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Further Constructions of Strict Lyapunov Functions for Time-Varying Systems

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Abstract— **We announce new methods for explicitly constructing strict input-to-state stable (ISS) Lyapunov functions for time-varying nonlinear systems. Our constructions are expressed in terms of nonstrict ISS Lyapunov functions which we assume are given. The nonstrict Lyapunov functions can in turn be constructed by known methods for many systems of interest. We also provide a new method for explicitly constructing input-to-output stable (IOS) Lyapunov functions for time-varying systems with outputs. We illustrate our results using a tracking problem for a rotating rigid body.**

Key Words: Lyapunov functions, time-varying systems with outputs, input-to-state and input-to-output stability

I. INTRODUCTION

Strict Lyapunov functions provide the foundation for much of current nonlinear control analysis and controller design (see [6], [10], [11], [17]). Starting from strict Lyapunov functions, one can design feedbacks that render systems asymptotically stable to actuator errors and observation noise, develop necessary and sufficient conditions for many types of stability, construct state estimators, track reference state trajectories, and much more. In many applications, it is necessary to have explicit formulas for strict Lyapunov functions. This is the case in the design of stabilizing feedbacks, which are usually expressed in terms of Lie derivatives of Lyapunov functions in the directions of the vector fields that define the systems (see [10], [11], [18]).

On the other hand, the known strict Lyapunov functions provided by converse Lyapunov theory are usually expressed as optimal control value functions, in which cost functions are maximized over infinitely many solution paths (see [2], [19], [22]). While value functions can sometimes be characterized as unique solutions of Hamilton-Jacobi equations and computed using numerical PDE methods, such methods can be difficult to implement and therefore are not always suitable for computing Lyapunov functions in practice. This has led to a great deal of current research devoted to finding new ways of constructing strict Lyapunov functions.

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This note continues the search (started in [12]) for methods of constructing strict Lyapunov functions for time-varying systems. Unlike the value function approach to Lyapunov functions, the constructions in [12] apply an integral smoothing technique to known *non*strict Lyapunov functions. This smoothing method has its origins in Lyapunov theory for time delay systems. For many systems of interest, nonstrict Lyapunov functions can in turn be constructed by backstepping or other known methods (see [5], [13]), so [12] leads to a complete method for explicitly constructing strict Lyapunov functions for time-periodic systems with no controls.

The systems in [12] are assumed to satisfy a nonstrict generalization of global asymptotic stability (GAS) for which the Lyapunov function nonstrictly decays along the trajectories of the system. Hence, [12] allows the gradient of the Lyapunov function along trajectories of the system to be zero at some points outside the origin. A natural and widely used generalization of GAS for control systems is the so-called input-to-state stable (ISS) property, as introduced by Sontag in his seminal paper [16]. For ISS systems, the magnitude of the state decays to zero, locally uniformly in the initial state, but with an overshoot depending on the magnitude of the input; see Section II for the precise ISS definition. More recently, ISS theory has been extended to systems with outputs and measurement errors in the controllers; see for example [7], [10], [11], [18], [20], [21].

The ISS framework has formed the basis for significant advances in controller design and control analysis (such as [10], [18], [19]). Many of these developments are based on the ISS Lyapunov function existence theory from [19]; see also [4] for analogues of [19] for time-varying systems, and Section II below for the relevant definitions. However, as in the case of no controls, the ISS Lyapunov functions from the existence theory are optimal control value functions and so do not lend themselves to explicit feedback design. Moreover, while most theoretical developments for ISS systems deal with time-invariant systems, it is sometimes more natural to consider perturbed *time-varying systems*, e.g., for tracking problems.

This motivates the search for explicit strict ISS Lyapunov function constructions for time-varying systems, in terms of known nonstrict ISS Lyapunov functions, which is the focus of this note. In Section II, we provide the definitions of nonstrict and strict ISS Lyapunov functions and corresponding nonstrict and strict versions of ISS. In nonstrict

ISS, the dissipation rate depends on a nonnegative time-dependent decay parameter that can be zero along intervals of positive length. As in [12], this allows the gradient of the Lyapunov function along trajectories to take the value zero at some points outside the origin. However, when the decay parameter is identically one, our nonstrict ISS property agrees with the usual ISS condition.

We announce our new strict ISS Lyapunov function constructions in Section III. In Section IV, we provide Lyapunov characterizations for nonstrict ISS for time-varying systems. In Section V, we present a general method for explicitly constructing strict input-to-output stable (IOS) Lyapunov functions for time-varying systems with outputs. We close in Section VI by applying our constructions to a tracking example for a rotating rigid body. This example also shows how to construct the required nonstrict Lyapunov functions. While our discussions of ISS systems will be mainly conceptual, we refer the reader to [9] where complete proofs of our main ISS results can be found. However, to our knowledge, our strict IOS Lyapunov function construction appears here for the first time.

