2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Robust Capacity of a Gaussian Noise Channel with Channel and Noise Uncertainty

Charalambos D. Charalambous, Stojan Z. Denic, and Seddik M. Djouadi

***Abstract*— This paper is concerned with the problem of defining, and computing the capacity of a continuous-time additive Gaussian noise communication channel when the true frequency response of the channel, and the power spectral density of the noise are not perfectly known, and the transmitted signal is a wide-sense stationary process constrained in power. The uncertainties of a true channel frequency response and power spectral density of the noise are described by weighted balls in the H^∞ space. In that way two sets are defined that describe the set of all possible channel frequency responses, and the set of all possible power spectral densities of the noise. The ball radii depend on the degree of uncertainty that one has about the true channel frequency response, and power spectral density of the noise. The channel capacity is defined as the max-min-min of a mutual information rate between transmitted, and received signals, where the first minimum is taken over the set of all possible noises, the second minimum is taken over the set of all possible channel frequency responses, and maximum is over the set of all possible power spectral densities of transmitted signal with constrained power. It is shown that such defined channel capacity, called robust capacity, is equal to the operational capacity that represents the theoretical maximum of all attainable rates over a given channel.**

I. INTRODUCTION

In the classical information, and communication theory, it is assumed very often that the communication channel is perfectly known to a transmitter and receiver. That means that both transmitter, and receiver are perfectly aware of all channel parameters such as the parameters of the channel frequency or impulse response, and the statistic of the noise. Although this may be true for some communication channels when it is possible to measure a channel with high accuracy, there are many situations when the channel is not perfectly known to the transmitter, and receiver, which affects the performance of a communication system. Some examples of communication systems with channel uncertainty include wireless communication systems, communication networks, communication systems in the presence of jamming.

This work was supported by the European Commission under the project ICCCSYSTEMS and the NSERC under an operating grant T-810-289-01.

C. D. Charalambous is with the School of Information Technology and Engineering, University of Ottawa, 161 Louis Pasteur, A519, Ottawa, Ontario K1N 6N5, Canada. Also with Electrical and Computer Engineering Department, University of Cyprus, 75 Kallipoleos Avenue, Nicosia, Cyprus, chadcha@site.uottawa.ca

S. Z. Denic is with the School of Information Technology and Engineering, University of Ottawa, 800 King Edward Ave., Ottawa, Ontario K1N 6N5, Canada, sdenic@site.uottawa.ca

S. M. Djouadi is with the Electrical and Computer Engineering Department, University of Tennessee, 414 Ferris Hall, Knoxville, TN 37996-210, USA, djouadi@ece.utk.edu

Here, we will mention just a few papers from the large body of the papers that were published on the topic of communications under uncertainties. Some of the earliest papers include [5], [4], [7]. Blackwell *et al.* [5] determined the capacity of compound discrete memoryless channels. Blachman [4], and Dobrushin [7] were first to pose the channel capacity (in presence of channel uncertainty) problem in game theoretic framework, taking mutual information as a pay-off function. In the seventies, Ahlswede [1] worked on the problem of an arbitrary varying Gaussian channel when the noise is an i.i.d. sequence, where the noise variance varies but does not exceed a certain bound. In the eighties, McEliece [8] considered the existence of saddle points and optimal transmitter and jammer's strategies for continuous discrete time communication channels for mutual information as a pay-off function. Hughes, and Narayan, [9], defined several problems depending on the constraints imposed on the transmitted signal, and unknown interference for Gaussian arbitrary varying channels (GAVC). Basar and Wu [3] employed a game theoretic approach but for mean-square error as a pay-off function. Deggavi, and Cover [6] considered vector channels with noise covariance matrix constraint. They proved that the worst additive noise in the class of lag p covariance constraints is the maximum entropy noise, i.e., the Gauss-Markov noise. For an example of capacity of MIMO channels see also [10].

