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Information Flow and its relation to the stability of the motion of vehicles in a rigid formation

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***Abstract*— It is known in the literature on Automated Highway Systems that information flow can significantly affect the propagation of errors in spacing in a collection of vehicles. This paper investigates this issue further for a homogeneous collection of vehicles, where in the motion of each vehicle is modeled as a point mass. The structure of the controller employed by the vehicles is as follows: $U_i(s) = C(s)\sum_{j \in S_i}(X_i - X_j - \frac{L_{ij}}{s})$, where $U_i(s)$ is the (Laplace transformation of) control action for the i^{th} vehicle, X_i is the position of the i^{th} vehicle, L_{ij} is the desired distance between the i^{th} and the j^{th} vehicles in the collection, $C(s)$ is the controller transfer function and S_i is the set of vehicles that the i^{th} vehicle can communicate with directly. This paper further assumes that the information flow is undirected, i.e., $i \in S_j \iff j \in S_i$ and the information flow graph is *connected*. We consider information flow in the collection, where each vehicle can communicate with a maximum of $q(n)$ vehicles, such that $q(n)$ may vary with the size n of the collection. We first show that $C(s)$ cannot have any zeroes at the origin to ensure that relative spacing is maintained in response to a reference vehicle making a maneuver where its velocity experiences a steady state offset. We then show that if the control transfer function $C(s)$ has one or more poles located at the origin of the complex plane, then the motion of the collection of vehicles will become unstable if the size of the collection is sufficiently large. These two results imply that $C(0) \neq 0$ and $C(0)$ is well defined. We further show that if $\frac{q(n)^3}{n^2} \to 0$ as $n \to \infty$, then there is a low frequency sinusoidal disturbance of at most unit amplitude acting on each vehicle such that the maximum errors in spacing response increase at least as $O\left(\sqrt{\frac{(n^2)}{q(n)^3}}\right)$. A consequence of the results presented in this paper is that the maximum of the error in spacing and velocity of any vehicle can be made insensitive to the size of the collection only if there is at least one vehicle in the collection that communicates with at least $O(n^{2/3})$ other vehicles in the collection.**

I. Introduction

Recent advances in a variety of technologies such as communication, computation, sensing and actuation have enabled the development and increased the possibility of deployment of collections of Unmanned Vehicles (UVs) (or simply vehicles) for a wide variety of tasks. UVs are central to automating driving tasks in an Automated Highway System (AHS) [1], the dynamic positioning of mobile offshore bases for creating a runway for large aircrafts [2] and for information gathering in dangerous environments [13]. There seem to be potentially many advantages to deploying UVs in collections for certain tasks: flexibility, ease of reconfiguration and lower cost of deploying collections of smaller UVs as compared to deploying a larger UV being some of them. In order to realize these potential advantages, the problem of coordinating the motion of the collection of vehicles must be addressed and this paper is devoted to an analysis of this problem.

It is conceivable that a collection of vehicles will be required to maintain (or remain close to) specified discernible geometric patterns during its motion. We call such a collection of vehicles as a formation if every vehicle aids in the maintenance of the specified geometric pattern by coordinating its motion through communication with or sensing other vehicles in the collection. The desired motion of every vehicle in a formation is determined by the desired motion of a few vehicles in the collection so that the specified geometric pattern is maintained. Since vehicles in a formation are coupled dynamically by feedback, errors in spacing and velocity (defined as the deviation in the position and velocity from their respective desired values) of a vehicle propagate from one vehicle in the formation to the other.

Of recent interest in the research community is the rigid formation of vehicles, where it is desired that the distance between any two vehicles remain constant throughout the motion. In an AHS, such rigid formations (referred to as a platoons) are desired from the viewpoint of maintaining safety and enhancing the throughput of vehicles on a section of a congested highway [9]. A rigid formation is helpful for localization in partially known environments in the case of mobile robots [7], and in drag reduction via close formation flight [4], [5].

An issue with the design of controllers for vehicles in a collection is that of *collective stability* of the controlled motion of vehicles. This issue arises because errors in spacing and velocity of a vehicle propagate to others in the collection. Intuitively, the collective stability requires the following: With a specified controller and with the vehicles starting at their desired positions and velocities, for any given bound, ε, is there a bound, δ, *independent of the size of the collection*, on the magnitude of *any* force disturbance that can act on *any* vehicle, so that as the errors propagate with the choice of controllers, they always remain smaller than ε ? The requirement of the independence of δ from the size of the collection captures the scalability of the stability of motion with the specified controllers. We will say a controller is scalable if the above requirement of collective stability of controlled motion is met. Since no formation can ever be rigid, we will say that an "approximately rigid formation" can be synthesized if one can synthesize a scalable controller.