II. Definitions and Standing Assumptions

We let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0,\infty) \to [0,\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. Note that $\mathcal{K}_\infty$ is closed under inverse and composition; i.e., if $\rho_1, \rho_2 \in \mathcal{K}_\infty$, then $\rho_1^{-1}, \rho_1 \circ \rho_2 \in \mathcal{K}_\infty$. We let $\mathcal{KL}$ denote the class of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (3) $\beta(s,t) \to 0$ as $t \to +\infty$ for each $s \geq 0$. When we say that a function ρ is *smooth* (a.k.a. C^1), we mean it is continuously differentiable, in which case we write $\rho \in C^1$. (For functions ρ defined on $[0,\infty)$, we interpret $\rho'(0)$ as a one-sided derivative, and continuity of ρ' at 0 as one-sided continuity.)

We study the stability properties of the fully nonlinear nonautonomous system

$$\dot{x} = f(t,x,u), \quad t \geq 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{1}$$

where we always assume f is locally Lipschitz in (t,x,u) (but see Section V for the extension to systems with outputs). Following [12], we also assume f is *periodic in* t, i.e., there exists a constant $T > 0$ such that

$$f(t+T,x,u) = f(t,x,u)\ \ \forall t \geq 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m.$$

However, our periodicity assumption can be relaxed to the uniform local boundedness condition from [1] (see [9]). The *control functions* (a.k.a. *inputs*) for the system (1) comprise the set of all measurable locally essentially bounded functions $\mathbf{u} : [0,\infty) \to \mathbb{R}^m$; we denote this set by $\mathcal{U}$. We let $|\mathbf{u}|_I$ denote the essential supremum of any control $\mathbf{u} \in \mathcal{U}$ restricted to any interval $I \subseteq [0,\infty)$. For each $t_o \geq 0$, $x_o \in \mathbb{R}^n$, and $\mathbf{u} \in \mathcal{U}$, we let

$$I \ni t \mapsto \phi(t; x_o, t_o, \mathbf{u}) \tag{*}$$

denote the unique trajectory of (1) for the input $\mathbf{u}$ satisfying $x(t_o) = x_o$ and defined on its maximal interval $I \subseteq [t_o, \infty)$. We let $T_{t_o,x_o,\mathbf{u}}$ denote the supremum of I. This trajectory is denoted by $h \mapsto \phi(t_o + h)$ for brevity when this would not lead to confusion. We say that f is *forward complete* provided each trajectory (*) is defined on $[t_o, \infty)$; i.e., $T_{t_o,x_o,\mathbf{u}} \equiv +\infty$.

A C^1 function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$ is said to be of *class* UPPD (written $V \in$ UPPD) provided it is *uniformly proper and positive definite*, which means there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)\ \&\ |\nabla V(t,x)| \leq \alpha_3(|x|). \tag{2}$$

We say that V *has period* τ *in* t provided there exists a constant $\tau > 0$ such that $V(t+\tau, x) = V(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$; in this periodic case, the bound on ∇V in (2) is redundant. We can always assume α_1 and α_2 in (2) are C^1 by taking

$$\alpha_2(s) = \int_0^s \alpha_3(r)dr$$

and minorizing α_1 by a C^1 function of class $\mathcal{K}_\infty$. For any C^1 function $V : [0,\infty) \times \mathbb{R}^n \to [0,\infty)$, we set

$$\dot{V}(t,x,u) := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) \cdot f(t,x,u).$$

Throughout this note, we simplify notation whenever no confusion can arise. For instance, we may denote $\dot{V}(t,x,u)$ by $\dot{V}$. If $V \in$ UPPD and $\chi \in \mathcal{K}_\infty$, then

$$s \mapsto \sup\{|\dot{V}(t,x,u)| : t \geq 0, |x| \leq \chi(s), |u| \leq s\} + s \tag{3}$$

is of class $\mathcal{K}_\infty$. We let $\mathcal{P}$ denote the set of all continuous $p : \mathbb{R} \to [0,\infty)$ admitting constants $\tau, \varepsilon, \bar{p} > 0$ for which

$$\int_{t-\tau}^{t} p(s)ds\ \geq\ \varepsilon\ \text{ and }\ p(t) \leq \bar{p} \quad \forall t \geq 0. \tag{4}$$

