A Survey of Consensus Problems in Multi-agent Coordination

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***Abstract*— As a distributed solution to multi-agent coordination, consensus or agreement problems have been studied extensively in the literature. This paper provides a survey of consensus problems in multi-agent cooperative control with the goal of promoting research in this area. Theoretical results regarding consensus seeking under both time-invariant and dynamically changing information exchange topologies are summarized. Applications of consensus protocols to multi-agent coordination are investigated. Future research directions and open problems are also proposed.**

I. Introduction

Cooperative control for multi-agent systems can be categorized as either formation control problems with applications to mobile robots, unmanned air vehicles (UAVs), autonomous underwater vehicles (AUVs), satellites, aircraft, spacecraft, and automated highway systems, or non-formation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, timing, and search. The cooperative control of multi-agent systems poses significant theoretical and practical challenges. For cooperative control strategies to be successful, numerous issues must be addressed, including the definition and management of shared information among a group of agents to facilitate the coordination of these agents.

In cooperative control problems, shared information may take the form of common objectives, common control algorithms, relative position information, or a world map. Information necessary for cooperation may be shared in a variety of ways. For example, relative position sensors may enable vehicles to construct state information for other vehicles [1], knowledge may be communicated between vehicles using a wireless network [2], or joint knowledge might be pre-programmed into the vehicles before a mission begins [3]. For cooperative control strategies to be effective, a team of agents must be able to respond to unanticipated situations or changes in the environment that are sensed as a cooperative task is carried out. As the environment changes, the agents on the team must be in agreement as to what changes took place. A direct consequence of the assumption that shared information is a necessary condition for coordination is that cooperation requires that the group of agents reach consensus on the coordination data. In other words, the instantiation of the coordination data on each agent must asymptotically approach a sufficiently common value.

Convergence to a common value is called the *consensus* or *agreement* problem in the literature. Although consensus problems have a history in computer science (e.g. [4]), we will focus on their applications in cooperative control of multi-agent systems in this paper. The main purpose of this paper is to summarize the recent progress of consensus problems in the cooperative control community with the goal to facilitate research in this area.

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II. Background and Problem Statement

A. Graph Theory

It is natural to model information exchange between agents in a cooperative team by directed/undirected graphs (e.g. [5]). A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite nonempty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \cdots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined analogously, where (v_{i_j}, v_{i_k}) implies (v_{i_k}, v_{i_j}). A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node, except the root, has exactly one parent. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a spanning tree if there exists a spanning tree that is a subset of the graph. Note that the condition that a digraph has a spanning tree is equivalent to the case that there exists a node having a directed path to all other nodes.

The adjacency matrix $A = [a_{ij}]$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The Laplacian matrix of the weighted digraph is defined as $L = [\ell_{ij}]$, where $\ell_{ii} = \sum_j a_{ij}$ and $\ell_{ij} = -a_{ij}$ where $i \neq j$. For an undirected graph, the Laplacian matrix is symmetric positive semi-definite.

B. Matrix Theory

Below we summarize some notation from nonnegative matrix theory (c.f. [6], [7]) which are important for studying consensus problems.

Let $M_n(I\!R)$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(I\!R)$, the digraph of A, denoted by $\Gamma(A)$, is the digraph on n vertices v_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from v_j to v_i if and only if $a_{ij} \neq 0$ (c.f. [7]).

A matrix is nonnegative (positive) if all its entries are nonnegative (positive). A vector is nonnegative (positive) if all its elements are nonnegative (positive). Furthermore, if all its row sums are $+1$, the matrix is said to be a (row) stochastic matrix [7]. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $\lim_{k\to\infty} P^k = \mathbf{1}\nu^T$, where $\mathbf{1}$ is a column vector of all ones and ν is some column vector [8].

The well-known Perron-Frobenius Theorem states that if a nonnegative matrix A is irreducible, that is, the digraph of matrix A is strongly connected, then $\rho(A)$ is a simple

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eigenvalue of A associated with a positive eigenvector, where $\rho(\cdot)$ denotes the spectral radius of a matrix.