In this paper, we are interested in the synthesis of scalable controllers which take into account an additional consideration - that of spatial shift-invariance (i.e. controller is not dependent on the index of the vehicle or the size of the collection). From a practical viewpoint, such a controller will be simple to develop and implement on every vehicle. This is important for applications such as the Adaptive Cruise Control (ACC) System for ground vehicles, because one will not know *a priori* how many vehicles with an ACC System will be placed in succession in traffic. In [10], [6], controllers that used the information about the index of the vehicle in the collection were synthesized; however, for them to achieve an approximately rigid linear formation, the control gains had to increase with the index of the vehicle in a geometric manner and from a practical viewpoint, this is unrealistic since it will lead to saturation of control effort even with small errors in spacing and velocity. For this reason and for the simplicity of treatment, we only consider the restricted class of controllers for

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further investigation.

The synthesis of an "approximate rigid formation" is strongly influenced by the communication pattern between the vehicles. If the formation has the knowledge of the information of a reference vehicle in the collection, then errors in the spacing and velocity resulting from a disturbance acting on a vehicle can made to attenuate as it propagates from one vehicle to another [3], [8]. To date, it is believed that the information of one vehicle must be available to almost all the vehicles in the formation if one were to construct approximate rigid formations. The results in [3], [8] and even in this paper point in this direction.

The following question naturally arises and is the focus of investigation in this paper: How does a pattern of communication amongst vehicles affect the propagation of errors? Specifically, with a specified pattern of communication amongst them, can an approximately rigid formation be synthesized? If the answer to the latter question is in the affirmative, one can employ the same controller in each of the vehicles irrespective of the size of the collection, i.e., one can design a "scalable" control system with the given information flow.

The main results of this paper concerns the necessary conditions on the information structure for the synthesis of approximately rigid formations and are as follows: If the motion of each vehicle can be represented as the motion of a unit mass under the action of a control force and a disturbance and that the information flow graph is undirected, we show that there is no "scalable" control system if every vehicle can only communicate with at most $q(n)$ vehicles, where n is the size of the collection and $q(n)$ satisfies

$$\lim_{n\to\infty} \frac{q(n)^3}{n^2} = 0.$$

We show this result by constructing a sinusoidal disturbance of atmost unit magnitude acting on each vehicle at an appropriately chosen low frequency that results in a maximum error in spacing of at least $O(\sqrt{\frac{n^2}{q^3(n)}})$. A consequence of this result is that at least one vehicle in the collection must communicate with at least $O(n^{2/3})$ other vehicles in the collection for a "scalable" controller to exist. We also show that if the controller incorporates an integral action, the motion of the collection is necessarily unstable for all sizes of the collection greater than a critical value.

The paper is organized as follows: In Section II, we formulate the problem precisely for one-dimensional formations and prove the results stated above. In Section III, we provide corroborating simulations. In Section IV, we summarize the results of this paper.

II. Problem Formulation for a string of vehicles travelling in a straight line

In this section, we will consider a string of vehicles moving in a straight line. The first vehicle, which we call reference vehicle, executes maneuvers with bounded velocity and acceleration. The reference vehicle is referred to as lead vehicle in the AHS literature. For each $i \geq 2$, the i^{th} vehicle desires to maintain a fixed following distance $L_{i,i-1}$ from its predecessor. Initially, all vehicles are assumed to be at their desired position and the velocity of all the vehicles are identical.

A. Model of a Vehicle

We will treat every vehicle as a point mass subjected to a controlled force, $u(t)$ and a disturbance $d(t)$. If $x(t)$ is the position of a vehicle measured from the origin of an inertial reference frame, then one may express the Laplace transformation, $X(s)$, of $x(t)$ in terms of the Laplace transformations, $U(s)$ and $D(s)$ of $u(t)$ and $d(t)$ respectively:

$$X(s) = \frac{1}{s^2}[U(s)+D(s)] + \frac{sx(0)+\dot{x}(0)}{s^2}. \quad (1)$$

B. Further Assumptions and Formulation of the problem

We make the assumption that the information flow graph is undirected; if a vehicle A transmits the information concerning its state directly to a vehicle B, then vehicle B transmits the information concerning its state directly to vehicle A. Therefore, if S_i is the set of vehicles the i^{th} vehicle in the collection can communicate directly with, this assumption implies that $j \in S_i \Rightarrow i \in S_j$. If the i^{th} vehicle, V_i and the j^{th} vehicle, V_j are in direct communication with each other, we refer to the ordered pair (i,j) as a communication link. We particularly assume that the information available to the i^{th} vehicle in the collection is $x_i(t) - x_j(t) - L_{ij}$, where $j \in S_i$ and L_{ij} is the desired distance to be maintained between the i^{th} and the j^{th} vehicles. We restrict the size of S_i (given by $|S_i|$) to be atmost $q(n)$, which may be a function of the size of the collection.