We write $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$ to indicate that (i) $p \in \mathcal{P}$ and (ii) $\tau, \varepsilon, \bar{p} > 0$ are constants such that (4) holds. In particular, any continuous periodic function $p : \mathbb{R} \to [0,\infty)$ that is not identically zero admits constants $\tau, \varepsilon, \bar{p} > 0$ satisfying (4). On the other hand, (4) also allows nonperiodic p with arbitrarily large null sets (see [9]). The following basic properties are easily checked (see [9]):

($\mathcal{P}_1$) for all $t \geq 0$,

$$\begin{aligned} \int_{t-\tau}^{t}\left(\int_{s}^{t} p(r)dr\right)ds &= \int_{t-\tau}^{t}(r-t+\tau)p(r)dr \\ &\leq \frac{\tau^2\bar{p}}{2}. \end{aligned} \tag{5}$$

($\mathcal{P}_2$) the continuous function

$$h \mapsto \underline{p}(h) = \inf\left\{\int_{t_o}^{t_o+h} p(r)dr : t_o \geq 0\right\} \tag{6}$$

is nondecreasing and unbounded on $[0,\infty)$.

Property $(\mathcal{P}_1)$ is a consequence of Fubini's Theorem. The elements of $\mathcal{P}$ serve as decay rates for our Lyapunov functions as follows:

Definition 1: Let $p \in \mathcal{P}$. A function $V \in$ UPPD is called a *nonstrict ISS Lyapunov function* for (1) and p, a.k.a. an *ISS(p) Lyapunov function*, provided there exist functions $\chi \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ such that

$$|x| \geq \chi(|u|) \;\; \Rightarrow \;\; \dot{V}(t,x,u) \;\leq\; -p(t)\mu(|x|) \; \forall t \geq 0. \quad (7)$$

An ISS(p) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a *strict ISS Lyapunov function*.

Notice that (7) allows $\dot{V}(t,x,u) = 0$ at those times t where $p(t) = 0$. This corresponds to allowing V to *nonstrictly* decrease along the trajectories of f. The assumption that $\mu \in C^1 \cap \mathcal{K}_\infty$ is not essential for Definition 1 since we can always minorize a function $\mu \in \mathcal{K}_\infty$ satisfying (7) by a smooth $\mathcal{K}_\infty$ function that again satisfies (7).

Definition 2: Let $p \in \mathcal{P}$. We say that (1) is *input-to-state stable (ISS) with decay rate* p, a.k.a. *ISS(p)*, provided there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$ and $h \geq 0$, the corresponding trajectories for f satisfy

$$|\phi(t_o+h)| \leq \beta\left(|x_o|, \int_{t_o}^{t_o+h} p(s)ds\right) + \gamma\left(|\mathbf{u}|_{[t_o,t_o+h]}\right).$$

If (1) is ISS(p) with $p \equiv 1$, then we say that (1) is *ISS*.

Notice that ISS(p) systems are automatically forward complete. Moreover, by causality, we can replace the argument of γ in the ISS(p) decay estimate by $|\mathbf{u}|_{[t_o,\infty)}$. We also study dissipation-type decay conditions as follows:

Definition 3: Let $p \in \mathcal{P}$. A function $V \in$ UPPD is called a *nonstrict dissipative Lyapunov function* for (1) and p, a.k.a. a *DIS(p) Lyapunov function*, provided that there exist $\Omega \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}_\infty \cap C^1$ such that for all $t \geq 0$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$, we have

$$\dot{V}(t,x,u) \;\leq -p(t)\mu(|x|) + \Omega(|u|). \quad (8)$$

A DIS(p) Lyapunov function for (1) and $p(t) \equiv 1$ is also called a *strict DIS Lyapunov function*.

Under our periodicity assumption on f, one can check (see [9, Section 3]) that a function $V \in$ UPPD is a strict DIS Lyapunov function for (1) if and only if it is a strict ISS Lyapunov function for the system; this follows because the functions (3) are in $\mathcal{K}_\infty$ when $\chi \in \mathcal{K}_\infty$. One can also check (see Section IV below) that ISS(p) and ISS are equivalent conditions for any $p \in \mathcal{P}$. Our main contributions in this note are simple direct constructions for strict ISS Lyapunov functions for (1) in terms of given ISS(p) or DIS(p) Lyapunov functions (but see Section V for an extension for systems with outputs).