The papers that are related to our work are those of Root and Varaiya [17], Baker and Chao [2], as well as the work of Gallager [15]. Root and Varaiya proved the coding theorem for the class of Gaussian channels but subject to white Gaussian noise. We proved that their approach still holds under certain conditions for colored Gaussian noise [16]. This enables the computation of the channel capacities when the channel, noise or both are uncertain. In [2], as opposed to our case, different constraints on the transmitted and noise signals (energy constraints in terms of covariance functions) are assumed, and derivation of the capacity formula relies on the saddle point. Further, [2] considers the channel capacity in the presence of just noise uncertainty, while we can deal with any combination of noise, and channel uncertainty.

The above discussion partially explains the importance of channel uncertainty in communications. An interested reader is referred to papers [11], [12], [13], that give an excellent overview of the topic and represent good sources of important references.

The channel capacity, called robust capacity, in the presence of channel frequency response uncertainty, and noise uncertainty will be defined and an explicit formula for robust capacity is derived. The problem of defining and computing the robust capacity is alleviated by using appropriate uncertainty models for the channel and noise. In this paper, a basic model borrowed from robust control theory is used [14]. In particular, the same additive uncertainty model of a transfer function is employed to model the uncertainty of the channel frequency response and the uncertainty of the power spectral density of the noise. This type of uncertainty modeling gives an explicit formula for robust capacity that describes how the channel capacity decreases when the uncertainty of the channel frequency response, and power spectral density of the noise increase. The other important result stemming from the robust capacity formula is the water-filling equation that shows the effect of uncertainty on the optimal transmitted power. Finally, it is shown that there exists a code that enables reliable transmission over the channel with uncertainty if the code rate is less than the robust capacity, and that the latter as defined in the paper is equal to the operational capacity. The derived formula for the robust capacity can be applied for any communication channel, which is characterized by its impulse response. It is important to notice that it can be applied for wireless channels as well, by randomizing the parameters of a frequency response, and then taking the expected value over the randomness induced by the frequency response random parameters.

II. COMMUNICATION SYSTEM MODEL

The model of communication system is depicted in Fig. 1. The transmitted signal $\mathbf{x} = \{x(t); -\infty < t < +\infty\}$, received signal $\mathbf{y} = \{y(t); -\infty < t < +\infty\}$, and noise signal $\mathbf{n} = \{n(t); -\infty < t < +\infty\}$ are wide-sense stationary processes with power spectral densities $S_{\mathbf{x}}(f)$, $S_{\mathbf{y}}(f)$, $S_{\mathbf{n}}(f)$, respectively. The transmitted signal $\mathbf{x}$ is constrained in power, and noise $\mathbf{n}$ is an additive Gaussian noise. The filter with frequency response $\tilde{W}(f)$ shapes the power spectral density of the noise $S_{\mathbf{n}}(f)$.

The uncertainty in the channel frequency response $\tilde{H}(f)$, and overall power spectral density of the noise $S_{\mathbf{n}}(f)|\tilde{W}(f)|^2$ is modeled through the additive uncertainty model of the filters $\tilde{H}(f)$, and $\tilde{W}(f)$. The additive uncertainty model of any transfer function $\tilde{G}(f)$ is defined by $\tilde{G}(f) = G_{nom}(f) + \Delta(f)W_1(f)$, where $G_{nom}(f)$ represents the nominal transfer function that can be chosen such that it reflects one's limited knowledge or belief regarding the transfer function $\tilde{G}(f)$. The second term represents a perturbation where $W_1(f)$ is a fixed known transfer function, and $\Delta(f)$ is unknown transfer function with $\|\Delta(f)\|_\infty \leq 1$. The norm $\|.\|_\infty$ is called the infinity norm, and it is defined as $\|\tilde{G}(f)\|_\infty = \sup_f |\tilde{G}(f)|$. The set of all stable transfer functions that have a finite $\|.\|_\infty$ norm is denoted as H^∞, and it can be proven that this space is a Banach space. All transfer functions mentioned until now belong to this normed linear space H^∞. It should be noted that uncertainty in the frequency response of the filter $\tilde{G}(f)$ can be seen as a ball in the frequency domain $|\tilde{G}(f) - G_{nom}(f)| \leq |W_1(f)|$, where the center of the ball is the nominal transfer function $G_{nom}(f)$, and the radius is defined by $|W_1(f)|$. Thus, the amplitude of uncertainty varies with the frequency, and it is determined by the fixed transfer function $W_1(f)$. The larger $|W_1(f)|$, the larger the uncertainty. The transfer functions $G_{nom}(f)$, and $W_1(f)$ can be determined from measured data. Based on this uncertainty model robust capacity is defined and computed in the following section.