We also assume that the information flow graph representing the communication pattern is *connected*. By connectedness, we mean that every vehicle in the collection *should be able* to communicate with every other vehicle in the collection, even if there are not communicating directly, through a sequence of already existing communication links. We further assume that the structure of the control law used by each vehicle, other than the reference vehicle, is the same. Specifically, we consider the following structure

$$U_i(s) = -C(s) \sum_{j \in S_i} (X_i(s) - X_j(s) - \frac{L_{ij}}{s}), \quad (2)$$

where $C(s)$ is a rational scalar transfer function.

Let $x_{ref}(t) \in \Re$ be the position of the reference vehicle at time t. The desired position $x_{i,des}(t)$ is related to the position of the reference vehicle x_{ref} through a constant offset L_i, i.e., $x_{i,des}(t) - x_{ref}(t) - L_i \equiv 0$. We define the error in spacing, $e_i(t)$ of the i^{th} vehicle to be the deviation of its position from the desired position, i.e.,

$$e_i(t) := x_i(t) - x_{i,des}(t) = x_i(t) - x_{ref}(t) - L_i.$$

Since the desired formation corresponds to the vehicles moving as a rigid body in a pure translational maneuver, the desired deviation $L_{ij} := x_{i,des}(t) - x_{j,des}(t)$ is constant throughout the motion and equals $L_i - L_j$.

Let $E_i(s)$ be the Laplace transformation of the error in spacing, $e_i(t)$ of the i^{th} vehicle. Let $x_{ref}(t)$ be the position of the reference vehicle at a time t and let $\bar{x}(t) := x_{ref}(t) - x_{ref}(0)$ be the displacement of the reference vehicle from its initial position at the time t. Then $X_{ref}(s) = \frac{x_{ref}(0)}{s} + \bar{X}(s)$. If all the initial positions of the vehicles were chosen to correspond to the rigid formation, then $x_i(0) - x_{ref}(0) - L_i \equiv 0$. With such a choice of initial conditions and the choice of control law given in equation (2) for the plant described by equation (1), evolution equations for the errors in spacing can be expressed compactly as:

$$[I_{n-1} + \frac{C(s)}{s^2}K_1]E(s) = \frac{1}{s^2}D(s) + \tilde{X}(s), \quad (3)$$

where $E(s)$ and $D(s)$ are the respective Laplace transformations of the vector of errors of the following vehicles and the disturbances acting on them. The term $\tilde{X}(s)$ is a vector of dimension $n-1$ and every element of this vector is $\bar{X}(s)$. The term I_{n-1} is an identity matrix of dimension $n-1$ and K_1 is the principal minor obtained by removing the first row and column of the Laplacian K of the information flow graph defined as follows: For $j \neq i$, $K_{ij} = -1$ if vehicles i and j communicate directly; otherwise $K_{ij} = 0$. The i^{th} diagonal element is defined as $K_{ii} = -\sum_{j \neq i} K_{ij}$. The Laplacian K is essentially the stiffness matrix obtained by connecting springs of unit spring constant between vehicles that communicate directly.

Fax and Murray [11] have considered a control law for the i^{th} vehicle of the following form, which is different from the control law considered in this paper in equation (2):

$$U_i(s) = -\frac{1}{|S_i|} C(s) \sum_{j \in S_i} (E_i(s) - E_j(s)) \tag{4}$$

This kind of control law for a vehicle essentially averages the feedback information from all the vehicles directly communicating with it. With this choice of control law and the plant described by Equation 1 the equations for errors in spacing can be written as:

$$E_i(s) = \frac{1}{s^2}[-\frac{1}{|S_i|} C(s) \sum_{j \in S_i} (E_i(s) - E_j(s)) + D_i(s)] + \bar{X}(s). \tag{5}$$

The corresponding error propagation equation may be compactly written as:

$$[I_{n-1} + \frac{C(s)}{s^2} L^{-1} K_1] E(s) = \frac{1}{s^2} D(s) + \tilde{X}(s), \tag{6}$$

where L is the diagonal of K_1.