III. Strict ISS Lyapunov Function Construction

In this section, we provide explicit formulas for strict ISS Lyapunov functions. Our strict Lyapunov functions are computed in terms of nonstrict Lyapunov functions, which we assume are given. Combined with the existing methods for constructing the required *non*strict Lyapunov functions (e.g., [5], [13]), this provides a complete strict Lyapunov function construction for a broad class of nonautonomous systems. Our constructions have the additional desirable property that if $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$ and our given DIS(p) Lyapunov function both have period τ in t, then the strict ISS Lyapunov function we construct also has period τ in t.

a) First Construction: Our main construction is as follows:

Theorem 4: Let $\tau, \varepsilon, \bar{p} > 0$ be constants, $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$, V be a DIS(p) Lyapunov function for the system (1), and $\alpha_2, \mu \in \mathcal{K}_\infty \cap C^1$ satisfy the UPPD and DIS(p) requirements (2) and (8) for some $\alpha_1 \in \mathcal{K}_\infty \cap C^1$ and $\alpha_3, \Omega \in \mathcal{K}_\infty$. Define $V^\sharp$ by

$$\begin{array}{l} V^\sharp(t,x) = V(t,x) \\ \quad + \left[\int_{t-\tau}^{t}\left(\int_s^t p(r)\,dr\right)ds\right] w(V(t,x)), \end{array} \quad (9)$$

where

$$w = \frac{1}{4\tau}\mu \circ \tilde{\alpha}_2^{-1}, \;\; \tilde{\alpha}_2(s) = \max\left\{\frac{\tau\bar{p}}{2}, 1\right\}(\alpha_2(s) + \mu(s) + s).$$

Then $V^\sharp$ is a strict ISS Lyapunov function for (1). If V and p have period τ in t, then so does $V^\sharp$.

Proof: With the choice $\tilde{\mu} := \mu \circ \tilde{\alpha}_2^{-1}$, we have

$$\dot{V}(t,x,u) \leq -p(t)\tilde{\mu}(V(t,x)) + \Omega(|u|) \quad (10)$$

for all $t \geq 0$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. This follows because

$$\tilde{\alpha}_2(s) \geq \alpha_2(s) \quad s \geq 0.$$

Also, $w \in \mathcal{K}_\infty \cap C^1$. In particular, $w'(s) \geq 0$ for all $s \geq 0$, and

$$\begin{aligned} w'(s) \;&=\; \frac{\mu'(\tilde{\alpha}_2^{-1}(s))}{4\tau\tilde{\alpha}_2'(\tilde{\alpha}_2^{-1}(s))} \\ &\leq\; \frac{\mu'(\tilde{\alpha}_2^{-1}(s))}{4\tau\max\{\frac{\tau\bar{p}}{2},1\}(\mu'(\tilde{\alpha}_2^{-1}(s))+1)} \;\leq\; \frac{1}{2\tau^2\bar{p}} \end{aligned}$$

for all $s \geq 0$. It follows from (5) that for all $t \geq 0$ and $x \in \mathbb{R}^n$, we have

$$1 + \left[\int_{t-\tau}^{t}\left(\int_s^t p(r)\,dr\right)ds\right] w'(V(t,x)) \in \left[1, \frac{5}{4}\right]. \quad (11)$$

Since $w = \frac{1}{4\tau}\tilde{\mu}$, it follows that if

$$|x| \geq \chi(|u|) := \alpha_1^{-1} \circ w^{-1}\left(\frac{5}{2\varepsilon}\Omega(|u|)\right), \quad (12)$$

then $\Omega(|u|) \le \frac{2\varepsilon}{5} w \circ \alpha_1(|x|)$, so

$$\begin{aligned}
\dot{V}^\sharp &= \left[1 + \left[\int_{t-\tau}^{t}\left(\int_{s}^{t} p(r)dr\right)ds\right]w'(V)\right]\dot{V} \\
&\quad + \left[\tau p(t) - \int_{t-\tau}^{t} p(r)dr\right]w(V) \\
&\le -p(t)\tilde{\mu}(V) \;+\; \frac{5}{4}\Omega(|u|) \;+\; \tau p(t)w(V) \\
&\quad - \left(\int_{t-\tau}^{t} p(r)dr\right)w(V) \\
&\le -\frac{3}{4}p(t)\tilde{\mu}(V) + \frac{5}{4}\Omega(|u|) - \left(\int_{t-\tau}^{t} p(r)dr\right)w(V) \\
&\le -\varepsilon w(\alpha_1(|x|)) + \frac{5}{4}\Omega(|u|) \;\; \text{(by (2) and (4))} \\
&\le -\frac{\varepsilon}{2}w(\alpha_1(|x|)) \;\; \text{(by (12))}.
\end{aligned}$$