III. ROBUST CAPACITY

Define the three following sets

$$\begin{aligned} A_1 := & \Big\{ S_{\mathbf{x}}(f) & ; \int S_{\mathbf{x}}(f) df \leq P \Big\} \\ A_2 := & \Big\{ \tilde{H}(f) & \in H^\infty; \tilde{H} = H_{nom} + \Delta_1 W_1; \\ & H_{nom} & \in H^\infty, W_1 \in H^\infty, \Delta_1 \in H^\infty, \\ & \|\Delta_1\|_\infty & \leq 1 \Big\} \end{aligned}$$

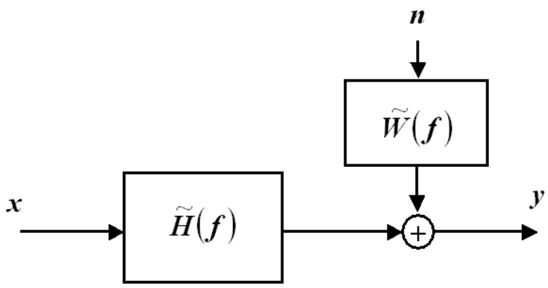

Fig. 1. Communication model

$$
\begin{aligned}
A_3 := \quad & \Big\{ \tilde{W}(f) && \in H^{\infty}; \tilde{W} = W_{nom} + \Delta_2 W_2; \\
& W_{nom} && \in H^{\infty}, W_2 \in H^{\infty}, \Delta_2 \in H^{\infty}, \\
& \|\Delta_2\|_{\infty} && \leq 1 \Big\}.
\end{aligned}
$$

The set A_1 is the set of all possible power spectral densities of transmitted signal. The sets A_2, and A_3 are the uncertainty sets that determine the set of all possible channel frequency responses $\tilde{H}(f)$, and the set of all possible overall power spectral densities $S_{\mathbf{n}}(f)|\tilde{W}(f)|^2$, respectively. The sizes of uncertainty sets are determined by the fact that $\|\Delta_1\|_{\infty}, \|\Delta_2\|_{\infty} \leq 1$.

Definition 3.1: The robust capacity of a continuous time additive Gaussian channel, when a transmitted signal $\mathbf{x}$ is subject to the power constraint $\int S_{\mathbf{x}}(f)df \leq P$, where the channel frequency response uncertainty is defined through the set A_2, and where the uncertainty of the power spectral density of the noise is determined by A_3, is defined by

$$
C_R := \sup_{S_{\mathbf{x}} \in A_1} \inf_{\tilde{H} \in A_2} \inf_{\tilde{W} \in A_3} \frac{1}{2} \int_{\infty}^{\infty} \log\left(1 + \frac{S_{\mathbf{x}}|\tilde{H}|^2}{S_{\mathbf{n}}|\tilde{W}|^2}\right) df. \quad (1)
$$

Clearly, the robust capacity definition is a variant of the Shannon capacity for additive Gaussian continuous time channels, subject to input power and frequency constraints [15], [17].

Theorem 3.2: Consider an additive uncertainty description of $\tilde{H}(f)$ and $\tilde{W}(f)$, and suppose that $\frac{(|H_{nom}(f)|+|W_1(f)|)^2}{S_{\mathbf{n}}(|W_{nom}(f)|-|W_2(f)|)^2}$ is bounded, integrable, and that $|H_{nom}(f)| \neq |W_1(f)|$, and $|W_{nom}(f)| \neq |W_2(f)|$. Then the following hold.