C. Problem Formulation

The following are the objectives of the control law given by equation (2):

1) In the absence of any disturbance on every vehicle in the formation, it is desired that for every $i \geq 2$, $\lim_{t \to \infty} e_i(t) = 0$, when the reference vehicle executes a maneuver where its speed asymptotically reaches a constant value.
2) In the presence of disturbances of atmost unit in magnitude, it is desirable that there exist a constant $M_R > 0$ such that $\max\{|e_i(t)|, |\dot{e}_i(t)|\} \leq M_R$ for every size of the collection and for every $t \geq 0$.

The second objective ensures that the control law given by equation (2) is scalable. In fact, the second objective may be analyzed for the case when the reference vehicle is stationary, because the collection of vehicles can be treated as an LTI system. The problem is to determine conditions on the information flow graph (through constraints on K_1) and on the controller (through constraints on $C(s)$) so that these two objectives are met.

D. Analysis

Let us analyze the first requirement: Since the speed of the reference vehicle reaches a constant value, say v_f asymptotically, we have: $\lim_{t \to \infty} \dot{\bar{x}}(t) = v_f = \lim_{s \to 0} s^2 \bar{X}(s)$. Therefore, we will have: $\lim_{s \to 0} s^3 \bar{X}(s) = 0$. Further, for the analysis of the requirement, we have $D(s) \equiv 0$. If $\det[I_{n-1} + \frac{C(s)}{s^2} K_1]$ is Hurwitz, we have:

$$\begin{aligned} \lim_{s \to 0} sE(s) &= \lim_{s \to 0} [I_{n-1} + \frac{C(s)}{s^2} K_1]^{-1} s\tilde{X}(s), \\ &= \lim_{s \to 0} [s^2 I_{n-1} + C(s) K_1]^{-1} \lim_{s \to 0} s^3 \tilde{X}(s) = 0. \end{aligned}$$

Therefore, the steady state error requirement is readily met if $\det[I_{n-1} + \frac{C(s)}{s^2} K_1]$ is Hurwitz, i.e., if the controlled motion of formations is stable. The second condition, in fact, concerns the stability of the controlled motion of formations.

We will prove the main result concerning the stability of the controlled motion by using the mechanical analogy between the Laplacian of the information flow graph and the stiffness matrix, which essentially provides a way to address the propagation of errors. A route to instability in structural mechanics, for systems that do not have a rigid body mode, is that the smallest eigenvalue of the stiffness matrix going to zero. In the context of vehicles, the smallest eigenvalue of the Laplacian K is zero, which corresponds to the rigid body mode, i.e., all vehicles have the same non-trivial displacement. A way to get a system without a rigid body mode is to ground one of the vehicles; in our case, for the sake of analysis of propagation of errors, there is no loss of generality in attaching the reference vehicle to the ground, that is, we set $\bar{X}(s) = 0$ from Equation 3.

The mechanical analogy indicates the following line of proof:

1) The smallest eigen value, λ, of K_1 goes to zero as $n \to 0$. Let v be the eigen vector of K_1 corresponding to the eigen value λ.
2) Let the inner product of vector of spacing errors with v be the signal $e_v(t)$. Its Laplace transformation, $E_v(s)$, is given by:

$$\begin{aligned} E_v(s) &= < v, E(s) > & (7) \\ &= < v, [I_{n-1} + \frac{C(s)}{s^2} K_1]^{-1} \frac{1}{s^2} D(s) > & (8) \\ &= \frac{1}{s^2 + \lambda C(s)} D_v(s), & (9) \end{aligned}$$

where $D_v(s) = < v, D(s) >$ and $d_v(t) = < v, d(t) >$, the component of the vector of disturbances acting along the eigen vector v. The mechanical analogy indicates the examination of $e_v(t)$ when $d_v(t)$ is a sinusoid at the first natural frequency or close to the first natural frequency.

1) Convergence of the smallest eigenvalue of K_1: Since K_1 is symmetric, we will use Rayleigh's inequality to get an upper bound for the smallest eigenvalue, λ. For that we construct an assumed mode, v_a in the following way: We keep the reference vehicle grounded and each vehicle to be displaced by one unit. Since, the assumed mode shape indicates the amount by which every mass is displaced, all the elements of v_a are equal. From the use of Rayleigh's inequality, it follows that:

$$\lambda \leq \frac{< v, K_1 v >}{< v, Lv >} \leq \frac{q_r}{n-1} \leq \frac{q(n)}{n-1}$$

In the latter part of the paper, various information flow graphs are considered, where the vehicle communication pattern is randomly assigned subject to the constraint that every vehicle can atmost communicate directly with a pre-specified number of vehicles. The numerical results obtained for them corroborate with the above bound.