Since

$$w \circ \alpha_1 \in C^1 \cap \mathcal{K}_\infty, \;\; \chi \in \mathcal{K}_\infty,$$

and $V^\sharp \in$ UPPD, it follows that $V^\sharp$ is the desired strict ISS Lyapunov function. The periodicity assertion is easily verified using Property $(\mathcal{P}_1)$ above (see [9]). ■

Remark 5: Since any ISS(p) Lyapunov function for (1) is also a DIS(p) Lyapunov function, the preceding theorem gives a method for converting a nonstrict ISS Lyapunov function into a strict one.

b) Second Construction: The preceding construction can be simplified if f takes the control affine form

$$\dot{x} = f(t, x, u) := h(t, x) + g(t, x)u, \tag{13}$$

as follows. We fix constants $\tau, \varepsilon, \bar{p} > 0$ and $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$, and we assume there exist a time-independent $W \in$ UPPD and a constant $\bar{g} > 0$ such that

$$\left|\frac{\partial W}{\partial x}(x)h(t,x)\right| \le \frac{\varepsilon}{\tau^2\bar{p}}W(x) \;\; , \quad \left|\frac{\partial W}{\partial x}(x)g(t,x)\right| \le \bar{g}$$

for all $t \ge 0$ and $x \in \mathbb{R}^n$.

Theorem 6: Let p, g, h, f, and W be as above. Assume that $V \in$ UPPD and $\chi \in \mathcal{K}_\infty$ are such that

$$|x| \ge \chi(|u|) \Rightarrow$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\left[h(t,x) + g(t,x)u\right] \le -p(t)W(x)$$

for all $t \ge 0$. Then

$$U(t,x) := V(t,x) + \frac{1}{\tau}\left[\int_{t-\tau}^{t}\left(\int_{s}^{t} p(r)\,dr\right)ds\right]W(x)$$

is a strict ISS Lyapunov function for (13). If p and V have period τ in t, then so does U.

We leave the proof of this theorem to the reader; see [9] for similar arguments.

IV. Nonstrict ISS Characterizations

We next relate the Lyapunov function and stability notions we introduced in Section II. For general $p \in \mathcal{P}$, we show that ISS(p) is equivalent to the existence of an ISS(p) Lyapunov function and to the existence of a strict ISS Lyapunov function. In particular, ISS and ISS(p) turn out to be equivalent. This extends the ISS Lyapunov characterizations [4], [19] which only cover the case where $p \equiv 1$. The proof of our equivalences is based on our strict ISS Lyapunov function constructions from the previous sections. For a partial generalization for systems with outputs, see below.

Theorem 7: Let $p \in \mathcal{P}$ and f be as above. Then the following are equivalent conditions for the system (1):

- (C_1) (1) admits an ISS(p) Lyapunov function.
- (C_2) (1) admits a strict ISS Lyapunov function.
- (C_3) (1) admits a DIS(p) Lyapunov function.
- (C_4) (1) admits a strict DIS Lyapunov function.
- (C_5) ISS(p).
- (C_6) ISS.

The proof of Theorem 7 proceeds by showing: $(C_1) \Rightarrow (C_2) \Rightarrow (C_4) \Rightarrow (C_1)$, $(C_3) \Leftrightarrow (C_4)$, $(C_2) \Leftrightarrow (C_6)$, and $(C_5) \Leftrightarrow (C_6)$. The equivalence $(C_5) \Leftrightarrow (C_6)$ is a consequence of Property $(\mathcal{P}_2)$ from Section II, and can be shown as follows.

Pick $p \in \mathcal{P}(\tau, \varepsilon, \bar{p})$. If (C_6) holds, then we can find $\beta \in \mathcal{KL}$ such that for all $t_o \ge 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$, and $h \ge 0$,

$$\begin{aligned}
|\phi(t_o + h)| &\le \beta(|x_o|, \bar{p}h) + \gamma(|\mathbf{u}|_{[t_o, t_o+h]}) \\
&\le \beta(|x_o|, \textstyle\int_{t_o}^{t_o+h} p(s)ds) + \gamma(|\mathbf{u}|_{[t_o, t_o+h]})
\end{aligned}$$

where ϕ is the corresponding trajectory of f we defined in Section II. Therefore, f is ISS(p) so $(C_6) \Rightarrow (C_5)$. Conversely, if f is ISS(p), then we can find $\beta \in \mathcal{KL}$ with the property that for all $t_o \ge 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$, and $h \ge 0$,

$$\begin{aligned}
|\phi(t_o + h)| &\le \beta\left(|x_o|, \textstyle\int_{t_o}^{t_o+h} p(s)ds\right) + \gamma(|\mathbf{u}|_{[t_o, t_o+h]}) \\
&\le \beta(|x_o|, \underline{p}(h)) + \gamma(|\mathbf{u}|_{[t_o, t_o+h]}).
\end{aligned}$$

By $(\mathcal{P}_2)$, $\hat{\beta}(s,t) := \beta(s, \underline{p}(t)) \in \mathcal{KL}$, so $(C_5) \Rightarrow (C_6)$, as desired.