1) The robust capacity of an additive Gaussian continuous time channel with additive uncertainty employed to model the channel frequency response uncertainty, and the uncertainty of the power spectral density of the noise is given parametrically by

$$
C_R = \frac{1}{2} \int \log\left(\frac{\nu^*(|H_{nom}| - |W_1|)^2}{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}\right) df, \quad (2)
$$

where ν^* is a Lagrange multiplier found via

$$
\int \left(\nu^* - \frac{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}{(|H_{nom}| - |W_1|)^2}\right) df = P, \quad (3)
$$

subject to the condition

$$
\begin{aligned}
&\nu^*(|H_{nom}| - |W_1|)^2 - S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2 > 0 \\
&\qquad\qquad\qquad\qquad\qquad \nu^* > 0, \quad (4)
\end{aligned}
$$

in which the integrations in (2), and (3) are over the frequency interval over which the condition (4) holds.

2) The infimum over the noise uncertainty in (1) is achieved at

$$
\begin{aligned}
\Delta_2^* &= \exp[-j\arg(W_2) + j\arg(W_{nom})] \\
&\qquad\qquad \|\Delta_2^*\|_{\infty} = 1,
\end{aligned}
$$

and the resulting mutual information rate after minimization is given by

$$
\begin{aligned}
\inf & \int \log\left(1 + \frac{S_{\mathbf{x}}|\tilde{H}|^2}{S_{\mathbf{n}}|W_{nom} + \Delta_2 W_2|^2}\right) df \\
&= \int \log\left(1 + \frac{S_{\mathbf{x}}|\tilde{H}|^2}{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}\right) df,
\end{aligned}
$$

where the infimum is over $\|\Delta_2\|_{\infty} \leq 1$.

3) The infimum over the channel frequency uncertainty in (1) is achieved at

$$
\begin{aligned}
\Delta_1^* &= \exp[-j\arg(W_1) + j\arg(H_{nom}) + j\pi] \\
&\qquad\qquad \|\Delta_1^*\|_{\infty} = 1,
\end{aligned}
$$

and the resulting mutual information rate after minimization is given by

$$
\begin{aligned}
\inf & \int \log\left(1 + \frac{S_{\mathbf{x}}|H_{nom} + \Delta_1 W_1|^2}{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}\right) df \\
&= \int \log\left(1 + \frac{S_{\mathbf{x}}(|H_{nom}| - |W_1|)^2}{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}\right) df,
\end{aligned}
$$

where the infimum is over $\|\Delta_1\|_{\infty} \leq 1$.

4) The supremum over A_1 yields the water-filling equation

$$
S_{\mathbf{x}}^* + \frac{S_{\mathbf{n}}(|W_{nom}| + |W_2|)^2}{(|H_{nom}| - |W_1|)^2} = \nu^*. \quad (5)
$$

Proof. The condition that $\frac{(|H_{nom}(f)|+|W_1(f)|)^2}{S_{\mathbf{n}}(f)(|W_{nom}(f)|-|W_2(f)|)^2}$ is bounded, and integrable is necessary to provide the existence of the integral in (1) for each $\tilde{H} \in A_2$, and each $\tilde{W} \in A_3$ (see Lemma 8.5.7, [15], page 423). Further, the infimum over the set A_2 is obtained as follows. It is possible to prove that

$$\inf_{\tilde{H}\in A_2} \frac{1}{2}\int \log\left(1+\frac{S_{\mathbf{x}}(f)|\tilde{H}(f)|^2}{S_{\mathbf{n}}(f)|\tilde{W}(f)|^2}\right)df$$
$$\geq \frac{1}{2}\int \log\left(1+\frac{S_{\mathbf{x}}(f)(|H_{nom}(f)|-|W_1(f)|)^2}{S_{\mathbf{n}}(f)|\tilde{W}(f)|^2}\right)df, \quad (6)$$

as well as the opposite inequality implying that (6) holds with equality. The infimum over A_3 is resolved in a similar manner. The problem of finding the supremum is exactly the same as in the case without uncertainty [15]. So, by applying the calculus of variation the capacity formula (2) is obtained accompanied by the modified water-filling formula (5) and power constraint (3). ν^* is a positive Lagrange multiplier, and is obtained as a solution of (3). The integrations in previous formulas are defined over the frequencies for which (4) holds.