If a nonnegative matrix A is primitive, that is, A is irreducible and $\rho(A)$ is a unique eigenvalue of maximum modulus, then $\lim_{k\to\infty}[\rho(A)^{-1}A^k] \to w\nu^T$, where ν and w are left and right positive eigenvectors of matrix A associated with eigenvalue $\rho(A)$ satisfying $w^T\nu = 1$ [7].

The classical result in Markov chains states that if a stochastic matrix A satisfies $A^m > 0$ for some positive integer m, then $\lim_{k\to\infty} A^k \to \mathbf{1}\nu^T$, where ν is a positive column vector satisfying $\mathbf{1}^T\nu = 1$ [9]. In fact, the condition $A^m > 0$ for some positive integer m is equivalent to the condition that A is irreducible and $\rho(A)$ is a unique eigenvalue of maximum modulus.

C. Consensus Protocols

Let x_i be the information state of the i^{th} agent. The information state represents information that needs be coordinated between agents. The information state may be agent position, velocity, oscillation phase, decision variable, and so on.

As described in [2], [10], [11], [12], [13], [14], a continuous-time consensus protocol can be summarized as

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{J}_i(t)} \alpha_{ij}(t)(x_i(t) - x_j(t)), \qquad (1)$$

where $\mathcal{J}_i(t)$ represents the set of agents whose information is available to agent i at time t and $\alpha_{ij}(t)$ denotes a positive time-varying weighting factor. In other words, the information state of each agent is driven toward the states of its (possibly time-varying) neighbors at each time. Note that some agents may not have any information exchange with other agents during some time intervals. The continuous-time linear consensus protocol (1) can be written in matrix form as $\dot{x} = -Lx$, where L is the graph Laplacian and $x = [x_1, \cdots, x_n]^T$.

Correspondingly, a discrete-time consensus protocol as proposed in [10], [15], [13] can be summarized as

$$x_i[k+1] = \sum_{j\in\mathcal{J}_i[k] \quad \{i\}} \beta_{ij}[k]x_j[k], \qquad (2)$$

where $\sum_{j\in\mathcal{J}_i[k] \quad \{i\}} \beta_{ij}[k] = 1$, and $\beta_{ij}[k] > 0$ for $j \in \mathcal{J}_i[k]\bigcup\{i\}$. In other words, the next state of each agent is updated as the weighted average of its current state and the current states of its (possibly time-varying) neighbors. Note that an agent simply maintains its current state if it has no information exchange with other agents at a certain time step. The discrete-time linear consensus protocol (2) can be written in matrix form as $x[k+1] = D[k]x[k]$, where $D[k]$ is a stochastic matrix with positive diagonal entries.

Consensus is said to be achieved for a team of agents if $\|x_i - x_j\| \to 0$ as $t \to \infty$, $\forall i \neq j$.

III. Theoretical Aspects of Consensus Problems

In this section, we review recent theoretical progress of consensus problems for multi-agent systems.

A. Convergence Analysis for A Time-invariant Information Exchange Topology

Under a time-invariant information exchange topology, it is assumed that if one agent can access another agent's information at one time, it can obtain information from that agent all the time.

For the continuous-time consensus protocol (1), it is straightforward to see that $L\mathbf{1} = 0$ and all eigenvalues of the Laplacian matrix L have non-negative real parts from Gershgorin's disc theorem. If zero is a simple eigenvalue of L, it is known that x converges to the kernel of L, that is, span$\{\mathbf{1}\}$, which in turn implies that $\|x_i - x_j\| \to 0$.

It is well-known that zero is a simple eigenvalue of L if the graph of L is strongly connected [16]. However, this is only a sufficient condition rather than a necessary one. We have the formal statement that zero is a simple eigenvalue of the Laplacian matrix if and only if its digraph has a spanning tree. This conclusion was shown in [17] by an induction approach while the same result is proven independently in [18] by a constructive approach. As a result, under a time-invariant information exchange topology, the continuous-time protocol achieves consensus asymptotically if and only if the information exchange topology has a spanning tree.