Remark 8: One of the novel features of Theorem 7 is that it applies to time-varying systems. The (strict) ISS property for time-varying systems was covered in [4]. In fact, the implication $(C_6) \Rightarrow (C_2)$ was announced in [4, Theorem 1] and can also be deduced using the existence theory for time-varying Lyapunov functions for set valued dynamics from [2]; see [9] for details.

Remark 9: Our proof of Theorem 7 shows that if V is a strict ISS Lyapunov function for f, then V is also a strict DIS Lyapunov function for f. This implication is no longer true if our periodicity condition on f is dropped, as illustrated in [4].

For the complete proof of Theorem 7, see [9].

V. Strict IOS Lyapunov Function Construction

The ISS property estimates the decay of the state in terms of an overshoot that depends on the magnitude of the control. However, in many applications, the current state may be difficult if not impossible to measure. Instead, only *output measurements* are available, giving rise to the model

$$\dot{x} = f(t,x,u)\ ,\ y = H(x) \tag{14}$$

where f is as before and H is locally Lipschitz. We assume for simplicity in this section that f is forward complete.

Many generalizations of ISS for time-invariant systems with outputs have been proposed; see [7], [18], [20], [21] for discussions. It is natural to generalize the ISS condition by positing a decay of the *output* (instead of the state) with an overshoot depending as before on the magnitude of the input. This is made precise in the following definitions, which generalize the corresponding definitions for time-invariant systems from [21]. In what follows, we set $y(t_o+h;x_o,t_o,\mathbf{u}) = H(\phi(t_o+h;x_o,t_o,\mathbf{u}))$ for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$, and $h \geq 0$.

Definition 10: We say that (14) is *input-to-output stable (IOS)* provided there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|y(t_o+h;x_o,t_o,\mathbf{u})| \quad \leq \quad \beta(|x_o|,h) \ + \ \gamma\left(|\mathbf{u}|_{[t_o,t_o+h]}\right)$$

for all $t_o \geq 0$, $x_o \in \mathbb{R}^n$, $\mathbf{u} \in \mathcal{U}$ and $h \geq 0$.
The corresponding Lyapunov function notion is as follows:

Definition 11: A smooth $V : [0,\infty)\times\mathbb{R}^n \to [0,\infty)$ is called a *(strict) IOS Lyapunov function* for (14) provided there exist functions $\alpha_1,\alpha_2,\chi \in \mathcal{K}_\infty$ and $\kappa \in \mathcal{KL}$ such that the following two conditions hold for all $t \geq 0$:

$$\alpha_1(|H(x)|) \leq V(t,x) \leq \alpha_2(|x|) \ \ \forall x \in \mathbb{R}^n \tag{15}$$

$$V(t,x) \geq \chi(|u|) \ \Rightarrow \ \dot{V}(t,x,u) \ \leq \ -\kappa(V(t,x),|x|) \tag{16}$$

For the equivalence of the IOS property to the existence of an IOS Lyapunov function for a class of time-invariant systems, see [21, Theorem 1.2]. Our methods from Section III can be used to construct strict IOS Lyapunov functions. A first result in this direction is as follows, in which $\mathrm{sat}\{q\}$ denotes the usual projection of $q \in \mathbb{R}$ onto $[-1,+1]$.

Theorem 12: Let f and H be as above and assume $p \in \mathcal{P}(\tau,\varepsilon,\bar{p})$. Let $U : [0,\infty)\times\mathbb{R}^n \to [0,\infty)$ be a C^1 function that admits $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\chi} \in \mathcal{K}_\infty$ and $\hat{\kappa} \in C^1 \cap \mathcal{K}_\infty$ that satisfy

$$\hat{\alpha}_1(|H(x)|) \leq U(t,x) \leq \hat{\alpha}_2(|x|) \ \ \forall x \in \mathbb{R}^n \tag{17}$$

$$U(t,x) \geq \hat{\chi}(|u|) \ \Rightarrow \ \dot{U}(t,x,u) \ \leq \ -p(t)\hat{\kappa}(U(t,x)) \tag{18}$$

for all $t \geq 0$. Define $w : [0,\infty) \to [0,\infty)$ by

$$w(r) = \frac{1}{\tau^2\bar{p} + 2\tau}\int_0^r \mathrm{sat}\{\hat{\kappa}'(s)\}ds.$$

Then

$$V(t,x) = U(t,x) + \left[\int_{t-\tau}^{t}\left(\int_s^t p(l)dl\right)ds\right] w(U(t,x))$$

is a strict IOS Lyapunov function for (14).