The formula (2) shows how the channel frequency response uncertainty, and noise uncertainty affect the capacity of the communication channel. To understand this point better assume that the noise $\mathbf{n}$ is a white Gaussian noise with $S_{\mathbf{n}}(f) = 1W/Hz$ over all frequencies such that the overall power spectral density of the noise is $|\tilde{W}|^2$. It can be seen that the capacity depends on the two fixed, and known transfer functions $W_1(f)$, and $W_2(f)$, first determining the size of the channel frequency response uncertainty set, and second determining the size of the noise uncertainty set. From (2), one could conjecture that the channel capacity decreases when the size of uncertainty sets $W_1(f)$, and $W_2(f)$ increase. This is an intuitive result because the channel capacity should be determined by the worst case channel and noise. This follows from the definition of the channel capacity which determines a single code that should be good for each channel, and the noise from uncertainty sets. Thus, a single code should be good for the worst channel, and the noise, as well. But, it cannot be guaranteed that the code, which is good for the worst channel, and the noise will also be good for the rest of the channels, and noises. That's why the robust channel capacity could be less than the channel capacity of the worst channel.

Here, we make some comments on the relations between $|H_{nom}|$, and $|W_1(f)|$, and between $|W_{nom}|$, and $|W_2(f)|$. It is reasonable to assume that in practical cases, $|H_{nom}| \geq |W_1(f)|$, and $|W_{nom}| \geq |W_2(f)|$, because the uncertainty could represent the errors in channel, and noise estimations. Thus, the second term in logarithm could go to zero, implying zero capacity, just if the channel and/or noise estimation are very poor.

IV. CHANNEL CODING AND CONVERSE TO CHANNEL CODING THEOREM

In this section, it is shown that under certain conditions the coding theorem, and its converse hold for the set of communication channels with uncertainties defined by sets A_2, and A_3. It means that there exists a code, whose code rate R is less than the robust capacity C_R given by (2), for which the error probability is arbitrary small over the sets A_2, and A_3. This result is obtained in [16], by combining two approaches found in [15], and [17].

First define the frequency response of the equivalent communication channel by $G(f) = \left(\frac{S_{\mathbf{x}}(f)|\tilde{H}(f)|^2}{S_{\mathbf{n}}(f)|\tilde{W}(f)|^2}\right)^{1/2}$, and denote its inverse Fourier transform by $g(t)$. Further define two sets $\mathcal{A}$ and $\mathcal{B}$ as follows $\mathcal{A} := \{G(f); \tilde{H}(f) \in A_2, \tilde{W}(f) \in A_3\}$, $\mathcal{B} := \{g(t); G(f) \in \mathcal{A}, g(t)$ satisfies 1),2),3)$\}$ where

1) $g(t)$ has finite duration δ,
2) $g(t)$ is square integrable ($g(t) \in L_2$),
3) $\int_{-\infty}^{-\alpha} |G(f)|^2 df + \int_{\alpha}^{+\infty} |G(f)|^2 df \rightarrow 0$ when $\alpha \rightarrow +\infty$.

The set of all $g(t)$ that satisfy these conditions is conditionally compact set in L_2 (see [17]), and this enables the proof of coding theorem, and its converse. Note that the condition 1) can be relaxed (see Lemma 4 [18]). Now, the definition of the code for the set of channels $\mathcal{B}$ is given as well as the definition of the attainable rate R, and operational capacity C. The channel code (M, ϵ, T) for the set of communication channels $\mathcal{B}$ is defined as the set of M distinct time-functions $\{x_1(t), ..., x_M(t)\}$, in the interval $-T/2 \leq t \leq T/2$, and the set of M disjoint sets $\{D_1, ..., D_M\}$ of the space of a received signal $\mathbf{y}$ such that

$$\frac{1}{T}\int_{-T/2}^{T/2} x_k(t)dt \leq P$$

for each k, and such that the error probability for each codeword is $\Pr\big(\mathbf{y} \in D_k^c | x_k(t) \text{ sent}\big) \leq \epsilon$, $k = 1, ..., M$, for all $g(t) \in \mathcal{B}$. The positive constant R is called attainable coding rate if there exists a sequence of codes $\{(M, \epsilon_n, T_n)\}$, $M = \exp[T_n R]$, such that when $n \rightarrow +\infty$, then $T_n \rightarrow +\infty$, and $\epsilon_n \rightarrow 0$ uniformly for all $g(t) \in \mathcal{B}$. Here ϵ_n is the codeword probability of error as previously defined, and T_n is a codeword time duration. The operational capacity C represents the supremum of all attainable rates R [17].