Remark 1. *The same bound holds even for the (combinatorial) Laplacian ($L^{-1}K_1$) considered by Fax and Murray [11]. We will start by noting that the eigenvalues of $L^{-1}K_1$ are the same as that of $L^{-0.5}K_1L^{-0.5}$. Since L is a diagonal, positive definite matrix, let $L^{0.5}v = w$. The proof is as follows:*

$$\begin{aligned} \lambda &\leq \frac{< w, L^{-0.5} K_1 L^{-0.5} w >}{< w, w >} \leq \frac{< v, K_1 v >}{< v, Lv >} \\ &\leq \frac{q_r}{n-1} \leq \frac{q(n)}{n-1}. \end{aligned}$$

The second inequality follows from the first because $< v, Lv > \geq < v, v >$ by virtue of the information flow graph being connected and therefore, every diagonal entry of L is greater than or equal to 1.

Now that we have determined an upper bound on the convergence of λ of K_1 to 0, we shall make use of it, to analyze the propagation of errors due to disturbances acting on the vehicles.

2) Analysis of the propagation of errors: We will focus on showing the following: Since $\lambda \to 0$ as $n \to \infty$,

1) If $C(s)$ does not have a pole at zero, there exists a sinusoidal disturbance acting on each vehicle of at most unit amplitude and of frequency proportional to $\sqrt{\lambda}$ that results in amplitudes of errors in spacing of the order of $O\left(\sqrt{\frac{(n-1)^2}{q(n)^3}}\right)$.
2) If $C(s)$ has a pole at zero, then there is a critical size N^* of the collection such that for all $n > N^*$, at least one root of the equation
$$1 + \frac{C(s)}{s^2}\lambda = 0$$
has a positive real part; in other words, the controlled motion of the collection is unstable.

Lemma 1. *If $C(s)$ has a pole at the origin and if $\lambda \to 0$ as the size of the collection, n, goes to ∞, then there exists a critical size N^* of the formation, such that for any size $n > N^*$ of the formation, the motion of the formation will be unstable.*

Proof: For the problem considered in this section, if $C(s)$ has l poles at zero, it can be factored as $C(s) = \frac{L(s)}{s^l}$, $(l > 0)$ for some $L(s)$ that does not have any poles at the origin. We can write the closed loop characteristic equation $\Delta(s)$ as,

$$\Delta(s) := s^{l+2} + \lambda L(s) = 0.$$

We first note that $\Delta(s)$ is Hurwitz only if $L(0) \neq 0$. We further note that $\Delta(s)$ is Hurwitz iff $s^m \Delta(1/s)$ is Hurwitz, where m is the degree of the polynomial $\Delta(s)$. We will now analyze the root locus of $\delta(s) := 1 + \frac{K}{L(1/s)s^{l+2}}, = 1 + \frac{\tilde{L}(s)K}{s^{l+2}}$, where $K := \frac{1}{\lambda}$ and $\tilde{L}(s) = \frac{1}{L(1/s)}$. Since, $\tilde{L}(s)$ is always proper, it is clear that the root locus of $\delta(s)$ has at least $l+2$ asymptotes. Thus, as $K \to \infty$, $(l+2)$ root loci move along lines that make the following angles with the positive real axis.

$$\phi_j = \frac{180^o + 360^o(j-1)}{l+2}, \qquad j = 1, 2, ..., l+2$$

Since $l \geq 1$, it is clear that at least one asymptote, along which one encounters a RHP pole, resulting in the instability of the closed loop as K increases. Hence, if $C(s)$ has more than a pole at origin, it is evident that there exists a critical size N^* of the formation, such that for any size $n > N^*$ of the formation, the motion of the formation will be unstable.

Remark 2. *In the above lemma, if the $C(s)$ does not have any poles at origin i.e., $l = 0$, and if $L(0)$ is negative, as $|s| \to 0$, there is at least one root of $\Delta(s)$ with positive real part. Hence, the motion of the formation will become unstable. Hence, even for $l = 0$ we require $L(0)$ must be positive so as to avoid instability of motion of the formation.*

The following theorem addresses the main result for platoons and it relates the propagation of errors in a platoon due to a disturbance of at most unit magnitude acting on each vehicle.

Theorem 1.
If $C(s)$ does not have a pole at the origin and if $C(0)$ is positive, then errors in spacing grow at least as $O\left(\sqrt{\frac{n^2}{q(n)^3}}\right)$; in other words, no control law of the type considered in this paper is scalable to arbitrarily large collections if $\frac{q(n)^3}{(n)^2} \to 0$ as $n \to \infty$.