For the discrete-time consensus protocol (2), it can be shown that all eigenvalues of D that are not equal to one are within the open unit circle from Gershgorin's disc theorem. If one is a simple eigenvalue of D and all other eigenvalues have modulus less than one, it is known that $\lim_{k\to\infty} D^k \to \mathbf{1}\nu^T$, where ν is a column vector. This implies that $\|x_i - x_j\| \to 0$.

The well-known Perron-Frobenius theorem states that one is a simple eigenvalue of a stochastic matrix if the graph of the matrix is strongly connected. Similar to the continuous-time case, this is only a sufficient condition rather than a necessary one. Ref. [13] shows that for a nonnegative matrix with identical positive row sums, the row sum of the matrix is a simple eigenvalue if and only if the digraph of the matrix has a spanning tree. In other words, a matrix may be reducible but retains its spectral radius as a simple eigenvalue. Furthermore, if the matrix has a spanning tree and positive diagonal entries, it is shown that the spectral radius of the matrix is the unique eigenvalue of maximum modulus. We have the formal statement that one is a unique eigenvalue of modulus one for the stochastic matrix D if and only if its digraph has a spanning tree [13]. As a result, under a time-invariant information exchange topology, the discrete-time protocol achieves consensus asymptotically if and only if the information exchange topology has a spanning tree.

B. Equilibrium State Under a Time-invariant Topology

Now that we know under what conditions the consensus protocols converge, the next step is to find the equilibrium state to which the consensus protocols converge.

In the case that the information exchange topology has a spanning tree, we know that $\lim_{t\to\infty} e^{-Lt} \to \mathbf{1}\nu^T$ and $\lim_{k\to\infty} D^k \to \mathbf{1}\mu^T$, where $\nu = [\nu_1, \cdots, \nu_n]^T$ and $\mu = [\mu_1, \cdots, \mu_n]^T$ are non-negative left eigenvectors of L and D corresponding to eigenvalues 0 and 1 respectively satisfying $\sum \nu_j = \sum \mu_j = 1$. As a result, $x(t) \to \sum \nu_j x_j(0)$ and $x[k] \to \sum \mu_j x_j[0]$. That is, the final equilibrium state is a weighted average of each agent's initial condition. However, it is not clear whether each agent will contribute to the final equilibrium state.

In the case that the information exchange topology is strongly connected, we know that all ν_j and μ_j are positive [7]. Therefore, each agent's initial condition contributes to the final consensus equilibrium in this case. Furthermore, if $\nu_i = \nu_j = \frac{1}{n}$ and $\mu_i = \mu_j = \frac{1}{n}$, where $i \neq j$, the final consensus equilibrium will be the average of each agent's initial condition, which is called "*average consensus*" in [11]. As shown in [11], average consensus is achieved if the information exchange topology is both strongly connected and balanced. In the case that the information exchange topology has a spanning tree, the final consensus value is equal to the weighted average of initial conditions of those agents that have a directed path to all the other agents [17]. While the requirement of having a spanning tree is less stringent than being strongly connected and balanced, the final consensus value may be in favor of some agents and may not be an average.

C. Convergence Analysis for Dynamic Information Exchange Topologies

Although consensus problems are significantly simplified by assuming a time-invariant information exchange topology, the information exchange topology between agents may change dynamically in reality. For instance, communication links between agents may be unreliable due to disturbances and/or subject to communication range limitations. If information is being exchanged by direct sensing, the locally visible neighbors of an agent will likely change over time.

Many research efforts on the coordination of multiple autonomous agents under switching information exchange topologies are motivated by Viscek's model [19]. Viscek's model can be thought of as a special case of a distributed behavioral model proposed in [20], where computer animations are used to generate the aggregate motions of a group of animals.