Theorem 4.1: The operational capacity C for the set of communication channels $\mathcal{B}$ is given by (2), and is equal to robust capacity C_R.

Proof. The proof follows from [15], and [17]. For details see [16].

V. EXAMPLES

A. Noise Uncertainty

In the first example, we assume that channel is completely known, $|W_1| = 0$, and $S_{\mathbf{n}} = 1W/Hz$. The noise uncertainty set is defined by $\tilde{W}(f) = \frac{\xi_p(f)}{j2\pi f/\beta+1}$, where $\xi_p(f) = \xi + \Delta_2(f)\delta\xi$, $\xi = \alpha/\beta$, $0 \leq \delta < 1$, and $W_{nom}(f) = \frac{\xi}{j2\pi f/\beta+1}$. Thus, $|\tilde{W}(f) - W_{nom}(f)| = |\frac{\Delta_2(f)\delta\xi}{j2\pi f/\beta+1}| \leq |\frac{\delta\xi}{j2\pi f/\beta+1}| = |W_2(f)|$, and the uncertainty set is the ball in frequency domain centered at $|W_{nom}(f)|$, and with radius $W_2(f)$. The radius, i.e., the size of uncertainty set is determined by parameter δ. The channel is modelled as a second order transfer function $H(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $s = j\omega = j2\pi f$. The parameters are chosen as follows, $\alpha = 1$, $\beta = 1000$ rad/s, $\xi = 0.3$, and $P = 0.01W$. Fig. 2 shows that the robust capacity indeed decreases with the size of uncertainty set determined by δ, where the slope is larger for small uncertainty. Fig. 3, shows the optimal power spectral density of transmitted signal $S^*_{\mathbf{x}}(f)$ for different values of δ. It seems that the transmitter tries to fight against uncertainty by reducing the bandwidth, and in the same time by regrouping the power towards the lower frequencies.

B. Channel - Noise Uncertainty

To illustrate the effect of the channel, and noise uncertainty on the capacity, we consider the following example. The channel is modelled by a second order transfer function $H(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $s = j\omega = j2\pi f$, where it is assumed that the damping ratio ξ is unknown (ξ can take values between 0, and 1), whose value is within certain interval, $\xi_{low} \leq \xi \leq \xi_{up}$. This set will be roughly approximated by using the following procedure. We choose the natural frequency to be $\omega_n = 900$ rad/s, nominal damping ratio $\xi_{nom} = 0.3$, and $0.25 \leq \xi \leq 0.4$. Further, the size of uncertainty set is defined by $|W_1| = |H_{nom}| - |H_{low}|$, where $H_{low}(s) = \frac{\omega_n^2}{s^2+2\xi_{up}\omega_n s+\omega_n^2}$, $H_{nom}(s) = \frac{\omega_n^2}{s^2+2\xi_{nom}\omega_n s+\omega_n^2}$. The values of $\xi_{low} = 0.25$, and $\xi_{up} = 0.4$ are deliberately chosen such that $|H_{nom}| + |W_1|$ is a good approximation of $H_{up}(s) = \frac{\omega_n^2}{s^2+2\xi_{low}\omega_n s+\omega_n^2}$. Thus, the frequency response uncertainty set is roughly described by $|H_{nom}| \pm |W_1|$. But, $|W_1| = |H_{nom}| - |H_{low}|$ implies $|H_{low}| = |H_{nom}| - |W_1|$.