Proof:

1) Consider the transfer function that relates E_v to D_v.
$$\frac{E_v}{D_v}(s) \quad = \quad \frac{1}{s^2 + \lambda C(s)}. \qquad (10)$$

Let $C(s) = \frac{N_c}{D_c}(s)$. Since $C(s)$ does not have a pole at zero, $C(0) \neq 0$. Consider a modal disturbance $\tilde{d_v}(t)$ to be a sinusoid of unit amplitude and of frequency $w = \sqrt{\lambda C(0)} rad/s$, then the amplitude of the modal response $\tilde{e}_v(t)$ is given by the magnitude of the following complex number:

$$\frac{1}{\underbrace{w^2\,(1 - \frac{C(jw)}{C(0)})}_{\theta(w)}}.$$

Since $\theta(w)$ defined above has a root at zero, let $|\theta(w)| = w^p|\tilde{\theta}(w)|$, where $\tilde{\theta}(0) \neq 0$ and $p \geq 1$. Therefore, the amplitude ratio is

$$\frac{1}{(\sqrt{\lambda C(0)})^{p+2}} |\frac{1}{\tilde{\theta}(w)}|.$$

As $\lambda \to 0$, the amplitude ratio grows to infinity as

$$\frac{1}{|\tilde{\theta}(0)|} \frac{1}{(\sqrt{\lambda})^{p+2}},$$

where $p \geq 1$. Since $p \geq 1$ as $\lambda \to 0$, $e_v(t)$ grows at least as

$$\frac{1}{|\tilde{\theta}(0)|} \frac{1}{(\sqrt{\lambda})^3}.$$

Since $e_v(t) =< v, e(t) >$, we may express $e_v(t)$ as: $e_v = q_{11}e_1(t) + \ldots + q_{1n}e_n(t)$, for some $q_{11}, \ldots, q_{1n}$. Since v is an eigenvector, we may assume without any loss of generality that $< v, v >= 1$, i.e., $q_{11}^2 + q_{12}^2 + \ldots + q_{1n}^2 = 1$. Each of the errors in spacing is a sinusoid of the frequency, $w = \sqrt{\lambda L(0)}$. Hence, $e_j(t)$ may be expressed as $A_j cos(wt) + B_j sin(wt)$; one may write $e_v = (\sum_{j=1}^n q_{1j}A_j) cos(wt) + (\sum_{j=1}^n q_{1j}B_j) sin(wt)$. It means that either the coefficient of $cos(wt)$ or $sin(wt)$ must increase as $O(\frac{1}{(\sqrt{\lambda})^3})$. Without any loss of generality, let us say that $(\sum_{j=1}^n q_{1j}A_j)$increases in that fashion. Since

$$\begin{aligned} (\sum_{i=1}^n q_{1i}A_i) & \leq (\textstyle\sum_{i=1}^n |q_{1i}|) max_{0<i<n+1}|A_i| \\ \Rightarrow max_{0<i<n+1}|A_i| & \geq O(\tfrac{1}{(\sqrt{\lambda})^3}) \tfrac{1}{||v||_1}. \end{aligned}$$

Since $v \in \Re^{n-1}$, it is true for finite dimensional vectors that $||v||_1 \leq \sqrt{n-1}\ ||v||_\infty \leq \sqrt{n}\ ||v||_\infty$. Since, $||v||_\infty \leq 1$, it follows that $||v||_1 \leq \sqrt{n}$. Therefore, the maximum amplitude of the errors in spacing over all the vehicles for sufficiently large size of the formation is of $O(\frac{1}{(\sqrt{\lambda})^3}) \frac{1}{\sqrt{n}} = O(\frac{1}{(\sqrt{(n)\lambda^3})})$. By Equation (10) we have, $\lambda \leq \frac{q(n)}{n-1}$. Therefore, the errors in the spacing increase as $O\left(\sqrt{\frac{(n)^2}{q(n)^3}}\right)$. Hence, a scalable control algorithm requires an information flow graph, where at least one vehicle in the collection communicates directly with at least $O(n^{\frac{2}{3}})$.