One approach to tackling switching topologies is the algebraic graph, which typically associates graph topologies with the algebraic structure of the corresponding matrices of those graphs. Notice that the solution of the discrete-time and continuous-time consensus protocols can be written as $x[k] = D[k]\cdots D[1]D[0]x[0]$ and $x(t) = \Phi(t,0)x(0)$ respectively, where $\Phi(t,0)$ is the transition matrix corresponding to $-L(t)$. Consensus can be reached if $\lim_{k\to\infty} D[k]\cdots D[1]D[0] \to \mathbf{1}\nu^T$ and $\lim_{t\to\infty}\Phi(t,0) \to \mathbf{1}\mu^T$, where ν and μ are column vectors. In the special case that $L(t)$ is piecewise constant with dwell times $\tau_j = t_{j+1} - t_j$, consensus can be reached if $\lim_{t\to\infty} e^{-L(t_j)(t-t_j)}e^{-L(t_{j-1})\tau_{j-1}}\cdots e^{-L(t_0)\tau_0} \to \mathbf{1}\mu^T$. Equivalently, we can study the property of infinite products of stochastic matrices.

The classical result in [8] demonstrates the property of the infinite products of certain categories. The main result of [8] can be summarized as follows. Let $\mathcal{S} = \{S_1, S_2, \cdots, S_k\}$ be a finite set of SIA matrices with the property that for each sequence $S_{i_1}, S_{i_2}, \cdots, S_{i_j}$ of positive length, the matrix product $S_{i_j}S_{i_{j-1}}\cdots S_{i_1}$ is SIA. Then for each infinite sequence $S_{i_1}, S_{i_2}, \cdots$ there exists a column vector ν such that $\lim_{j\to\infty} S_{i_j}S_{i_{j-1}}\cdots S_{i_1} = \mathbf{1}\nu^T$. From the concluding remarks in [8], we see that in the case that $\mathcal{S}$ is an infinite set, $\lambda(W) < 1$, where $W = S_{k_1}S_{k_2}\cdots S_{k_{N_t+1}}$, $\lambda(W) = 1 - \min_{i_1,i_2}\sum_j \min(w_{i_1j}, w_{i_2j})$, and N_t is defined as the number of different types of all $n\times n$ SIA matrices. Furthermore, if there exists a constant $0 \le d < 1$ satisfying $\lambda(W) \le d$, then the above limit result of an infinite sequence also holds.

In the case that the union of undirected information exchange graphs across a bounded time interval is connected, the product of D matrices across such an interval is SIA [10]. Using the above result for finite $\mathcal{S}$ in [8], Ref. [10] provides a theoretical explanation for consensus of the heading angles of a group of agents using nearest neighbor rules under undirected switching information exchange topologies. It is shown that consensus is achieved asymptotically if the union of the information exchange graphs for the team is connected most of the time as the system evolves. This result is further discussed in [21] and [22]. Taking into account the fact that sensors may have a limited field of view, the authors in [12] use digraphs to derive consensus seeking results under switching information exchange topologies. It is shown that consensus using the continuous-time linear protocol can be achieved if in each uniformly bounded time interval there exists at least one piecewise constant switching topology being strongly connected. Ref. [13] further extends the previous results to the case that consensus can be achieved asymptotically if the union of the directed information exchange graphs for the group has a spanning tree frequently enough as the system evolves.

A common feature in the above analysis is that $L(t)$ is assumed to be piecewise constant for the continuous-time consensus protocol. However, it is possible that $L(t)$ may be time-varying to reflect the relative confidence of each agent about its information state, that is, the weighting factors α_{ij} may be time-varying. In fact, in the case that $L(t)$ is piecewise continuous and each nonzero entry ℓ_{ij}, where $i \neq j$, is uniformly lower and upper bounded, consensus is reached asymptotically using the continuous-time consensus protocol if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the information exchange graph across each such interval has a spanning tree [23].