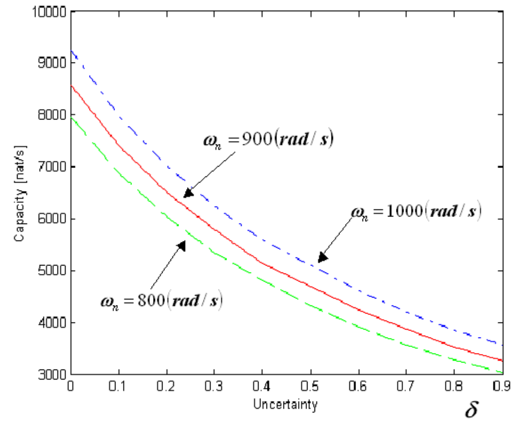

Fig. 2. Robust capacity - noise uncertainty

That means that the robust capacity is determined by the transfer function H_{low}. In the same way, two other uncertainty sets are modeled, for $\xi_{low} = 0.2$, and $\xi_{up} = 0.5$, and $\xi_{low} = 0.18$, and $\xi_{up} = 0.6$, where the rest of the parameters keep their previous values. The uncertainty sets are identified by the range of the damping ratios, $\Delta\xi = \xi_{up} - \xi_{low}$, $\Delta\xi = 0.15$, $\Delta\xi = 0.30$, $\Delta\xi = 0.42$, for the first, second, and third uncertainty set, respectively. Notation $\Delta\xi = 0$ stands for the nominal channel model. The noise uncertainty set is the same as in the case of single noise uncertainty. The power is constrained to P =0.01 W. Fig. 4 depicts the effect of noise uncertainty for different sizes of channel frequency response uncertainty sets. Similarly to the previous example, the channel uncertainty tends to affect the capacity more for lower values of uncertainty. For instance, the distance between the curves $\Delta\xi = 0$, and $\Delta\xi = 0.15$ is larger than the distance between the curves $\Delta\xi = 0.15$, and $\Delta\xi = 0.30$. Fig. 5 reveals that, at least for this set of parameters (P =0.01 W, δ =0.1, $\omega_n = 900$ rad/s, $\alpha = 1$, $\beta = 1000$ rad/s), and fixed noise uncertainty, the channel uncertainty has a little effect on the optimal power spectral density $S^*_{\mathbf{x}}(f)$.

VI. CONCLUSION

This paper considers the capacity problem of a continuous time additive Gaussian noise channels when the true frequency response of the channel, and power spectral density of the noise is not completely known. The channel capacity, called robust capacity, is defined as the max-min-min of a mutual information rate between the transmitted, and received signals, and explicitly computed. Also, a modified water-filling equation is derived showing how the