Remark 3. *This theorem may be viewed as a generalization of Theorem 2.3 in Seiler's doctoral thesis [8]. Theorem 2.3 considers* *a string of vehicles moving in a straight line, where each vehicle may only communicate with its neighbors.*

Remark 4. *If the errors were governed by the Equation (5), then the propagation of errors can be analyzed as follows: Since, $L = L^T$ and $K_1 = K_1^T$, we find Q such that $Q^T L Q = I; Q^T K_1 Q = \Lambda$. The simultaneous diagonalization of two symmetric positive definite matrices is dealt in vibrations, where L is commonly referred to as the mass matrix and K_1 is referred to as the stiffness matrix. Let $E_Q(s) = QE(s)$ and similarly, $D_Q(s) = QD(s)$. Then:*

$$Q(L + K_1 \frac{1}{s^2} C(s)) Q^T E(s) = \frac{1}{s^2} Q L Q^T D(s)$$

By the orthogonality relationship, we have:

$$(I + \Lambda \frac{1}{s^2} C(s)) E_Q(s) = \frac{1}{s^2} D_Q(s).$$

Now that the equations are decoupled, let us call the first element of $E_Q(s)$ be $E_v(s)$. This will enable us to write the above equation as:

$$\tilde{E}_v(s) \quad = \quad \frac{\frac{1}{s^2}}{1 + \lambda_i \frac{1}{s^2} C(s)} \tilde{D}_v(s),$$

where D_v is the first element of D_Q. But, we have shown in Theorem 1 that for equation in above form, $|e_v(t)|$ is of $O(\frac{1}{\sqrt{\lambda}^{l+1}})$. Since $e_v(t) =< v, e(t) >$, we may write it as

$$\begin{aligned} e_v(t) \quad &= \quad < L^{0.5} v, L^{-0.5} e(t) > \\ &\leq \quad ||L^{0.5} v||_2 ||L^{-0.5} e(t)||_2 \\ &\leq \quad ||L^{-0.5} e(t)||_2 \quad (v^T L v = 1) \\ &\leq \quad \bar{\sigma}(L^{-0.5}) ||e(t)||_2 \\ &\leq \quad p\sqrt{n-1} ||e(t)||_\infty < p\sqrt{n} ||e(t)||_\infty. \end{aligned}$$

where, v is the first column of Q and $p = \bar{\sigma}(L^{-0.5}) = \frac{1}{\sqrt{min_i |S_i|}}$, $i = \{1, 2...n-1\}$. Since we are considering information flow graphs which are connected, p is well-defined and $p \leq 1$. Therefore, $||e(t)||_\infty$ increases at least as $O(\frac{1}{\sqrt{n}} \frac{1}{\sqrt{\lambda}^3}) = O(\sqrt{\frac{n^2}{q^3(n)}})$, for considerable large collections. Hence, it is evident from here that at least one vehicle in the connection should communicate at least $O(n^{2/3})$, for a scalable controller to exist.

III. Simulations

For the purposes of numerical simulation, we consider the motion of collection of vehicles moving in a straight line. Each vehicle is assumed to be a point mass. As mentioned earlier , the control law used is as follows: $U_i(s) = \sum_{j \in S_i} C(s)(x_i - x_j - L_{ij})$, where $j \in S_i$ implies that there exists a communication link between i^{th} vehicle and j^{th} vehicle. We consider a string of vehicles moving in a straight line trying to maintain constant distance amongst them. We describe the corresponding results below:

String of Vehicles

We consider six vehicles, indexed from 1 to n. The set of vehicles that the first vehicle communicates with directly is the second vehicle, i.e. $S_1 = \{2\}$. For $i = 2, \ldots, n-1$, the set S_i of vehicles the i^{th} vehicle communicates with directly is $\{i-1, i+1\}$ and $S_n = \{n-1\}$. A lag controller is used for feeding back the error in spacing and is given by $C(s) = \frac{3s+2}{0.01s+1}$. Figure 1 shows the convergence of λ to 0 as the length of the string increases. Figure 3 shows the propagation of errors in spacing in a string of six vehicles. It shows how errors amplify in response to a sinusoidal

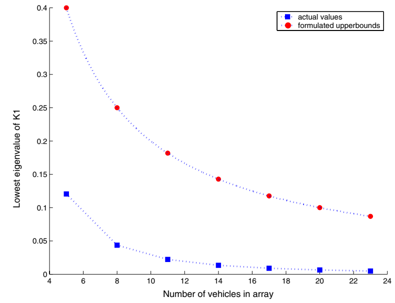

Fig. 1. The variation of λ (lowest eigenvalue of K_1) with n, for a string of n vehicles with each vehicle connected to the vehicles directly behind and ahead of it.