Proof: Suppressing arguments as before gives

$$\begin{aligned}\dot{V} \ &= \ \left[1+\left(\int_{t-\tau}^{t}\left(\int_s^t p(l)dl\right)ds\right)w'(U(x,t))\right]\dot{U} \\ &\quad + \left[\tau p(t) - \int_{t-\tau}^{t} p(l)dl\right] w(U(t,x)).\end{aligned}$$

Since

$$\tau w(r) \leq \frac{1}{2}\hat{\kappa}(r) \ \ \forall r \geq 0, \tag{19}$$

and $w' \geq 0$, it follows that if $U(t,x) \geq \hat{\chi}(|u|)$, then

$$\begin{aligned}\dot{V}(t,x,u) \ &\leq \ -p(t)\hat{\kappa}(U(t,x)) \\ &\qquad + \left[\tau p(t) - \int_{t-\tau}^{t} p(l)dl\right] w(U(t,x)) \\ &\leq \ p(t)\left[-\hat{\kappa}(U(t,x)) + \tau w(U(t,x))\right] \\ &\qquad -w(U(t,x))\int_{t-\tau}^{t} p(l)dl \\ &\leq \ -\frac{1}{2}p(t)\hat{\kappa}(U(t,x)) - \varepsilon w(U(t,x)) \\ &\leq \ -\varepsilon w(U(t,x).\end{aligned} \tag{20}$$

Recalling Property $(\mathcal{P}_1)$ for $p \in \mathcal{P}$ from Section II and the structure of V, and noting that $w(r) \leq \frac{r}{\tau^2\bar{p}}$ for all $r \geq 0$, it follows that

$$U(t,x) \leq V(t,x) \leq \frac{3}{2}U(t,x) \tag{21}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$. Therefore, if $V(t,x) \geq \frac{3}{2}\hat{\chi}(|u|)$, then $U(t,x) \geq \hat{\chi}(|u|)$, so the calculation (20) gives

$$\dot{V}(t,x,u) \ \leq \ -\varepsilon w(U(t,x)) \ \leq \ -\varepsilon w(2V(t,x)/3)$$

for all $t \geq 0$. Moreover, $\hat{\alpha}_1(|H(x)|) \leq V(t,x) \leq \frac{3}{2}\hat{\alpha}_2(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$, by (17) and (21). We conclude that V satisfies the strict IOS Lyapunov function requirements with

$$\alpha_1 = \hat{\alpha}_1, \ \ \alpha_2 = \frac{3}{2}\hat{\alpha}_2, \ \ \chi = \frac{3}{2}\hat{\chi}, \ \ \kappa(r,s) \equiv \varepsilon\frac{w(2r/3)}{(1+s)}$$

which proves the theorem. ■

VI. Tracking Example

We next use our results to construct a strict ISS Lyapunov function for a tracking problem for a rotating rigid body; see [3], [14], [15] for the background and motivation for this problem. Following Lefeber [8, p.31], we only consider the dynamics of the velocities, namely,

$$\begin{aligned}&\dot{\omega}_1 = \tfrac{I_2-I_3}{I_1}\omega_2\omega_3 + d_1 + u_1 \\ &\dot{\omega}_2 = \tfrac{I_3-I_1}{I_2}\omega_3\omega_1 + d_2 + u_2, \quad \dot{\omega}_3 = \tfrac{I_1-I_2}{I_3}\omega_1\omega_2\end{aligned} \tag{22}$$

where the ω_i's are the angular velocities, and $I_1 > I_2 > 0$ and $I_3 > 0$ are the principal moments of inertia. The change of feedback and change of coordinate

$$\begin{aligned}&\delta_1 := \tfrac{I_2-I_3}{I_1}\omega_2\omega_3 + d_1, \ \ \delta_2 := \tfrac{I_3-I_1}{I_2}\omega_3\omega_1 + d_2, \\ &Z_3 := \tfrac{I_3}{I_1-I_2}\omega_3\end{aligned}$$

transform (22) as follows:

$$\dot{\omega}_1 = \delta_1 + u_1, \ \ \dot{\omega}_2 = \delta_2 + u_2, \ \ \dot{Z}_3 = \omega_1\omega_2. \qquad (23)$$

