \\ 0 & (\gamma - \mu) I & \tilde{D}_i^T \\ \tilde{C}_i & \tilde{D}_i & -\gamma I \end{bmatrix} > 0 \tag{24}$$

With computationally efficient means of determining upper bounds on these two norms for polytopic systems, the designer can determine the class of disturbances or reference changes that will result in acceptable values of u and y, and ensure that $\rho = f(u, y)$ stays within an acceptable region in the scheduling space.

V. Example Problem

To demonstrate the proposed method for stability analysis, a nonlinear mass-spring-damper system is used. The governing equations are given in 25. The damping coefficient and the spring constant vary as a function of the displacement. The root locus for the system is shown in Fig. 5 for varying y. The scheduling variable is selected as $\rho = y$, and design points are chosen at several points along the one-dimensional scheduling space. $\mathcal{H}_\infty$ controllers are designed for each point using frequency dependent performance weightings on the tracking error and controller effort. The LCN is formed using the weighting functions shown in Fig. 4. Similarly, a LMN representation of the plant is formed using identical weighting functions and blending fixed plant models for $c \in [0.3,\ 0.7]$, $k \in [0.15,\ 0.05]$, and $m = 1$.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -k(y) & -c(y) \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + w) \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned} \tag{25}$$

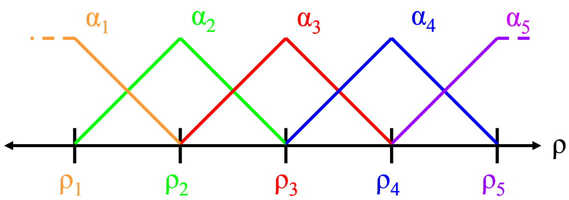

Figure 4: Weighting Functions

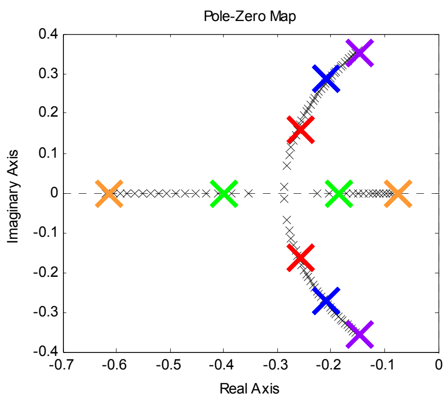

Figure 5: Root Locus for values of *y* with five design points

Table 1: Norm Bounds for the Gain-Scheduled Mass-Spring-Damper

	LCN/LMN	**LQN/LSN (depends on choice of K_0)**
$\sup_{0<\|w_1\|_2<\infty}\left(\frac{\|y\|_\infty}{\|w_1\|_2}\right)$	< 0.78	$<[0.46, 0.65]$
$\sup_{0<\|w_2\|_2<\infty}\left(\frac{\|y\|_\infty}{\|w_2\|_2}\right)$	< 2.46	$<[0.94, 0.95]$
$\sup_{0<\|w_1\|_\infty<\infty}\left(\frac{\|y\|_\infty}{\|w_1\|_\infty}\right)$	< 23.95	$<[1.69, 1.91]$
$\sup_{0<\|w_2\|_\infty<\infty}\left(\frac{\|y\|_\infty}{\|w_2\|_\infty}\right)$	< 302.2	$<[1.63, 3.07]$

Using the LCN as the framework for scheduling our controllers, performance bounds are found using the LMI machinery outlined previously as shown in Table 1. Thus the gain-scheduled system is internally stable, and the peak deviation in the scheduling space can be bounded for disturbances with finite energy or finite peak magnitude.

The value of these bounds can be decreased significantly using the alternative LQN/LSN framework. For example, selecting $K_0 = K_i$, the analysis for the Youla gain-scheduled systems yields the bounds of shown in Table 1 Note that the bound varies depending on the choice of K_0, indicating some flexibility in the design process.

VI. CONCLUSIONS

This paper explores guaranteeing global stability of a gain-scheduled system for exogenous and endogenous scheduling with arbitrarily fast changes in the scheduling parameter. Using Local Model Networks and Local Controller Networks, bounds on the peak deviation in the scheduling space for disturbances with finite energy or finite peak magnitude can be efficiently determined using LMIs. A modified formulation of the gain-scheduled system using the Youla parameterization is shown to result in significantly lower bounds, and an element of design freedom that can be exploited when designing the gain-scheduled controller.

REFERENCES

[1] D.J. Leith and W.E. Leithead, "Survey of gain-scheduling analysis and design," *International Journal of Control*, vol. 73, no. 11, pp. 1001-25, Jul 20, 2000.
[2] W.J. Rugh and J.S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401-25, Oct, 2000-.
[3] J.S. Shamma and M. Athans, "Gain scheduling: Potential hazards and possible remedies," in *Proceedings of the 1991 American Control Conference*, vol. 1, 1991, pp. 516-521.
[4] D.J. Stilwell and W.J. Rugh, "Stability preserving interpolation methods for the synthesis of gain scheduled controllers," *Automatica*, vol. 36, no. 5, pp. 665-671, 2000.
[5] D.J. Stilwell, "J-Q interpolation for gain scheduled controllers," in *Proceedings of 1999 Conference on Decision and Control*, vol. 1, 1999, pp. 749-54.
[6] H. Niemann and J. Stoustrup, "An architecture for implementation of multivariable controllers," in *Proceedings of the 1999 American Control Conference*, vol. 6, 1999, pp. 4029-33.
[7] P. Apkarian and R.J. Adams, "Advanced gain-scheduling techniques for uncertain systems," *IEEE Transactions on Control Systems Technology*, vol. 6, no. 1, pp. 21-32, 1998.
[8] D.J. Stilwell and W.J. Rugh, "Interpolation of observer state feedback controllers for gain scheduling," *IEEE Transactions on Automatic Control*, vol. 44, no. 6, pp. 1225-9, Jun, 1999-.
[9] Rhong Zhang, A.G. Alleyne, and D.E. Carter, "Robust gain scheduling control of an earthmoving vehicle powertrain," in *Proceedings of 2003 American Control Conference*, vol. 6, 2003, pp. 4969-74.
[10] H. Buschek, "Full envelope missile autopilot design using gain scheduled robust control," *Journal of Guidance, Control, and Dynamics*, vol. 22, no. 1, pp. 115-22, Jan, 1999-.
[11] H. Niemann and J. Stoustrup, "Gain scheduling using the Youla parameterization," in *The 38th IEEE Conference on Decision and Control*, vol. 3, 1999, pp. 2306-2311.
[12] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modelling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. SMC-15, no. 1, pp. 116-32, Jan, 1985.
[13] Tay, T.-T., Mareels, I., and Moore, J. B. High Performance Control, Ann Arbor, MI: Birkhauser, 1998.
[14] Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. Linear Matrix Inequalities in System and Control Theory, Philadelphia, Pa: SIAM, 1994.
[15] El Ghaoui, L. and Niculescu, S. Advances in Linear Matrix Inequality Methods in Control, Philadelphia, Pa: SIAM, 2000.