In contrast to the algebraic graph approach, nonlinear tools are used by some other researchers to study consensus problems. For the discrete-time consensus protocol, a set-valued function V is defined as $V(x_1,\cdots,x_n) = (\text{conv}\{x_1,\cdots,x_n\})^n$, where $\text{conv}\{x_1,\cdots,x_n\}$ denotes the convex hull of x_i, $i = 1,\cdots,n$ [15]. It is shown that under some conditions V is non-increasing over time and V indeed approaches a singleton set $x_1(t) = \cdots = x_n(t) =$ constant, which implies that consensus is reached. Using the set-valued Lyapunov theory, Ref. [15] shows that the discrete-time linear protocol is uniformly globally attractive with respect to the collection of equilibrium solutions $x_1(t) = \cdots = x_n(t) =$ constant if and only if there exists a $T \ge 0$ such that there is a node that has a directed path to all the other nodes across each interval of length T. For the continuous-time consensus protocol, a Lyapunov function candidate is proposed as $V(x) = \max\{x_1,\cdots,x_n\} - \min\{x_1,\cdots,x_n\}$ in [14]. It is shown that $V(x)$ decreases over a sufficient length of time intervals. In the case that $L(t)$ is piecewise continuous and each nonzero entry ℓ_{ij}, where $i \neq j$, is uniformly lower and upper bounded, the equilibrium set $x_1 = \cdots = x_n =$ constant is uniformly exponentially stable if there is an index $k \in \{1, \cdots, n\}$ and an interval length $T > 0$ such that for all t the digraph of $\int_t^{t+T}(-L(s))ds$ has the property that node k has a directed path to all the other nodes.

In addition, in [24], nonlinear contraction theory is used to study synchronization and schooling applications, which are related to the consensus problem. In particular, the continuous-time consensus protocol is analyzed under undirected switching information exchange topologies. It is shown that consensus is reached asymptotically if there exists an infinite sequence of bounded intervals such that the union of the graphs across each interval is connected, which recovered the result shown in [10].

D. Relative Information Uncertainty

In practical applications of multi-agent systems, there are many cases where some individuals on the team will have access to better information than others. In cases like these the consensus algorithm needs to be biased to favor agents with better information. As a result, weighting factors in Eqs. (1)-(2) should be adjusted to reflect agent certainty about its information. For example, large α_{ij} and β_{ij} indicates that agent j has a high degree of confidence about its information while small α_{ij} and β_{ij} indicates low confidence.

Certainty information can be encoded into each agent by means of covariance matrix $P_i = E[(x_i - x^f)(x_i - x^f)^T]$, where x^f represents the final consensus value. Motivated by the continuous-time Kalman filter, the weighting factors can be updated as follows [23]:

$$\dot{P}_i = -P_i \underset{j\in\mathcal{J}_i(t)}{\qquad} (P_j + \Omega_{ij})^{-1} \quad P_i + Q, \quad \alpha_{ij} = P_i(P_j + \Omega_{ij})^{-1},$$

where Q denotes the covariance of the process noise and Ω_{ij} denotes the covariance of transmission or communication noise between agent j and agent i.

E. Communication Delays

In the case that information is exchanged between agents through communications, time delays of the communication channels need to be considered.

Let τ_{ij} denote the time delay for information communicated from agent j to agent i. The continuous-time consensus protocol is now denoted by $\dot{x}_i = \sum_{i\in\mathcal{J}_i(t)} \alpha_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})]$. In the simplest case where $\tau_{ij} = \tau$ and the communication topology is fixed, undirected, and connected, average-consensus is achieved if and only if $\tau \in [0, \frac{\pi}{2\lambda_{max}(L)})$, where L is the graph Laplacian matrix [11].

Consider another case where the time delay only affects the information state that is being transmitted. The continuous-time consensus protocol is denoted as $\dot{x}_i = \sum_{i\in\mathcal{J}_i(t)} \alpha_{ij}[x_j(t - \tau_{ij}) - x_i(t)]$. In the case where $\tau_{ij} = \tau$ and the communication topology is directed and switching, the consensus result for switching topologies described previously is still valid for an arbitrary time delay τ [14].