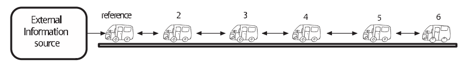

Fig. 2. Predecessor and follower based information flow pattern in the string

disturbance acting on the last vehicle along the string, as we move away from the reference vehicle (vehicle indexed 1). The maximum error in spacing increases as n^3 as the size n of the string increases. The Figure 4 shows an example of the effect stated in Theorem 1. This plot shows the disturbance to error gain as a function of frequency. As predicted, the steady state as well as the peak gain increases as N increases.The Figure 5 shows the same effect.

The above simulations are repeated with randomly generated information flow architectures. The convergence of λ to 0 for various random graphs with a maximal degree constraint of 4 is shown in Figure 6. It can be observed that though the information flow graphs are random, the upper bound on λ of K_1 seems to hold good for all the cases even for a small size of the collection.

To illustrate the limitations in the sizes of collection that can be considered when an integral action is included in the controller, we consider a controller described by the following transfer function *e.g* $C(s) = \frac{3s^2+2s+1}{s(0.01s+1)}$. However, this strategy will not assure the stability of the motion of the collection of vehicles as shown in Lemma 1. Figure 7 shows the migration of dominant pole to the right half plane as the size of the collection of vehicles increases.

IV. Conclusions

In this paper, we have considered information flow graphs for a collection of vehicles, where there is a constraint on the

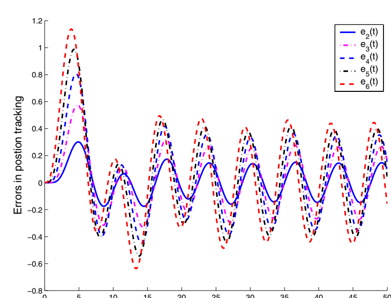

Fig. 3. Propagation of the errors along the string

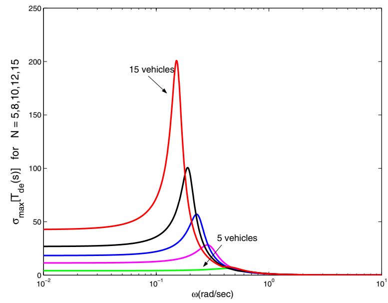

Fig. 4. Variation of the maximum singular value ($\bar{\sigma}$) of the error to disturbance transfer function matrix vs. the size of the string

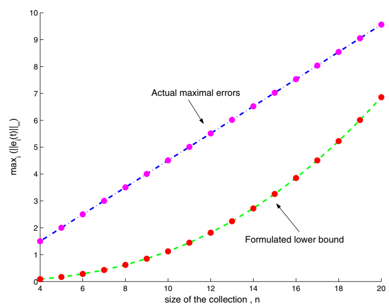

Fig. 5. Variation of maximum spacing error with the size of the string

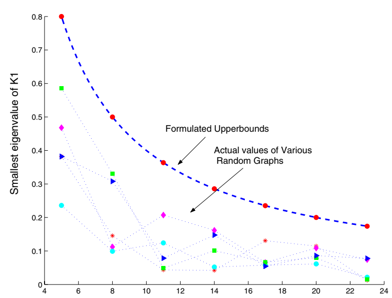

Fig. 6. Variation of λ with n , for a string of n vehicles, connected in a random fashion to a maximum of 4 other vehicles

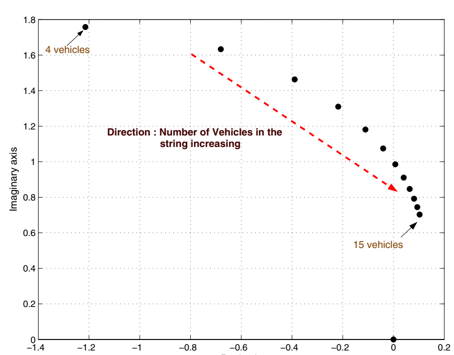

Fig. 7. Plot showing the migration of the dominant pole towards Imaginary axis with increase in the number of vehicles

maximum number of vehicles in the collection every vehicle can communicate with directly. We have related how the smallest eigenvalue λ of a principal minor of the Laplacian of information flow graph goes to zero. We then showed that the motion of vehicles is unstable if the controller transfer function $C(s)$ had one or more poles at the origin and that it must have no zeroes at the origin to track ramp inputs resulting from the reference vehicle moving at a constant velocity. We further showed that if $\lambda \to 0$, there is a disturbance of sufficiently low frequency acting on each vehicle of at most unit magnitude which results in errors in spacing of $O\left(\sqrt{\frac{(n)^2}{q(n)^3}}\right)$.The extension of the above results for higher dimensional formations and the problem of synthesis of scalable control laws for maintenance of rigid formations is currently underway and will be reported in [14].

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