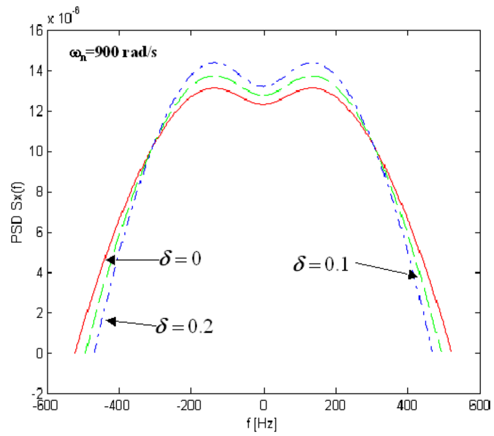

Fig. 3. Optimal psd - noise uncertainty

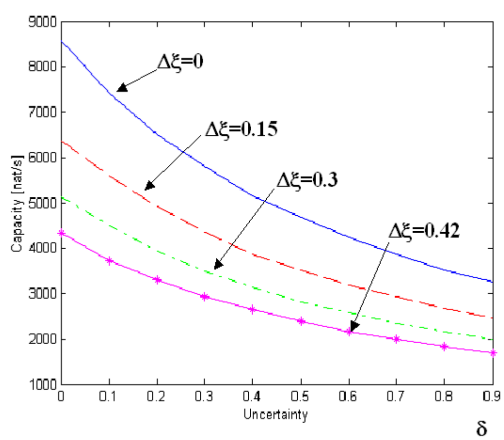

Fig. 4. Robust capacity - channel-noise uncertainty

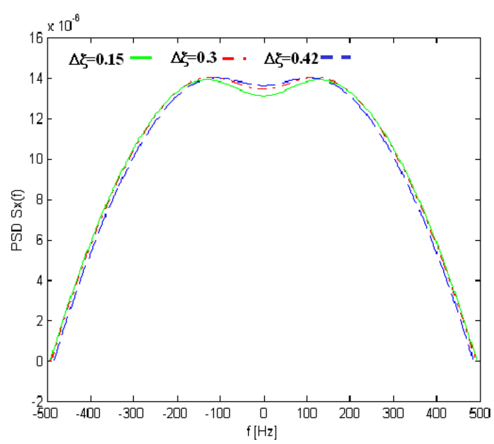

Fig. 5. Optimal psd - channel-noise uncertainty

optimal transmitted power changes with the uncertainty. It is shown that the robust capacity as introduced in the paper is equal to the operational capacity, i.e., the channel coding theorem, and its converse hold.

REFERENCES

[1] Ahlswede, R., "The capacity of a channel with arbitrary varying Gaussian channel probability functions," *Trans. 6th Prague Conf. Information Theory, Statistical Decision Functions, and Random Processes*, pp. 13-31, Sept. 1971.

[2] Baker, C. R., Chao, I.-F., "Information capacity of channels with partially unknown noise. I. Finite dimensional channels," *SIAM J. Appl. Math.*, vol. 56, no. 3, pp. 946-963, June 1996.

[3] Basar, T., "A complete characterization of minimax, and maximin encoder-decoder policies for communication channels with incomplete statistical description," *IEEE Transactions on Information Theory*, vol. 31, pp. 482-489, Jan., 1985.

[4] Blachman, N. M., "Communication as a game," IRE Wescon 1957 Conference Record, vol. 2, pp. 61-66, 1957.

[5] Blackwell, D., Breiman, L., Thomasian, A. J., "The capacity of a class of channels," *Ann. Math. Stat.*, vol. 30, pp. 1229-1241, 1959.

[6] Diggavi, S. N., Cover, T. M., "The worst additive noise under a covariance constraint", *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 3072-3081, November, 2001.

[7] Dobrushin, L., "Optimal information transmission through a channel with unknown parameters", *Radiotekhnika i Electronika*, vol. 4, pp. 1951-1956, 1959.

[8] McElice, R. J., "Communications in the presence of jamming An information theoretic approach", in *Secure Digital Communications*, G. Longo, ed., Springer-Verlang, New York, 1983, pp. 127-166.

[9] Hughes, B., Narayan P., "Gaussian arbitrary varying channels," *IEEE Transactions on Information Theory*, vol. 33, no. 2, pp. 267-284, Mar., 1987.

[10] Vishwanath, S., Boyd, S., Goldsmith, A., "Worst-case capacity of Gaussian vector channels," *Proceedings of 2003 Canadian Workshop on Information Theory*, 2003.

[11] Medard, M., "Channel Uncertainty in Communications," *IEEE Information Society Newsletter*, vol. 53, no. 2, p. 1, pp. 10-12, June 2003.

[12] Biglieri, E., Proakis., J., Shamai, S., "Fading Channels - Information-Theoretic and Communications Aspects," *IEEE Trans. Info. Theo.*, vol. 44, no. 6, pp. 2619-2692, Oct. 1998.

[13] Lapidoth, A., Narayan, P., "Reliable Communication Under Channel Uncertainty," *IEEE Trans. Info. Theo.*, vol. 44, no. 6, pp. 2148-2177, Oct. 1998.

[14] Doyle, J. C., Francis, B. A., Tannenbaum, A. R., *Feedback Control Theory*. New York: Mcmillan Publishing Company, 1992.

[15] Gallager, R., *Information Theory and Reliable Communication*. New York: Wiley, 1968.

[16] Denic, S.Z., Charalambous, C.D., Djouadi, S.M., "Robust Capacity for Additive Colored Gaussian Uncertain Channels," *preprint, http://www.uottawa.ca/ sdenic/colored*

[17] Root, W. L., Varaiya, P. P., "Capacity of Classes of Gaussian Channels," *SIAM J. Appl. Math.*, vol. 16, no. 6, pp. 1350-1393, November 1968.

[18] Forys, L.J., Varaiya, P.P., "The ϵ-Capacity of Classes of unknown channels," *Information and control*, vol. 44, pp. 376-406, 1969.