F. Consensus Synthesis

In the above discussions, we analyze the properties of consensus protocols given beforehand. On some occasions, we may want to generate consensus protocols that satisfy certain properties or optimize some performance criteria.

For example, in a network with a large number of agents, it may be desirable to solve the fastest distributed linear averaging (FDLA) problem defined as follows [25].

Let $W = [W_{ij}] \in M_n(I\!R)$ such that $W_{ij} = 0$ if there is no information exchange between agent i and agent j. Given $x[k+1] = Wx[k]$, the FDLA problem is to find the weight matrix W that guarantees the fastest convergence speed to the average consensus value. In contrast to discrete-time consensus protocol (2), the weighting factors W_{ij} above are allowed to be negative. In fact, as shown in [25], the optimal weighting factors may sometimes be negative. With an additional constraint that $W_{ij} = W_{ji}$, the FDLA problem can be simplified as a semi-definite program and solved accordingly [25].

More generally, consider an interconnected network of n agents with dynamics given by $\dot{x}_i = \sum_{j=1}^n A_{ij}x_j + B_i^{(1)}w_i + B_i^{(2)}u_i$, where $x_i \in I\!R^n$ denotes the state, $w_i \in I\!R^m$ denotes the disturbance, and $u_i \in I\!R^r$ denotes the control input with $i = 1, \cdots, n$. Ref. [26] focuses on synthesizing a state feedback controller that guarantees consensus for the closed loop system without disturbance as well as a state feedback controller that achieves not only consensus but optimal $\mathcal{H}_2$ performance for disturbance attenuation. Necessary and sufficient convex conditions are derived for the existence of such state feedback controllers.

G. Other Issues

Under certain circumstances, it is desirable to construct nonlinear consensus protocols as shown in [11], [27], [24].

Agreement problems are also studied from a stochastic point of view in [28], which relies on graph theoretic results developed in [29].

Furthermore, dynamic consensus problems are studied in [30], which focuses on how to achieve and analyze tracking of linear consensus on time-varying inputs.

IV. Multi-agent Coordination via Consensus Schemes

In this section, we investigate applications of the consensus schemes to multi-agent coordination.

A. Vehicle Formations

Consensus schemes have been extensively applied to achieve vehicle formations. In [2], information exchange techniques are studied to improve stability margins and vehicle formation performance. The authors also derive a Nyquist-like stability criterion for formation stabilization. It is argued that vehicle formations can be achieved through reaching consensus on the center point of the formation. In [31], a decentralized control strategy is implemented to maintain multiple robot formations, where each robot only needs position information of its two neighbors. Ref. [18] studies the formation stabilization problem of multiple unicycles using a consensus scheme, where results are given for formation stabilization of the unicycles to a point, a line, and a general formation pattern. In addition, the simplified pursuit strategy for wheeled-vehicle formations in [32] can be thought of as a special case of the continuous-time linear consensus protocol, where the information exchange topology is a uni-directional ring. In [33], a network of vehicles are required to achieve a polygonal formation by exchanging information among themselves, where stochastic disturbance is introduced to the communication graph and the effect of stochastic loss of information is studied by Monte Carlo simulations. In [34], a general framework is provided to analyze both consensus problems and formation stabilization problems, where the effect of formation switching is addressed for both single integrator dynamics and general plant dynamics. In addition, double integrator dynamics are applied in [35] to study formation stabilization problems in the general framework of sensing/communication architectures, which is related to the consensus problems.

B. Attitude Alignment

In [36], formation control laws are presented for maintaining attitude alignment among a group of spacecraft, where the attitude of each spacecraft is synchronized with its two adjacent neighbors via a bi-directional communication topology. Similarly, the bi-directional communication topology is used in [37] to synchronize instantiations of group level information, the formation state, among multiple spacecraft. Ref. [38] considers the attitude alignment problem for a team of UAVs using nonlinear decentralized consensus protocols, where input constraints are also taken into account.