We consider the reference state trajectory

$$\omega_{1r}(t) = \cos^2(t), \ \ \omega_{2r}(t) = Z_{3r}(t) = 0 \qquad (24)$$

but our method applies to more general reference trajectories as well; see [9]. The substitutions

$$\begin{aligned}
&\tilde{\omega}_1(t) := \omega_1(t) - \omega_{1r}(t), \ \ \tilde{\omega}_2(t) := \omega_2(t) - \omega_{2r}(t) \\
&\tilde{Z}_3(t) := Z_3(t) - Z_{3r}(t) \\
&k_1(t) := \delta_1(t) + 2\cos(t)\sin(t), \ \ k_2(t) := \delta_2(t) \ \ .
\end{aligned}$$

transform (23) into the error equations

$$\dot{\tilde{\omega}}_1 = k_1 + u_1, \ \dot{\tilde{\omega}}_2 = k_2 + u_2, \ \dot{\tilde{Z}}_3 = (\tilde{\omega}_1 + \cos^2(t))\tilde{\omega}_2 \ \ .$$

Consider the control laws

$$\begin{array}{l} k_1 = -\tilde{\omega}_1 - \frac{1}{2}[\tilde{\omega}_2^2 + 2\tilde{Z}_3\tilde{\omega}_2], \\ k_2 = -\cos^2(t)(\tilde{\omega}_2 + \tilde{Z}_3) \end{array} \qquad (25)$$

and set $Q(\tilde{\omega}_2, \tilde{Z}_3) := \tilde{\omega}_2^2 + \tilde{Z}_3^2 + \tilde{\omega}_2\tilde{Z}_3$ and $R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) := Q(\tilde{\omega}_2, \tilde{Z}_3) + \tilde{\omega}_1^2$. Along the trajectories of the error equations, our control laws (25) give

$$\begin{aligned}
\dot{Q} &= -\cos^2(t)Q(\tilde{\omega}_2, \tilde{Z}_3) + [\tilde{\omega}_2^2 + 2\tilde{Z}_3\tilde{\omega}_2]\tilde{\omega}_1 \\
&\quad +[\tilde{Z}_3 + 2\tilde{\omega}_2]u_2 \\
\dot{R} &\leq -\cos^2(t)R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) + [\tilde{Z}_3 + 2\tilde{\omega}_2]u_2 + 2\tilde{\omega}_1 u_1
\end{aligned}$$

Setting

$$V(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) := \sqrt{R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) + 1} - 1, \ \ \tilde{\mu}(s) := \frac{s}{2},$$

choosing $\Omega(s) := 2s$ and $p(t) := \cos^2(t)$, and noting that

$$Q(\tilde{\omega}_2, \tilde{Z}_3) = \frac{1}{4}(\tilde{Z}_3 + 2\tilde{\omega}_2)^2 + \frac{3}{4}\tilde{Z}_3^2 \ \ \geq \ \ \frac{1}{2}(\tilde{\omega}_2^2 + \tilde{Z}_3^2)$$

everywhere gives

$$\begin{aligned}
\dot{V} \ &\leq \ -\cos^2(t)\tfrac{R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3)}{2\sqrt{R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3)+1}} \\
&\quad + \tfrac{\tilde{Z}_3 + 2\tilde{\omega}_2}{2\sqrt{R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3)+1}}u_2 + \tfrac{2\tilde{\omega}_1}{2\sqrt{R(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3)+1}}u_1 \\
&\leq \ -\tfrac{1}{2}\cos^2(t)V(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) + |u_1| + |u_2| \\
&\leq \ -p(t)\tilde{\mu}(V(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3)) + \Omega(|u|)
\end{aligned}$$

for all $u \in \mathbb{R}^2$, $t \geq 0$, and $(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) \in \mathbb{R}^3$. This is the DIS(p) condition (8) for the error equations in closed-loop with the control laws (25). Therefore, V is a DIS(p) Lyapunov function for the closed loop system, where $p \in \mathcal{P}(\pi, \frac{\pi}{2}, 1)$. Applying our strict ISS Lyapunov function construction method gives the following strict ISS Lyapunov function for the closed-loop system (see [9]):

$$V^\sharp = \Big(S(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) - 1\Big)\left(1 + \frac{\pi}{32} + \frac{1}{32}\sin(2t)\right),$$

where

$$S(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{Z}_3) := \sqrt{\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{Z}_3^2 + \tilde{\omega}_2\tilde{Z}_3 + 1}.$$

This illustrates how a time-invariant nonstrict Lyapunov function can give a time-varying strict Lyapunov function.

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