C. Rendezvous Problem

The rendezvous problem requires that each agent arrive at a location simultaneously. In [39], consensus seeking ideas are applied to a rendezvous problem for a group of mobile autonomous agents, where both the synchronous case and the asynchronous case are considered. In [17], the "meet for dinner" problem is addressed in the context of consensus seeking. A similar idea is extended for multiple UAVs to converge on the boundary of a radar detection area simultaneously to maximize the element of surprise [40].

D. Coordinated Decision Making

In multi-agent systems, distributed decision making has an advantage over centralized decision making in the sense that a decision maker is not required to access information from all the other decision makers. In [27], a distributed consensus protocol is introduced to coordinate orders of a network of buyers. The authors prove that distributed protocols can achieve the same coordination as the centralized decision making process. Ref. [41] considers a scenario of multiple distributed noisy sensors observing a single event, where those sensors need to reach consensus about the event after exchanging messages through a network. The authors apply belief propagation as a message passing strategy to solve a distributed hypothesis testing problem for a prespecified network connectivity.

E. Flocking

In [42], [43], [44], the authors study the flocking phenomenon observed in [20] by constructing local control laws that allow a group of mobile agents to align their velocities, move with a common speed and achieve desired inter-agent distances while avoiding collisions with each other under a fixed topology and switching topologies respectively. These results extend some results reported in [45].

F. Coupled Oscillators

Ref. [46] studies connections between phase models of coupled oscillators and kinematic models of groups of self-propelled particles. The authors develop analysis and design tools for stabilization of collective motions based on previous results for coupled oscillators. In [47], the authors analyze the stability of classic Kuramoto model of coupled nonlinear oscillators. It is proved that for couplings above a critical value, all the oscillators synchronize in the case of identical and uncertain natural frequencies.

G. Robot Position Synchronization

In [48], a linear continuous-time consensus-like scheme is proposed to solve the problem of position synchronization of multiple cooperative robot systems, where only position measurements are required and the controller is shown to be semi-globally exponentially stable.

V. Conclusions and Future Directions

We have reviewed consensus protocols in the current literature. Since much research on consensus problems is ongoing, this survey is by no means complete.

In the current literature, most consensus problems are studied in the context of single integrator dynamics. Some results from single integrator dynamics imply that the same results may be extended to double integrator dynamics or more complicated dynamics. As a result, the same framework of consensus-seeking for single integrator dynamics may be applied to decentralized robot, spacecraft, and UAV formation flying scenarios, where the communication topologies between spacecraft/UAV could be switching with time. The study of consensus problems for a team of agents with more complicated nonlinear dynamics and a team of heterogenous agents is an interesting topic for future research.

Most research in consensus problems assumes that the final consensus value to be reached is inherently constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics, as happens in some formation control problems where the formation is moving through space. It will be interesting to study consensus problems where the final consensus value evolves with time or as a function of vehicle/environmental dynamics.

Furthermore, consensus problems may be studied from a stochastic point of view to take into account the case that at each time instance the existence of an information exchange link between agents may be probabilistic.

In the current literature, most research activities focus on theoretical study of consensus problems and most results are demonstrated via simulations except for a few experimental results of multiple mobile robot coordination with strongly connected time-invariant sensing/communication topologies. Experimental implementation of consensus schemes for multiple agent systems is a key element of research in the future. Furthermore, issues like disturbances, time-delay, communication/sensor noise, and model uncertainties should also be taken into account. Future research may be involved in studying how communication noise and inconsistent time-delay from different neighboring agents affect consensus for the whole system under dynamically changing information exchange topologies.

Several other key problems that need to be addressed are as follows.

1) How does a group of agents form consensus when the data is (a) continuous versus discrete in time, (b) quantized in amplitude, or (c) originates from sources with variable reliability?
2) How does one design consensus protocols that not only account for control input constraints but also make use of system dynamics to converge on an optimal solution with respect to an objective?
3) How do we make the team objectives invariant with respect to the consensus seeking problem? In other words, as consensus is being formed, the agents must act on the best information available to them at the time. One way of viewing this is that the individuals understand the team objectives differently. Under what conditions will the "design" objectives be satisfied?

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